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# TRANSACTIONS

OF THE

# AMERICAN MATHEMATICAL SOCIETY

EDITED BY

ELIAKIM HASTINGS MOORE

ERNEST WILLIAM BROWN THOMAS SCOTT FISKE

WITH THE COÖPERATION OF

EDWARD BURR VAN VLECK HENRY SEELY WHITE
CHARLES LEONARD BOUTON LEONARD EUGENE DICKSON JOHN IRWIN HUTCHINSON
EDWARD KASNER EDWIN BIDWELL WILSON

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# ON THE LINEAR TRANSFORMATIONS OF A QUADRATIC FORM INTO ITSELF*

BY

PERCEY F. SMITH

The problem of the determination † of all linear transformations possessing an invariant quadratic form, is well known to be classic. It enjoyed the attention of EULER, CAYLEY and HERMITE, and reached a certain stage of completeness in the memoirs of FROBENIUS,‡ VOSS,§ LINDEMANN|| and LOEWY.¶ The investigations of CAYLEY and HERMITE were confined to the general transformation, FROBENIUS then determined all proper transformations, and finally the problem was completely solved by LINDEMANN and LOEWY, and simplified by VOSS.

The present paper attacks the problem from an altogether different point, the fundamental idea being that of building up any such transformation from simple elements. The primary transformation is taken to be *central reflection* in the quadratic locus defined by setting the given form equal to zero. This transformation is otherwise called in three dimensions, point-plane reflection,—point and plane being pole and polar plane with respect to the fundamental quadric. In this way, every linear transformation of the desired form is found to be a product of central reflections. The maximum number necessary for the most general case is the number of variables. VOSS, in the first memoir cited, proved this theorem for the general transformation, assuming the latter given by the equations of CAYLEY. In the present paper, however, the theorem is derived synthetically, and from this the analytic form of the equations of transformation is deduced.

* Presented to the Society December 29, 1903. Received for publication, July 2, 1904.

† The results of §§ 1, 2 were communicated to the American Mathematical Society in December, 1901.

‡ FROBENIUS, *Ueber linear substitutionen und bilinear Formen*, Crelle, vol. 84 (1878), pp. 1–63.

§ VOSS, *Zur theorie der orthogonalen substitutionen*, Mathematische Annalen, vol. 13 (1878), pp. 320–374; *Ueber die cogrediente Transformation der bilinearen formen in sich selbst*, Münchener Berichte (1896), pp. 1–23.

|| LINDEMANN, *Vorlesungen über Geometrie*, vol. 2 (1891), p. 356; *Ueber die linearen Transformationen einer quadratischen Mannigfaltigkeit in sich*, Münchener Berichte (1896), pp. 31–66.

¶ LOEWY, *Ueber die Transformationen einer quadratischen Form in sich selbst*, Leop. Nova Acta, Halle, vol. 65 (1896), pp. 1–66.



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The theory of central reflections developed in § 1 is so simple, and the analytic representation derived therefrom so direct that the present discussion presents a complete solution of the problem which may be regarded as elementary. Furthermore, the equations (24) found in explicit form for every such transformation have not been given elsewhere, as far as the author's knowledge goes.

In §§ 5, 6 are given some applications, and in § 7 the solution of the corresponding problem for the alternating bilinear form. The results of the first six sections apply, of course, to the symmetrical bilinear form.

## § 1. *The theory of central reflections.*

Let

$$f(x) = \sum a_{ik} x_i x_k, \qquad a_{ik} = a_{ki} \qquad (i, k = 1, 2, \cdots, n), \tag{1}$$

be the given quadratic form. The quadratic locus $f(x) = 0$ in linear space $R_{n-1}$ of $n-1$ dimensions may be called after Cayley the "absolute." Any group of $r$ independent sets of coördinates

$$(a_1^{(1)}, a_2^{(1)}, \cdots, a_n^{(1)}), \cdots, (a_1^{(r)}, a_2^{(r)}, \cdots, a_n^{(r)}),$$

determine a linear manifold $M_{n-r-1}$ of $n-r-1$ dimensions, the coördinates $x$ of any point of which are linearly derived from the $a$'s:

$$x_i = \lambda^{(1)} a_i^{(1)} + \lambda^{(2)} a_i^{(2)} + \cdots + \lambda^{(r)} a_i^{(r)} \qquad (i = 1, 2, \cdots, n).$$

The involutory transformation which forms the basis of the discussion is the "central reflection" of Voss (loc. cit.), or the point-plane reflection, as it is often called, point and plane standing in the relation of pole and polar with respect to $f(x) = 0$. I shall denote by $\{\mathbf{a}\}$ the reflection in the point $a$ and its polar plane, and by $x\{\mathbf{a}\}x'$ the fact of $x$ transforming into $x'$, from which also, since the transformation is involutory, will follow $x'\{\mathbf{a}\}x$. From the definition of the transformation $x'$, $x$ and $a$ are collinear, hence $x' = x + \lambda a$, and since by hypothesis, $f(x') \equiv f(x)$, we find for the equations of $\{\mathbf{a}\}$

$$x' = x - \frac{2f(x, a)}{f(a)} a, \qquad \text{or} \qquad x = x' - \frac{2f(x', a)}{f(a)} a, \tag{2}$$

where, of course, $f(a) \neq 0$ (and may be taken equal to 1 when desired), because *the center $a$ must not lie on the absolute.*

Evidently the center $a$ and each point in the polar $M_{n-2}$ of $a$, viz., $f(x, a) = 0$ are invariant under $\{\mathbf{a}\}$.

The product $\{\mathbf{ab}\}$ of two reflections $\{\mathbf{a}\}$ and $\{\mathbf{b}\}$ comes out as

$$x' = x - \frac{2f(x, a)}{f(a)} a - \frac{2f(x, b)}{f(b)} b + \frac{4f(x, a) f(a, b)}{f(a) f(b)} b. \tag{3}$$

This is symmetrical in $a$ and $b$ when and only when $f(a, b) = 0$, i. e., we have the result:

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*Two central reflections* $\{\mathbf{a}\}$ *and* $\{\mathbf{b}\}$ *are commutative, when and only when* $a$ *and* $b$ *are conjugate points with respect to the absolute.**

The important question is now this: *When is* $\{\mathbf{ab}\} \equiv \{\mathbf{cd}\}$*?*

A *necessary* condition is expressed in

THEOREM 1. *If* $\{\mathbf{a}^{(1)}\mathbf{a}^{(2)}\mathbf{a}^{(3)}\cdots\mathbf{a}^{(r)}\} \equiv \{\mathbf{b}^{(1)}\mathbf{b}^{(2)}\mathbf{b}^{(3)}\cdots\mathbf{b}^{(r)}\}$, *then each of the centers* $b$ *is numerically derived from the centers* $a^{(1)}, a^{(2)}, \cdots, a^{(r)}$.†

Take $r = 2$. Then from (3), $\{\mathbf{ab}\} \equiv \{\mathbf{cd}\}$ gives as a necessary and sufficient condition

$$(4)\quad \frac{f(x,a)}{f(a)}a + \left(\frac{f(x,b)}{f(b)} - \frac{2f(x,a)f(a,b)}{f(a)f(b)}\right)b$$
$$\equiv \frac{f(x,c)}{f(c)}c + \left(\frac{f(x,d)}{f(d)} - \frac{2f(x,c)f(c,d)}{f(c)f(d)}\right)d$$

for every value of $x$. If $c \not\equiv a$, take $f(x,c) = 0$, and $f(x,a) \neq 0$. Then $d$ is numerically derived from $a$ and $b$. Similarly for $c$.

The following proof involving the principles of GRASSMANN's *Ausdehnungslehre*‡ is general and very direct. Regarding $a, b, c, d$ as extensive magnitudes or complex quantities, and the coefficients in (4) as scalars, we notice that the coefficient of $b$ is of the same form as that of $a$ diminished by a multiple of the latter. Now write down (4) with $x$ replaced by $x'$, and multiply the corresponding members of the two equations together using the "combinatorial law" $ab = -ba$, $cd = -dc$; then, taking $f(a) = f(b) = f(c) = f(d) = 1$, we get easily

$$\begin{vmatrix} f(x,a) & f(x,b) \\ f(x',a) & f(x',b) \end{vmatrix} ab \equiv \begin{vmatrix} f(x,c) & f(x,d) \\ f(x',c) & f(x',d) \end{vmatrix} cd,$$

in which the scalar coefficients are determinants. In this form the theorem follows at once, and as the method is in no wise different for the general case, the truth of the proposition is established.

It is now necessary to add only the following theorem, after which a complete theory of central reflections may be developed by synthesis.

THEOREM 2. *The product* $\{\mathbf{ab}\}$ *of two central reflections may be resolved in* $\infty^1$ *ways into* $\{\mathbf{cd}\}$, *the centers* $c$ *and* $d$ *lying on the manifold* $M_1$ *derived from* $a$ *and* $b$. *Either* $c$ *or* $d$ *may be chosen arbitrarily on this line. The other is then uniquely determined.*

For if $f(a) = 1$, $f(b) = 1$, and $\lambda^2 + \mu^2 + 2\lambda\mu f(a,b) = 1$, then $\{\mathbf{ab}\} \equiv \{\mathbf{cd}\}$ if $c = \lambda a + \mu b$, and $d = -\mu a + [\lambda + 2\mu f(a,b)]\,b$, as direct substitution in

* Voss, loc. cit., p. 345.

† We assume $r \not> n$, and $a_1, a_2, \cdots, a_r$ independent.

‡ *Ausdehnungslehre* von 1862, p. 42.

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(4) will show.* Furthermore, we cannot write $\{\mathbf{ab}\} \equiv \{\mathbf{cd}\} \equiv \{\mathbf{cd}'\}$ unless $\{\mathbf{d}\} \equiv \{\mathbf{d}'\}$. For $\{\mathbf{ccd}\}$ would in this case be identical with $\{\mathbf{ccd}'\}$, and since $\{\mathbf{cc}\}$ is the identical transformation, we must also have $\{\mathbf{d}\} \equiv \{\mathbf{d}'\}$.

The invariant configuration of $\{\mathbf{ab}\}$ is made up of the points of intersection of the line $ab$ with the absolute, and every point of the $M_{n-3}$ common to the two polar manifolds $M_{n-2}$ of $a$ and $b$.†

It is to be noted also that $\lambda a + \mu b$ lies upon the absolute when

$$\lambda^2 + \mu^2 + 2\lambda\mu f(a, b) = 0,$$

which is the condition of coincidence of $c$ and $d$, i. e., in the resolution of $\{\mathbf{ab}\}$ the express hypothesis made in (2) that a center shall not lie upon the absolute is equivalent to coincidence of $c$ and $d$.

Pass on now to $r = 3$, and the proof of

THEOREM 3. *The product of three central reflections* $\{\mathbf{abc}\}$ *may always be resolved in* $\infty^3$ *ways into* $\{\mathbf{def}\}$. *The first center is any point in the plane of* $a, b, c$; *but the line joining* $e$ *and* $f$ *is then uniquely determined, though either* $e$ *or* $f$ *may be chosen at will upon this line.*

Given, then, the three centers $a, b, c$, and any center $d$ in their plane. If the point $b'$ of intersection of $ad$ and $bc$ is not on the absolute, then by theorem 2, $\{\mathbf{abc}\} \equiv \{\mathbf{ab'c'}\} \equiv \{\mathbf{def}\}$. Suppose, however, $f(b') = 0$, and let $a'$ be any center on $ab$, so that we may write $\{\mathbf{ab}\} \equiv \{\mathbf{a'b'}\}$. Then the lines $da'$ and $cb'$ are projectively related, and accordingly their point of intersection $b''$ describes a conic containing $d$ and $c$. Since $d$ and $c$ are not on the absolute, this conic cannot be the intersection of the plane of centers with the absolute, and $b''$ may be assumed a center. Hence we now have $\{\mathbf{abc}\} \equiv \{\mathbf{a'b'c}\} \equiv \{\mathbf{a'b''c'}\} \equiv \{\mathbf{def}\}$, and the first part of theorem 3 is proved.

For the rest, if $(\mathbf{def}) \equiv (\mathbf{de'f'})$, then $(\mathbf{ef}) \equiv (\mathbf{e'f'})$ and the second part follows at once from theorem 2.

It appears, therefore, that the transformation $\{\mathbf{abc}\}$ establishes in the plane of centers a point-line relation between the first center $d$ and the line $ef$ upon which the remaining two must then lie.

This relation is a *correlation.* For if $f$ is *any* center on $ef$, then $\{\mathbf{def}\} \equiv \{\mathbf{d_1e_1f_1}\}$; and if $de$ and $d_1e_1$ intersect in $d'$, we may write $\{\mathbf{d'e'f}\} \equiv \{\mathbf{d'e_1'f_1}\}$. Hence $e'$ and $e_1'$ are on $ef$, and $e'$ and $d'$ coincide with $e$ and $d$ respectively. Therefore as $f$ in $\{\mathbf{def}\}$ moves along $ef$, the corresponding line turns around $d$. Hence the transformation $\{\mathbf{abc}\}$ establishes a duality $\{\Gamma\}$ in the plane of centers, such that the second and third centers lie upon the line that corresponds under $\{\Gamma\}$ to the first center. Furthermore, since

* The equations simply indicate that $a, b$ and $c, d$ are corresponding points of a projectivity on the line of centers and with fixed points upon the absolute.

† The line joining $a$ and $b$ touches the absolute when $f(a)f(b) = [f(a, b)]^2$.

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$\{\mathbf{def}\} \equiv \{\mathbf{ed'f}\} \equiv \{\mathbf{efd''}\}$, $f$ lies on the correlate of $e$, and finally, as the preceding shows that a point and its line under $\{\Gamma\}$ are united only when the point is on the absolute, we have

THEOREM 4. *The transformation* $\{\mathbf{abc}\}$ *establishes in the plane of centers a duality which is characteristic of the transformation. In any resolution of* $\{\mathbf{abc}\}$, *the line corresponding in this duality to any center contains the following centers of the product. Coincidence of point and line under this duality occurs when and only when the point is on the absolute.*

The generalization of this theorem is at once accomplished by complete induction. Assume, then, that theorems 3 and 4 hold for the product of any $p$ central reflections $p < n$. Write the transformation

$$\{\mathbf{T}_{p+1}\} \equiv \{\mathbf{a}^{(1)}\mathbf{a}^{(2)}\cdots\mathbf{a}^{(p)}\mathbf{a}^{(p+1)}\} \equiv \{\mathbf{a}^{(1)}\mathbf{T}_p\}.$$

Let $b^{(1)}$ be any point in the $M_p$ numerically derived from $a^{(1)}, a^{(2)}, \cdots, a^{(p+1)}$, and suppose the line $a^{(1)}b^{(1)}$ intersects the locus of centers of $\{\mathbf{T}_p\}$ in $a^{(0)}$. Considerations analogous to the preceding permit us to assume $a^{(0)}$ not on the absolute, and we may by hypothesis write $\{\mathbf{T}_p\} \equiv \{\mathbf{a}^{(0)}\mathbf{T}_{p-1}\}$. Since also $\{\mathbf{a}^{(1)}\mathbf{a}^{(0)}\} \equiv \{\mathbf{b}^{(1)}\mathbf{b}^{(0)}\}$, we evidently have $\{\mathbf{T}_{p+1}\} \equiv \{\mathbf{b}^{(1)}\mathbf{T}'_p\}$. The remainder of the proof follows as easily and may be omitted, the statement following.

THEOREM 5. *The transformation* $\{\mathbf{a}^{(1)}\mathbf{a}^{(2)}\cdots\mathbf{a}^{(r)}\}$ *compounded of $r$ independent central reflections establishes within the $M_{r-1}$ numerically derived from the centers $a^{(1)}, a^{(2)}, \cdots a^{(r)}$, a duality which is characteristic of the transformation. We may resolve the transformation in $\infty^{\frac{1}{2}r(r-1)}$ ways, the first center being any point in $M_{r-1}$, and the following limited only by the condition that the $M_{r-2}$ corresponding to any one of them in the duality contains all the succeeding centers. Coincidence of point and corresponding $M_{r-2}$ occurs when and only when the point is on the absolute. The transformation depends upon $r(n-1) - \frac{1}{2}r(r-1) = \frac{1}{2}r(2n-r-1)$ essential parameters.*

From this result we may at once state

THEOREM 6. *Any product of reflections in centers lying in an $M_{r-1}$ may always be reduced to one of $r$ factors or fewer. The most general transformation compounded of central reflections may therefore be reduced to one of $n$ factors.*

The proof follows at once from theorem 5. For example, $\{\mathbf{a}^{(1)}\mathbf{a}^{(2)}\cdots\mathbf{a}^{(n)}\mathbf{a}\}$ may be written $\{\mathbf{b}^{(1)}\mathbf{b}^{(2)}\cdots\mathbf{b}^{(n-1)}\mathbf{a}\,\mathbf{a}\}$ or $\{\mathbf{b}^{(1)}\mathbf{b}^{(2)}\cdots\mathbf{b}^{(n-1)}\}$. For any, $\mathbf{T}_p = \{\mathbf{a}^{(1)}\mathbf{a}^{(2)}\cdots\mathbf{a}^{(p)}\}$ the polar $M_{n-p-1}$ with respect to $f = 0$ of the locus of centers is a locus of invariant points. All remaining invariant points lie in the locus of centers.

## § 2. *On the resolution of any orthogonal transformations into central reflections.*

The quadratic form of the preceding section is now assumed to be

$$\phi(x) = x_1^2 + x_2^2 + \cdots x_n^2.$$

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Let $\{\mathbf{T}\}$, defined by the equations

$$y_i = \sum_i \alpha_{ik} x_k \qquad (i, k = 1, 2, \cdots, n), \tag{4}$$

be any orthogonal transformation, i. e., $\phi(y) \equiv \phi(x)$. The question is to show that $\{\mathbf{T}\}$ is a product of central reflections. This is known to be the case for $n = 2$, for example. For if we interpret $x_1$ and $x_2$ as Cartesian rectangular coordinates in the plane, $\{\mathbf{T}\}$ is either a rotation around the origin or a reflection * in a line through the origin according as the determinant $|\alpha_{ik}|$ in (4) is 1 or $-1$. But in the case of a rotation, $\{\mathbf{T}\}$ is compounded of two line reflections.

Knowing, then, that the orthogonal transformation for $n = 2$ is always a product of central reflection, I shall prove the theorem true in general by establishing the truth of the proposition:

*An orthogonal transformation in n variables may always be resolved into one in $n - 1$ variables compounded with central reflections.*†

We may assume in (4) that $\alpha_{nn} \neq 0$. For if this were the case, suppose $\alpha_{nr} \neq 0$, which must be true for some value of $r$. But the reflection $\{\mathbf{a}\}$ in the centre $a_1 = a_2 = \cdots = a_{r-1} = 0$, $a_r = 1$, $a_{r+1} = \cdots = a_{n-1} = 0$, $a_n = -1$, merely interchanges $x_r$ and $x_n$, and, accordingly, composition of $\{\mathbf{T}\}$ and $\{\mathbf{a}\}$ will give us a transformation for which the hypothesis holds.

Now suppose $\alpha_{ns} \neq 0$. Consider a reflection $\{\mathbf{b}\}$ for which only $b_s$ and $b_r$ do not vanish. This has the form

$$\begin{aligned} x_k &= x'_k \qquad (k \neq s \neq n), \\ x_s &= \lambda x'_s + \mu x'_n, \\ x_n &= \mu x'_s - \lambda x'_n, \end{aligned}$$

where $\lambda^2 + \mu^2 = 1$. Then if $\lambda : \mu$ be chosen so that $\lambda\alpha_{ns} + \mu\alpha_{nn} = 0$, the compounded transformation $\{\mathbf{Tb}\}$, viz., $y_i = \sum_k \beta_{ik} x'_k$, is such that $\beta_{ns} = 0$, $\beta_{nn} \neq 0$, but otherwise $\beta_{nr} = \alpha_{nr}$. Hence we may multiply $\{\mathbf{T}\}$ by a product of central reflections such that finally $y_n = \gamma_{nn} x_n$. But for an *orthogonal* transformation‡ we must have $\gamma_{nn} = \pm 1$, and $x_n$ must disappear from the other equations, and therefore the proposition is established.

Referring to the preceding theorems we may now state the fundamental result.

THEOREM 7. *Every linear transformation of a general quadratic form in n variables into itself may be compounded of n central reflections or fewer, and theorems 1–5 hold for all such transformations.*

---

* For this interpretation of the variables, a central reflection becomes the usual reflection in a line through the origin.

† This theorem is given by KRONECKER, Berliner Monatsberichte (1890), p. 1071, whose proof is essentially identical with the following.

‡ Since $\sum_i \alpha_{in}\alpha_{is} = 0$ $(i, s = 1, 2, \cdots, n-1)$, and also $|\alpha_{in}| \neq 0$, we must have $\alpha_{in} = 0$, and from $\sum_i \alpha_{in}^2 + \gamma_{nn}^2 = 1$, $\gamma_{nn} = \pm 1$.

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As already remarked, VOSS has proved that the *general* orthogonal substitution is compounded of $n$ central reflections, and for this purpose he chooses the equations of the transformation in CAYLEY's form. He also points out that the determination of the centers is exactly analogous to that of finding a self-conjugate tetrahedron with respect to a quadric.* The association of the general transformation with a characteristic duality is implied in this statement, but very imperfectly. Moreover, since the formulas of CAYLEY apply only to *general proper* transformations, it is evident that the proof given by VOSS is incomplete.

§ 3. *Equations of the transformations† compounded of n independent central reflections.*

Since the general transformation is completely characterized by a correlation in which coincidence of point and corresponding $M_{n-2}$ arises when the point is on the absolute, we begin with the duality $\{\Gamma\}$ defined by

$$\sum_{i,k} \beta_{ik} x_i y_k = 0 \qquad (i, k = 1, 2, \cdots, n). \tag{5}$$

For $x = y$, this must reduce to (1), or, $f(x) = 0$; i. e., $\beta_{ik} + \beta_{ki} = 2\lambda a_{ik}$. Then if we set $\beta_{ik} - \beta_{ki} = 2\mu\alpha_{ik}$, we may write $\{\Gamma\}$ in the form

$$\sum_{i,k} (\lambda a_{ik} + \mu\alpha_{ik}) x_i y_k = 0 \qquad (a_{ik} = a_{ki},\ \alpha_{ik} = -\alpha_{ki}). \tag{6}$$

The equation (6) gives the most general form of the correlation in question, and shows that for a given $f(x)$, the variety is given by the number of the arbitrary $\alpha$'s, viz., $\frac{1}{2}n(n-1)$; and accordingly, the number of essential parameters in the most general Hermite transformation is $\frac{1}{2}n(n-1)$.

The duality

$$f(x, y) \equiv \sum a_{ik} x_i y_k = 0 \tag{7}$$

is evidently the polar reciprocation with respect to the absolute, while

$$w(x, y) \equiv \sum \alpha_{ik} x_i y_k = 0 \qquad (\alpha_{ik} = -\alpha_{ki}) \tag{8}$$

is a *nullsystem*, since point and corresponding $M_{n-2}$ lie in coincidence. The geometrical relation of the dualities (6), (7) and (8) is simply this. Under $\{\Gamma\}$ the point $x$ gives the two $M_{n-2}$'s, $M$ and $M'$ say, whose equations are $\lambda f(x, y) + \mu w(x, y) = 0$, $\lambda f(x, y) - \mu w(x, y) = 0$; and therefore $M$ and $M'$ are divided harmonically by the polar of $x$ with respect to the absolute and the correspondent of $x$ in the nullsystem.‡

* Loc. cit., p. 349.

† It will be convenient to designate any linear transformation of a quadratic form into itself, an *Hermite transformation*. (Cf. LOEWY, loc. cit.)

‡ Cf. *Vorlesungen über Geometrie*, CLEBSCH-LINDEMANN, vol. 2, p. 402, for the corresponding discussion in ordinary space.

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We proceed now to find the equations of the Hermite transformation $\{\mathbf{T}\}$ corresponding to the correlation $\{\Gamma\}$ defined by (6).

Let $x\{\mathbf{T}\}x'$, and suppose $\{\mathbf{T}\} \equiv \{\mathbf{a}^{(1)}\mathbf{a}^{(2)}\cdots\mathbf{a}^{(n)}\}$, according to theorem 7. Choose $a$ such that $a\{\Gamma\}\ M$, when $M$ is the polar of $x'$, i. e.,

$$\sum_i(\lambda a_{ik} + \mu\alpha_{ik})a_i \equiv \rho\sum_i a_{ik}x_i' \qquad (i, k=1, 2, \cdots, n). \tag{9}$$

Then $x'$ is an *invariant* point for $\{\mathbf{aT}\}$, and hence $x\{\mathbf{a}\}x'$. That is, we have to eliminate the $a$ from (9) and

$$x = x' - \frac{2f(x', a)}{f(a)}a. \tag{10}$$

Multiplying (9) by $a_k$ and summing with respect to $k$, gives $\lambda f(a) = \rho f(x', a)$; hence (10) becomes

$$x = x' - \frac{2\lambda}{\rho}a; \tag{11}$$

and now eliminating $a$ from (9) and (11) we obtain

$$\sum_i(\lambda a_{ik} - \mu\alpha_{ik})x_i' + \sum_s(\lambda a_{sk} + \mu\alpha_{sk})x_s = 0 \qquad (i, k, s=1, 2, \cdots, n). \tag{12}$$

The equations† of $\{\mathbf{T}\}$ are then found by solving (12) for $x'$. Before taking up this question, however, an interesting geometrical theorem is at once read out of (12), viz.,

THEOREM 8. *Corresponding points in any general Hermite transformation are points which give the same $M_{n-2}$ in the duality which is characteristic of the transformation.*

This theorem completes the connection between the transformation $\{\mathbf{T}\}$ and its corresponding duality, and would serve as basis of a purely geometrical discussion.

The equations of $\{\mathbf{T}\}$ in the form† (12) are very convenient for discussion as the following deductions will show.

Write (12) when solved for $x'$ in the form

$$x_i' = \sum_k c_{ik}x_k \qquad (i, k=1, 2, \cdots, n). \tag{13}$$

Then denoting the determinant $|\lambda a_{ik} + \mu\alpha_{ik}|$ by $\Delta(\lambda, \mu)$, we evidently have

$$|c_{ik}| \equiv (-1)^n\frac{\Delta(\lambda, \mu)}{\Delta(\lambda, -\mu)} = (-1)^n, \tag{14}$$

for $\Delta(\lambda, -\mu)$ is simply $\Delta(\lambda, \mu)$ with rows and columns interchanged. This

* Multiplying (12) by $x_k'$ and $x_k$, respectively, and summing with respect to $k$ gives $f(x) = f(x')$, a verification.

† I do not find the Hermite transformation so given elsewhere. See, for example, LOEWY, loc. cit., pp. 9 et seq.

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result is a verification, for the determinant of a central reflection is $-1$, and of course of $\{\mathbf{T}\}$, should be $(-1)^n$. Hence

THEOREM 9. *The equations* (12) *define a proper or improper Hermite transformation according as n is even or odd.*

The solution of (12) is obtained at once by first solving (9) for the $a$'s and then substituting in (11). Following this method we readily find for the $c$'s of (13) the equations:

$$(15)\qquad c_{ii} = \frac{\Delta(\lambda, \mu) - 2\lambda \sum_k a_{ki}\Delta_{ki}(\lambda, \mu)}{\Delta(\lambda, \mu)}, \qquad c_{is} = \frac{-2\lambda \sum_k a_{ks}\Delta_{ki}(\lambda, \mu)}{\Delta(\lambda, \mu)}$$

$$(i \neq s;\ i, k, s = 1, 2, \cdots, n).$$

The classical formulas of Cayley for an orthogonal substitution are of course obtained by setting $a_{ik} = 0$, $i \neq k$, $a_{ii} = 1$.

Associated with the same nullsystem $\alpha_{ik}$ are $\infty^1$ dualities determined by the ratio $\lambda : \mu$. An interesting result is obtained by putting $\mu = 0$: for then the duality becomes the polar reciprocation in the absolute. The centers of $\{\mathbf{T}\}$ are now conjugate in pairs and therefore reflections in these centers are commutative (§ 1). The equations (12) become simply $x_i' = -x_i$, and we get

THEOREM 10. *The transformation in $R_{n-1}$ compounded of n commutative central reflections in the absolute reduces to an identical point transformation.*

This result is obtained by VOSS (loc. cit., p. 345), who also gives interesting geometrical consequences for $n = 3$ and $n = 4$. Compounding $\{\mathbf{T}\}$ with $x_i' = -x_i$ gives a *proper* transformation for every $n$. If $n$ is odd, however, the product reduces down to one of $n - 1$ central reflections. In this case, therefore, the transformation compounded of $n$ and of $n - 1$ central reflections are essentially identical.

The unsolved form (12) of $\{\mathbf{T}\}$ is* very convenient for discussion of fixed elements. For, putting $x' = \rho x$, we find

$$(15a)\qquad \phi(\rho) \equiv |\lambda a_{ik}(\rho + 1) - \mu\alpha_{ik}(\rho - 1)| = 0,$$

the corresponding fixed point $x$ being given by

$$\sum_i [\lambda\alpha_{ik}(\rho + 1) - \mu\alpha_{ik}(\rho - 1)]\, x_i = 0.$$

Evidently $\phi(1) \neq 0$ if $|a_{ik}| \neq 0$, and we learn that *the characteristic equation*† *of the transformation* (15) *cannot have the root* $+1$ *if the absolute is non-degenerate.*

Geometrically, the determination of the fixed elements is the same as the

* Cf., e. g., LINDEMANN, Münchener Berichte (1896), p. 52.

† The "characteristic function" of (13), (15) is readily seen to be $\phi(\rho) + (-1)^n\Delta(\lambda, \mu)$, since for $\rho = 0$, the characteristic function must reduce to $|c_{ik}|$ or $(-1)^n$.

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question of coincident elements in the polar reciprocation (7) and the nullsystem (8).

§ 4. *The transformations compounded of n central reflections or fewer.*

The transformation $\{\mathbf{T}_{n-t}\} \equiv \{\mathbf{a}^{(1)}\mathbf{a}^{(2)}\cdots\mathbf{a}^{(n-t)}\}$, $t > 0$, becomes a $\{\mathbf{T}_n\}$ if multiplied by any $\{\mathbf{T}_t\}$ whose locus of centers does not intersect the locus of centers of $\{\mathbf{T}_{n-t}\}$. The correlation *within* the latter $M_{n-t-1}$ which is characteristic of $\{\mathbf{T}_{n-t}\}$ is therefore determined by a duality (6), in the sense that a point in $M_{n-t-1}$ corresponds to the $M_{n-t-2}$ determined by the intersection of $M_{n-t-1}$ and the $M_{n-2}$ corresponding to the point by (6). Furthermore, it is evident that the relation

$$\{\mathbf{T}_n\} \equiv \{\mathbf{T}_t\mathbf{T}_{n-t}\}$$

indicated above, implies that the locus of centers of $\{\mathbf{T}_t\}$ corresponds under the duality defining $\{\mathbf{T}_n\}$ to the locus of centers $M_{n-t-1}$ of $\{\mathbf{T}_{n-t}\}$. As remarked before, these must not intersect. The necessary and sufficient condition is found. For if $M_{n-t-1}$ is defined by

$$(16) \qquad (\nu^{(1)}x) = 0,\ (\nu^{(2)}x) = 0,\ \cdots,\ (\nu^{(t)}x) = 0,$$

then the corresponding $M_{t-1}$ under (6) is determined from

$$(17) \qquad \sum_i (\lambda a_{ik} + \mu \alpha_{ik})\, a_i = \sigma^{(1)}\nu_k^{(1)} + \sigma^{(2)}\nu_k^{(2)} + \cdots + \sigma^{(t)}\nu_k^{(t)} \quad (i, k = 1, 2, \cdots, n).$$

If then the $a$'s satisfy (16), the condition appears in the vanishing of the determinant formed by bordering $\Delta(\lambda, \mu)$ by the $\nu$'s. Denoting this bordered determinant by

$$\Delta^{(t)}(\lambda, \mu),$$

we have

THEOREM 11. *The necessary and sufficient condition for the intersection of an $M_{n-t-1}$ (16) with the $M_{t-1}$ corresponding to it under any duality is found by equating to zero the determinant of order $n + t$ formed by bordering the determinant of the duality with the $\nu$'s.*

In particular, $\Delta^{(1)}(\lambda, \mu) = 0$ defines in the coördinates $\nu_i$ the correlate of the absolute itself under $\{\Gamma\}$. That is, the envelope of the $M_{n-2}$'s which contain their corresponding points. This quadratic manifold is the second "fundamental locus" of the duality, the other being the absolute itself.*

Proceeding now to find the equations of $\{\mathbf{T}_{n-t}\}$, the locus of centers being defined by (16) and $\Delta^{(t)}(\lambda, \mu) \neq 0$, we may at once write down the equations of the most general duality within $M_{n-t-1}$ such that the $M_{n-t-2}$ determined by any point shall lie in the $M_{n-2}$ corresponding to that point under (6). These are

* LINDEMANN, *Vorlesungen* etc., vol. 2, p. 402; VOSS, Mathematische Annalen, vol. 13 (1878), p. 359.

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$$(18)\quad \sum_i (\lambda a_{ik} + \mu \alpha_{ik}) a_i + \sigma^{(1)} \nu_k^{(1)} + \sigma^{(2)} \nu_k^{(2)} + \cdots + \sigma^{(t)} \nu_k^{(t)} = \rho u_k \quad (i, k = 1, 2, \cdots, n).$$

The $a$'s satisfy (16) also, or

$$(19)\qquad (\nu^{(1)} a) = 0,\ (\nu^{(2)} a) = 0,\ \cdots,\ (\nu^{(t)} a) = 0.$$

The $n + t$ equations (18) and (19) may be solved for the $a$'s and $\sigma$'s, since $\Delta^t(\lambda, \mu) \neq 0$. The $a$'s depend only upon the *intersection* of $(ux) = 0$ and the $M_{n-t-1}$ (16).

As before, if $x \{ \mathrm{T}_{n-t} \} x'$, then choosing for $u$ the polar $M_{n-2}$ of $x'$ with respect to $f = 0$, i. e., taking in (18),

$$(20)\qquad u_k = \sum_i a_{ik} x_i',$$

and assuming the $a$ determined by (18) as the *first* center of $\{ \mathrm{T}_{n-t} \}$, then obviously $x \{ \mathrm{a} \} x'$, as in the general case, § 3. Multiplying (18) by $a_k$ and summing with respect to $k$, gives by (19) and (20) $\lambda f(a) = \rho f(x', a)$. Hence the equations of the $a$'s are

$$(21)\qquad x_i - x_i' + \frac{2\lambda}{\rho} a_i = 0.$$

The equations of $\{ \mathrm{T}_{n-t} \}$ are then found by the elimination of the $a$'s, the $u$'s, and the $\sigma$'s from (18), (19), (20), and (21). In fact substituting from (20) and (21) in (18) and (19), gives

$$(22)\qquad \sum_i (\lambda a_{ik} + \mu \alpha_{ik}) x_i + \sum_j (\lambda a_{jk} - \mu \alpha_{jk}) x_j' - \frac{2\lambda}{\rho} \sum_s \sigma^{(s)} \nu_k^{(s)} = 0$$

$$(i, j, k = 1, 2, \cdots, n;\ s = 1, 2, \cdots, t),$$

$$(23)\qquad (\nu^{(s)} x) - (\nu^{(s)} x') = 0.$$

Multiplying (22) by $x_k$ and $x_k'$ respectively and summing with respect to $k$ gives as before the verification

$$f(x) = f(x').$$

We may therefore state

THEOREM 12. *The equations* (22) *and* (23) *define in unsolved form a linear transformation of the quadratic form* $f(x)$ *into itself, and every such transfortion is given by these equations,* $t$ *having any value from* 0 *to* $n - 1$, *and the* $\alpha$'s *being elements of an arbitrary skew determinant.*

The form $x_i' = \sum c_{is} x_s$ for the transformation is readily obtained by solving (18), (19), (20) for $a_i$ and substituting in (21). This gives

$$(24)\qquad c_{is} = e_{is} - \frac{2\lambda \sum_k a_{ks} \Delta_{ki}^{(t)}(\lambda, \mu)}{\Delta^{(t)}(\lambda, \mu)},$$

where

$$e_{is} = 0\,(i \neq s),\ e_{ii} = 1 \qquad (i, k, s = 1, 2, \cdots, n).$$

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THEOREM 13. *The equations* (24) *give the form of the coefficient of every Hermite transformation.*

The determinant $|c_{is}| = (-1)^{n-t}$. This is easily seen by making a change of variable $y_s = (\nu^{(s)}x)$, $s = 1, 2, \cdots, t$, by which $n-t$ of the $x$'s remain unchanged, say $x_{t+1}, \cdots, x_n$, it being assumed that the determinant

$$|\nu_1^{(1)}\nu_2^{(2)}\cdots\nu_t^{(t)}| \tag{25}$$

of the matrix

$$\|\nu_1^{(1)}\nu_2^{(1)}\cdots\nu_t^{(1)}\| \tag{26}$$

does not vanish.

This transformation may readily be effected by first eliminating the $\sigma$'s, and then changing the variables.

The $\alpha$'s are still arbitrary in (24), but no longer essential. In fact, (18) and (19) defining a duality in a linear $M_{n-t-1}$, contain at most $\frac{1}{2}(n-t)(n-t-1)$ essential parameters. And indeed, if the $\sigma$'s and $a_1, a_2, \cdots, a_t$ be eliminated from (18) and (19) under the hypothesis (25) made above, there will remain $(n-t)$ equations for $a_{t+1}, \cdots, a_n$, of the form

$$\sum_j (\lambda a'_{t+j,\,t+k} + \mu \alpha'_{t+j,\,t+k})a_{t+j} = \rho u'_{t+k} \qquad (j, k = 1, 2, \cdots, n-t), \tag{27}$$

in which the $a'$'s and $\alpha'$'s are linear in the $a$'s and $\alpha$'s respectively, and at the same time elements of a symmetric and skew determinant respectively. The coefficients of the $a$'s and $\alpha$'s are quadratic in the determinants of order $t$ of the matrix (25).

Since the number of independent determinants is $t(n-t)$, we find for the total number of essential parameters in (24),

$$\tfrac{1}{2}(n-t)(n-t-1) + t(n-t) = \tfrac{1}{2}(n-t)(n+t-1),^*$$

which agrees with the statement of Theorem 5 for $r = n-t$.

Returning to (22) and (23) for discussion of fixed elements $x' = \rho x$, we find for the characteristic function of (24),

$$\phi(\rho) = \frac{(-1)^{n-t}(1-\rho)^t\phi^{(t)}(\rho)}{\Delta^{(t)}(\lambda, \mu)}, \tag{28}$$

where $\phi^{(t)}(\rho)$ is the determinant (15$a$) of the general case bordered with the $\nu$'s. It is easy to transform $\phi^{(t)}(\rho)$ into the determinant of order $n-t$,

$$\phi^{(t)}(\rho) = |\lambda a'_{t+j,\,t+k}(\rho+1) - \mu\alpha'_{t+j,\,t+k}(\rho-1)|, \tag{29}$$

in which $a'$ and $\alpha'$ have the same significance as above. The equations (28) and (29) give the theorem due to VOSS.†

* Cf. LINDEMANN, Münchener Berichte (1896), p. 66.

† VOSS, Münchener Berichte (1896), p. 14.

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*The characteristic function for any Hermite transformation* (24), *except for the factor* $(1-\rho)^t$, *always has the form*

$$|\lambda a'_{ik}(\rho+1)-\mu\alpha'_{ik}(\rho-1)| \qquad (i, k=1, 2, \cdots, n-t),$$

*in which the* $a'$*'s and* $\alpha'$*'s are elements of a symmetric and skew determinant respectively.*

Finally, (22) and (23) show that $\rho=1$ gives for the corresponding fixed point

$$\rho\sum_i \lambda a_{ik}x_i = \sum_s \sigma^{(s)}\nu_k^{(s)},$$

i, e., *every* point of the polar of the $M_{n-t-1}$ (16) with respect to the absolute is fixed. Any other fixed point lies in the $M_{n-t-1}$ (16), the discussion of their arrangement being precisely that of the general case if $n$ is replaced by $n-t$.

The problem of the determination of all linear transformations of a quadratic form into itself may therefore be regarded as completely solved in this and the preceding sections. The derivation of canonical forms for any given case is a matter of no great difficulty. The question evidently depends primarily upon the discussion of the dualities (7) and (8) of § 3.

## § 5. *Further resolution of the transformation into involutory transformations.*

The inverse $\{\mathbf{T}_{n-t}^{-1}\}$ of the transformation (22), (23) is found by changing $\mu$ to $-\mu$. Hence $\{\mathbf{T}_{n-t}\}$ is involutory when and only when $\mu=0$, i. e., when $\{\mathbf{T}_{n-t}\}$ is compounded of reflections in centers conjugate with respect to the absolute. Such a transformation depends upon $t(n-t)$ essential parameters, viz., the coördinates of the locus of centers.

We now state the theorem:

THEOREM 14. *Every linear transformation of a quadratic form into itself may be resolved into the product of two involutory transformations.*

Consider any $\{\mathbf{T}_t\}$, $t=2r$; then we shall prove

$$\{\mathbf{T}_t\} \equiv \{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_r\}\{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_r\}, \tag{30}$$

where the $a$'s and also the $b$'s are conjugate in pairs.

For simplicity of statement, take $r=2$. Then the locus of centers of $\{\mathbf{T}_4\}$ is ordinary space, $M_3$, the transformation being characterized by a duality $\{\Gamma\}$ in $M_3$, which is pointed out in § 3, is defined by means of a nullsystem $\{\mathbf{N}\}$ together with reciprocation in the absolute. From (6) and (8) § 3, it is readily seen that $\{\mathbf{a}_1\mathbf{a}_2\} \equiv \{\mathbf{a}_2\mathbf{a}_1\}$ when and only when $a_2$ lies in the plane corresponding to $a_1$ under $\{\mathbf{N}\}$. Thus the line $A$ joining $a_1$, $a_2$ is invariant under $\{\mathbf{N}\}$, that is, $A$ "belongs to $\{\mathbf{N}\}$," and the line $B$ determined by $b_1$, $b_2$ enjoys the

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same property.* Furthermore $A\{\Gamma\}B$, by theorem 5. Hence, since the $\infty^3$ lines of $\{\mathbf{N}\}$ transform by $\{\Gamma\}$ into the $\infty^3$ lines of a nullsystem $\{\mathbf{N}'\}$, it appears that the line $A$ of $\{\mathbf{N}\}$ becomes $B$ of $\{\mathbf{N}'\}$, that is, $B$ belongs to $\{\mathbf{N}\}$ and $\{\mathbf{N}'\}$. Through any point $b_1$ passes one line of the congruence common to $\{\mathbf{N}\}$ and $\{\mathbf{N}'\}$, hence if $b_1$ is chosen arbitrarily, $B$ is determined uniquely, and also $A$ from $A\{\Gamma\}B$, but $a_1$ is any point on $A$. Thus the resolution may be effected in $\infty^2$ ways.

The proof for $r > 2$ is precisely the same, and the degree of freedom is found to be $r$. For $t = 2r - 1$, it is only necessary to multiply $\{\mathbf{T}_t\}$ by an $\{\mathbf{a}\}$, and then apply the theorem, remembering that $a$ may be chosen arbitrarily. For $n = 2r - 1$, it has already been remarked that the transformation is not different from $t = 2r - 2$.

As a general theorem, it may be stated that the resolution†

$$\{\mathbf{T}_t\} \equiv \{\mathbf{T}_\lambda \mathbf{T}_\mu \mathbf{T}_\nu \cdots\}$$

into involutory transformations may be effected in

$$\tfrac{1}{2}t(t+1) - (\lambda^2 + \mu^2 + \nu^2 + \cdots)$$

ways.

## §6. *Application to the case* $n = 6$.

Special interest attaches to the case $n = 6$, for then the $x$'s satisfying $f(x) = 0$, $|a_{ik}| \neq 0$, may be assumed as line coördinates in ordinary space, $R_3$, central reflection becomes the transformation defined by a nullsystem in $R_3$, or, inversion in a linear line complex, while the transformations $\{\mathbf{T}_t\}$ in question are the collineations and correlations of projective geometry, according as $t$ is even or odd. In other words, we are concerned with line geometry in the sense of PLÜCKER. The involutory transformations $\{\mathbf{T}_2\}$ and $\{\mathbf{T}_3\}$ are respectively a skew reflection‡ and polar reciprocation in a quadric. Theorem 14 now gives the well known results that a general collineation is compounded of two polar reciprocations, and the general correlation of a skew reflection and a polar reciprocation. Resolving

$$\{\mathbf{T}_6\} \equiv \{\mathbf{T}_2\}\{\mathbf{T}_2'\}\{\mathbf{T}_2''\}, \tag{31}$$

gives the theorem due to WILSON, that the general collineation is compounded of three skew reflections. The resolution possesses nine degrees of freedom, and the discussion brings out some essential facts not given in the theorem of WILSON.

Let $\{\mathbf{T}_2\}$ in (31) be $\{\mathbf{ab}\}$. Then if $x$ and $x'$ are the directrices of $\{\mathbf{T}_2\}$,

* For discussion of the nullsystem of MÖBIUS, reference may be made to LINDEMANN, *Vorlesungen*, vol. 2, p. 52.

† VOSS has given theorem 14 for $t = n = 2r$; Mathematische Annalen, vol. 13, p. 343.

‡ Cf. WILSON, Transactions, vol. 1 (1900), pp. 193–196.

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$a = x + \lambda x'$, $b = x - \lambda x'$, and from the duality $\lambda f(x, y) + \mu w(x, y) = 0$ defining $\{T_6\}$, since $a$ and $b$ satisfy this, and also $f(a, b) = 0$, we find

$$\sum \alpha_{ik} x_i x'_k = 0 \qquad (\alpha_{ik} = -\alpha_{ki}), \tag{32}$$

and the directrices of each component of $\{T_6\}$ satisfy (32). Furthermore, it is readily found that the directrices of $\{T'_2\}$ and $\{T''_2\}$ belong to the congruence common to the complexes,

$$\lambda f(x, y) + \mu w(x, y) = 0, \qquad \lambda f(x', y) + \mu w(x', y) = 0, \tag{33}$$

and one line of this congruence may be chosen arbitrarily. The complete result therefore is

THEOREM 15. *A general collineation may be resolved in $\infty^9$ ways into the product of three skew reflections. The first pair of directrices satisfy* (32) *in three-dimensional space. Either one having been chosen, the other belongs to a linear line complex containing the first. The remaining four directrices then belong to the linear line congruence* (33), *and one having been chosen arbitrarily the other three are determined uniquely.*

## § 7. *Linear transformation of the alternating bilinear form into itself.*

The alternating form

$$w(x, y) \equiv \sum \alpha_{ik} x_i y_k = 0 \qquad (\alpha_{ik} = -\alpha_{ki}), \tag{34}$$

is invariant under the transformation obtained from (22) and (23) by changing the sign of the $x'$ in (22); viz.,

$$\sum_k (\lambda \alpha_{ik} + \mu \alpha_{ik}) x_i - \sum_i (\lambda \alpha_{ik} - \mu \alpha_{ik}) x'_i - \frac{2\lambda}{\rho} \sum_s \sigma^{(s)} \gamma_k^{(s)} = 0, \tag{35}$$

$$(\nu^{(s)} x) - (\nu^{(s)} x') = 0 \qquad (s = 1, 2, \cdots, t), \tag{36}$$

$$\alpha_{ik} = \alpha_{ki} \qquad (i, k = 1, 2, \cdots, n),$$

$x$ and $y$ being cogredient variables. For multiplying (35) by $y_k$ and summing up, with respect to $k$,

$$\lambda f(x, y) + \mu w(x, y) - \lambda f(x', y) + \mu w(x', y) - \frac{2\lambda}{\rho} \sum \sigma^{(s)} (\nu^{(s)} y) = 0. \tag{37}$$

In the same way are found three other equations, and from these is found by using (36),

$$w(x, y) = w(x', y').$$

The determinant of the solution of (35), (36) is $+1$, the reasoning being the same as in § 4. The characteristic function is therefore

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$$\phi(\rho) = \frac{(1-\rho)^t \phi^t(-\rho)}{\Delta^{(t)}(\lambda, \mu)},$$

$\phi^{(t)}(\rho)$ having same meaning as in (28).

The $a$'s in (35) are arbitrary, but not essential parameters. The number of the latter is readily found to be

$$\tfrac{1}{2}(n-t)(n-t+1) + t(n-t) = \tfrac{1}{2}(n-t)(n+t+1).$$

By the method of this section all linear transformations of the required type are found.*

*Cf. Voss, Münchener Berichte (1896), p. 20.

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# A SET OF POSTULATES FOR REAL ALGEBRA, COMPRISING POSTULATES FOR A ONE-DIMENSIONAL CONTINUUM AND FOR THE THEORY OF GROUPS*

BY

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The postulates for real algebra presented in this paper may be analysed into three groups: (1) propositions concerning the relation $<$, which, taken by themselves, form a set of independent postulates for a one-dimensional continuum, or a continuous scale; (2) propositions concerning the operation $+$, which, taken by themselves, form a set of independent postulates for the theory of groups; and (3) propositions connecting the two symbols $<$ and $+$. All these postulates taken together form a complete set of postulates for real ‡ algebra as given in §§ 1–4. Various modifications of this set are given in the later sections of the paper and in the appendix.

All these postulates are shown to be *independent*, that is, the list contains no redundancies; and the system which they determine is shown to be unique, §

* Presented to the Society at the St. Louis summer meeting, September 17, 1904, and to the Congress of Arts and Sciences, Section of Algebra and Analysis, St. Louis, September 22, 1904. Received for publication September 17, 1904.

† Received for publication October 3, 1904.

‡ A set of postulates for *complex* algebra will be presented in a subsequent paper.

§ A set of postulates having this property has been called a "categorical," as distinguished from a "disjunctive," set; see O. VEBLEN, Transactions, vol. 5 (1904), p. 346. Every proposition concerning a class $K$, a relation $<$, and an operation $+$, is either deducible from the postulates of this set, or in contradiction with them.



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that is, there is essentially only one system (namely the system of all real numbers) in which a relation $<$ and an operation $+$ are so defined as to satisfy all the postulates (see below, theorem 30.) The existence of the system of real numbers, as built up from the positive integers by the "genetic" method of successive generalization of the number concept, * proves, moreover, the *consistency* of the postulates.

The postulates for a continuum (§ 1) are the obvious ones (see, for example, SCHÖNFLIESS's *Bericht über Mengenlehre* †), but their independence has not heretofore been established.

The postulates for the theory of groups (§ 2 and § 5) carry the analysis farther, it is thought, than the earlier sets given by the writer ‡ and by E. H. MOORE.§

The postulates for real algebra (§ 3 and § 6) are more satisfactory than the writer's earlier set ‖ in several respects; first, the separation of the postulates concerning $<$ from those concerning $+$ is now complete; secondly, the individual postulates are more nearly simple statements (and are hence more numerous); and finally, in the statement of the postulates no assumption is made in regard to the existence of any kind of numbers. This last improvement was suggested by a recent paper by BURALI-FORTI, to which further reference will be made in § 3.

On the fundamental concepts involved, namely: *class*, and *element* of a class; *dyadic relation;* and *operation*, or *rule of combination;* see Transactions, vol. 5 (1904), p. 288–290; for further bibliographical references see also Transactions, vol. 3 (1902), p. 265, and vol. 4 (1903), p. 358; and the *Theoretische Arithmetik* of STOLZ and GMEINER (1901 —).

## § 1. A SET OF POSTULATES FOR A ONE-DIMENSIONAL CONTINUUM.

In this section we consider the conditions which must be imposed upon a *class* $K$, and a *dyadic relation* $<$ (read: "below," or "algebraically less than"),¶ in order that $K$ shall be a *one-dimensional continuum*, or a *continuous scale* with respect to $<$. These conditions are expressed in the following eight postulates: I, II, $R1$–$R6$ (the letter $R$ being prefixed to indicate that the postulates concern the Relation $<$).**

* Cf. D. HILBERT, *Über den Zahlbegriff*, Jahresbericht der deutschen Mathematiker-Vereinigung, vol. 8 (1900), part 1, pp. 180–184.

† *Ibid.*, vol. 8 (1900), part 2.

‡ E. V. HUNTINGTON, Bulletin of the American Mathematical Society, ser. 2, vol. 8 (1901–2), pp. 296–300; Transactions, vol. 4 (1903), pp. 27–30.

§ E. H. MOORE, Transactions, vol. 3 (1902), pp. 485–492; vol. 5 (1904), p. 549.

‖ Transactions, vol. 4 (1903), pp. 358–370.

¶ Since the notion of *quantity* is not involved here, the expressions "below" and "above" (compare a thermometer scale) are clearly preferable to "less than" and "greater than." The expressions "before" and "after" may also be used to advantage.

** A summary of these postulates will be given in § 4.

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The symbol $<$ here denotes a general dyadic relation, having no properties not expressly stated in the postulates. When, as in § 4, there is danger of confusion with the ordinary $<$ of arithmetic, we shall enclose the more general symbol in a circle, thus: $\textcircled{<}$; but the omission of the circle, whenever it is not absolutely necessary, will give the theorems a more familiar appearance, and thus facilitate the reading. Note that $a < b$ ($a$ "below" $b$) and $b > a$ ($b$ "above" $a$) denote the same relation between $a$ and $b$; and that $a \leqq b$ means: $a < b$ or $a = b$.

POSTULATE I. There is an entity which belongs to the class.

This postulate tells us that $K$ is not an "empty" class.

POSTULATE II. If $a$ is an element of the class, there is an element $b$ such that $a \neq b$.

This postulate excludes the trivial case of a class containing only a single element. — Postulates I and II together give us

*Theorem* 1. *There are at least two distinct elements in the class.*

POSTULATE $R1$. If $a$ and $b$ are elements of the class, and $a \neq b$, then either $a < b$ or $a > b$.

POSTULATE $R2$. If $a$ and $b$ are elements of the class, and $a \neq b$, then $a < b$ and $a > b$ cannot both be true.

That is, the relation $<$ is "non-symmetric" for every pair of distinct elements.

POSTULATE $R3$. If $a$ is an element of the class, then $a < a$ cannot be true.

That is, the relation $<$ is "non-reflexive" for every element of the class. — From postulates $R1$–$R3$ we have at once:

*Theorem* 2. *If $a$ and $b$ are elements of the class, then either*

$$a = b, \qquad a < b, \qquad \textit{or} \qquad a > b;$$

*and these three relations are mutually exclusive.*

POSTULATE $R4$. If $a$, $b$, $c$ are elements of the class, and $a \neq c$, then from $a < b$ and $b < c$ follows $a < c$.

In view of postulate $R2$, this postulate gives us

*Theorem* 3. *From $a < b$ and $b < c$ follows $a < c$, whenever $a$, $b$ and $c$ belong to the class.*

That is, the relation $<$ is "transitive."

The postulates $R1$–$R4$, with I and II, make the class a "scale" or a "simply ordered assemblage" with respect to the relation $<$; or, an assemblage possessing "*Grössencharakter*" in the sense defined by SCHÖNFLIESS (loc. cit.).

POSTULATE $R5$. If $a$ and $b$ are elements of the class such that $a < b$ or $a > b$, then: if $a \neq b$, there is either an element $x$ such that $a < x$ and $x < b$, or an element $y$ such that $a > y$ and $y > b$.

Such an element $x$ (or $y$) is said to lie "between" the elements $a$ and $b$; in view of theorems 1–3, we have

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*Theorem* 4. *If $a$ and $b$ are two distinct elements, there is at least one, and therefore an infinite number of elements lying "between" them; and hence the class $K$ itself is infinite.*

Thus postulate $R5$, taken with the preceding postulates, makes the class "dense" (*dicht, compact*) with respect to the relation $<$. (Without this postulate, the class might be infinite but "discrete," like the class of integral numbers, or, it might consist of merely a finite number of elements.) But a "dense" class is not necessarily continuous (as witness the system of all rational numbers with regard to $<$); we therefore add the following *postulate of continuity*, which may be stated in either of two forms, $R6$ or $R6'$:

POSTULATE $R6$. If $\Gamma$ is a non-empty sub-class in $K$, and if there is an element $c$ in $K$ such that every element of $\Gamma$ is $< c$, then there is an element $X$ in $K$ having the two following properties with regard to the sub-class $\Gamma$:

1°) if $\alpha$ is an element of $\Gamma$, then $\alpha \leqq X$; while

$\neq X$ and

2°) if $x$ is any element of $K$ which is $< X$, there is an element in $\Gamma$ which is $> x$.

This element $X$, which is readily seen to be uniquely determined by the sub-class $\Gamma$, is called the "upper limit" of the sub-class, or sometimes its "lowest upper bound." Thus, if a subclass $\Gamma$ has *any* "upper bound" $c$ in $K$, it will have a "*lowest* upper bound" $X$. If $X$ happens to belong to the sub-class $\Gamma$, it is its highest element; but a sub-class $\Gamma$ may have an upper limit and yet not have any highest element.

In place of postulate $R6$, and as equivalent to it, we might use

*Postulate* $R6'$: The same as postulate $R6$ with the symbols $<$ and $>$ interchanged.

In brief, if a sub-class $\Gamma'$ has *any* "lower bound" in $K$, it will have a "*highest* lower bound," called also its "lower limit." This element will be uniquely determined by the sub-class $\Gamma'$, and if it belongs to the sub-class it will be its lowest element.

Each of these postulates, $R6$ and $R6'$ can be deduced from the other (with the aid of the preceding postulates),* and from both together we have the following theorem:

*Theorem* 5. *If $\Gamma$ and $\Gamma'$ are two non-empty sub-classes in $K$, such that every element of $\Gamma$ is $<$ every element of $\Gamma'$, then there is at least one element $M$, which is $\geqq$ every element of $\Gamma$ and $\leqq$ every element of $\Gamma'$.*

* Thus, suppose, in the first place, that $R6$ is given, then the proof for $R6'$ is as follows: In the hypothesis, the sub-class $\Gamma'$ is a non-empty class, which has *some* lower bound, say $c'$; we are to prove that it has *a highest* lower bound, $X'$. To show this, we define a sub-class $\Gamma$ as the class of all those elements in $K$ which are below every element in $\Gamma'$. This class $\Gamma$ will be a non-empty class, since it contains the element $c'$, and it will have some upper bound, since any element of the non-empty class $\Gamma'$ will answer that purpose. Therefore, by postulate $R6$, the sub-class $\Gamma$ will have an upper limit, $X$. It is then easy to show that this element $X$ will be the required lower limit of the sub-class $\Gamma'$. — In like manner, $R6$ could be deduced from $R6'$.

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For, $\Gamma$ will have its upper limit, $X$, and $\Gamma'$ will have its lower limit, $X'$, by postulates $R6$ and $R6'$, and $X \leqq X'$. Then take $M = X$, or $X'$, or any element which may lie between $X$ and $X'$.

In particular, if the two sub-classes $\Gamma$ and $\Gamma'$ make up, together, the whole class $K$, the upper limit of $\Gamma$ will coincide with the lower limit of $\Gamma'$, there will be only one element $M$, and this will be either the highest element in $\Gamma$ or the lowest element in $\Gamma'$.*

Thus every system $(K, <)$ which satisfies postulates I, II, $R1$–$R6$ will have the property of continuity as defined by DEDEKIND in his *Stetigkeit und irrationale Zahlen*, and may be called a *one-dimensional continuum* or a *continuous scale*, with respect to the relation $<$. [See below, end of § 1.]

Moreover, these eight postulates I, II, $R1$–$R6$, as will be shown in § 4, are *independent* of each other; that is, no one of them can be deduced from the remaining seven.

---

In conclusion, we notice that a one-dimensional continuum, or continuous scale, as defined by postulates I, II, $R1$–$R6$, may be either limited or unlimited, in either direction. To complete the discussion of the present section, therefore, we add the two following postulates, although they will not be required in the remainder of this paper:

*Postulate R7.* If $a$ is an element of the class, and if there is any element $b \neq a$, then there is an element $x$ such that $x < a$;

*Postulate R8.* If $a$ is an element of the class, and if there is any element $b \neq a$, then there is an element $y$ such that $y > a$.

Hence, the class has neither a lowest nor a highest element, but is unlimited in both directions.

The ten postulates: I, II, $R1$–$R8$, form a set of independent postulates (see § 4), which define completely the properties of an *unlimited* continuous scale with respect to $<$. For, if $C$ and $C'$ are two systems $(K, <)$ which satisfy all these postulates, they clearly can be brought into one-to-one correspondence † in such a manner that if $a < b$ in $C$, then $a' < b'$ in $C'$, where $a'$ and $b'$ are the elements which correspond to $a$ and $b$ respectively. Hence the set is "categorical," ‡ that is, every proposition concerning $K$ and $<$ must either be deducible from the postulates of the set, or be in contradiction with them.

See footnote on p. 211 for correct of error.

[It should be noticed that these postulates, in spite of the fact that they form a categorical set, are hardly sufficient to define a continuum in the ordinary *geometric* sense of the word. For example, the class of positive real numbers

* Notice also that postulate $R6$ can be deduced from theorem 5 (with the aid of the preceding postulates), by taking as $\Gamma'$ the subclass of all the elements of $K$ which are $>$ every element of $\Gamma$.

† And that in an infinite number of ways.

‡ O. VEBLEN: loc. cit.

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between 0 and 3, omitting the numbers $x$ for which $1 < x \leqq 2$, is a class which satisfies all the conditions but is not a geometric continuum in any ordinary sense. All we can say is that it is *equivalent* to a continuum *as far as our relation* $<$ *is concerned.**]

## § 2. A SET OF POSTULATES FOR ABELIAN GROUPS.†

In this section we consider the condition which must be imposed upon a *class* $K$, and an *operation* or *rule of combination* $+$ (read: "plus"), in order that $K$ shall be an *abelian group* with respect to $+$. These conditions are expressed in the following eight postulates: I, II, $A1$–$A6$ (the letter $A$ being prefixed to indicate that the postulates concern Abelian groups, or if one will, the operation of Addition).‡

The symbol $+$ here denotes a general operation, having no properties not expressly stated in the postulates. When, as in § 4, there is danger of confusion with the ordinary $+$ of arithmetic, we shall enclose the more general symbol in a circle, thus: $\oplus$.

POSTULATE I. There is an entity which belongs to the class.

POSTULATE II. If $a$ is an element of the class, there is an element $b$ such that $a \neq b$.

These postulates serve to exclude the cases of an "empty" class, and a class containing a single element. Together they give

*Theorem* 1. *There are at least two distinct elements in the class.*

POSTULATE $A1$. If $a$ and $b$ are elements of the class, then $a + b$ is an element of the class.

This postulate states the fundamental property of a *group*, the element $a + b$ being called, for our present purpose,§ the "sum" of the elements $a$ and $b$.

POSTULATE $A2$. If $a$, $b$, $a + b$, and $b + a$ are elements of the class, then

$$a + b = b + a.$$

This is the "commutative law" for the operation $+$, and is the characteristic property of abelian groups as distinguished from groups in general.

* On G. CANTOR's definition of a continuum, which differs somewhat from DEDEKIND's definition, see B. RUSSELL, *Principles of Mathematics*, vol. 1 (1903), chap. 36.

† For a further analysis of the postulates for the theory of groups, see § 5.

‡ A summary of these postulates will be given in § 4.

§ If we interpreted the operation $\oplus$ as "multiplication," as one usually does in the general theory of abstract groups, we should call $a \oplus b$ the "product" of $a$ and $b$. So below, instead of the "zero-element" $(0)$, the "negative of $a$" $(-a)$, the operation of "subtraction" $(b - a)$, etc., we should speak of the "unit element" $(1)$, the "reciprocal of $a$" $(a^{-1})$, the operation of "division" $(b/a)$, etc. Again, if we were dealing with groups of transformations, we should call 1 the "identical transformation," or the "identity," and $a^{-1}$ the "inverse of $a$." Compare § 5.

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POSTULATE $A3$. If $a$, $b$, $c$, $a+b$, $b+c$, $(a+b)+c$, and $a+(b+c)$ are elements of the class, then

$$(a+b)+c=a+(b+c).$$

This is the "associative law" for the operation $+$. These three postulates give us

*Theorem* 6.* (*a*) *The operation* $+$ *is always possible within the class, and it obeys* (*b*) *the commutative law and* (*c*) *the associative law.*

POSTULATE $A4$. If $a, x, y, a+x$, and $a+y$ are elements of the class, then from $a+x=a+y$ follows $x=y$.

Hence, in view of this commutative law, we have

*Theorem* 7. *A change in either* $a$ *or* $b$ *alone produces a change in* $a+b$.

POSTULATE $A5$. If there is any element in the class, then there is an element 0 such that $0+0=0$.

It is easy to show that for any such element 0, and for every element $a$, $a+0=0+a=a$; and hence that the element 0 is uniquely determined.† Thus we have the theorem:

*Theorem* 8. (*a*) *There is a uniquely determined element* 0 *such that* $0+0=0$; (*b*) *for every element* $a$

$$a+0=a \qquad \textit{and} \qquad 0+a=a;$$

*and* (*c*) *if* $a+x=a$ *or* $x+a=a$, *then* $x=0$.

This element 0 may be called, for our present purpose, the "zero-element" of the class, the properties in theorem 8 being the "additive" properties of zero. When, as in § 4, there is any danger of confusing this symbol with the ordinary 0 of arithmetic, we shall enclose it in a circle, or (which is typographically more convenient) denote the element in question by $z$ (the initial letter of "zero"), or by $i$ (the initial letter of "identity").‡

POSTULATE $A6$. If there is a uniquely determined element 0 such that $0+0=0$, then for every element $a$ there is an element $\bar{a}$ such that $a+\bar{a}=0$.

In view of postulate $A4$, this element $\bar{a}$ is uniquely determined by $a$, and may be called the "negative of $a$." ‡

Further, if we take $x=\bar{a}+b$, we have

$$a+x=a+(\bar{a}+b)=(a+\bar{a})+b=0+b=b;$$

hence,

* The theorem-numbers 2–5 have been used in § 1.

† Thus, if $z+z=z$, we have $z+(a+z)=z+(z+a)=(z+z)+a=z+(a)$, whence, by postulate $A4$, $a+z=a$, and, by the commutative law, $z+a=a$. Hence, if $z_1$ and $z_2$ were two elements such that $z_1+z_1=z_1$ and $z_2+z_2=z_2$, then $a+z_1=a$ and $b+z_2=b$, for all values of $a$ and $b$; whence, taking $a=z_2$ and $b=z_1$, we should have $z_2+z_1=z_2$ and $z_1+z_2=z_1$; or, $z_1=z_2$.

‡ Compare footnote under postulate $A1$.

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*Theorem* 9. (*a*) *Every two elements*, $a$ *and* $b$, *determine uniquely a third element* $x$, *denoted by* $b - a$, *such that*

$$a + (b - a) = b;$$

(*b*) *in particular*,

$$a - a = 0;$$

*and* (*c*) *the element* $0 - a$ *is usually abbreviated into* $-a$, *so that*

$$a + (-a) = 0.$$

The element $b - a$ is called the "difference" $b$ minus $a$, and the operation of finding it, the "subtraction" of $a$ from $b$.*

From theorems 1, and 6–9, we see that *every system* $(K, +)$ *which satisfies the eight postulates*, I, II, $A1$–$A6$, *is an abelian group with respect to* $+$, according to the usual definition.† These eight postulates, as will be shown in § 4, are *independent* of each other; that is, no one of them can be deduced from the other seven. The complete theory of abelian groups would contain all the propositions which follow from these eight postulates by logical deduction.

In conclusion, it should be noticed that the eight postulates of § 2 form a "disjunctive," not a "categorical," set; ‡ for an abelian group may contain any finite number of elements, or be infinite; and even if the number of elements in two groups is the same, the groups are not necessarily isomorphic; hence there are many propositions concerning $K$ and $+$ which are neither deducible from these postulates, nor in contradiction with them.

## § 3. A SET OF POSTULATES FOR REAL ALGEBRA.

In this section, which forms the main part of the present paper, we consider the conditions which must be imposed upon a *class* $K$, with a *dyadic relation* $<$ (read: "below," or "algebraically less than"), and an *operation* $+$ (read: "plus"), in order that $K$ shall be the class of *all real variables* with respect to $<$ and $+$. (On the use of the symbols $<$ and $+$, or $\odot$ and $\oplus$, see the opening paragraphs in § 1 and § 2.) These conditions are expressed in sixteen postulates, § which fall into four groups as follows:

POSTULATES I–II. (As in § 1 or § 2.)

These postulates give the class at least two distinct elements.

* Compare footnote under postulate $A1$.

† H. WEBER, *Algebra*, vol. 2. Cf. E. V. HUNTINGTON, and E. H. MOORE, loc. cit.; and also G. A. MILLER, *Report on the groups of an infinite order*, Bulletin of the American Mathematical Society, vol. 7 (1900–1), pp. 121–130.

‡ O. VEBLEN, loc. cit.

§ A summary of these postulates is given in § 4.

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POSTULATES $R1$–$R6$. (As in § 1.)

These postulates, with I and II, make the class a *continuous scale*, or a *one-dimensional continuum*, with respect to the relation $<$.

POSTULATES $A1$–$A6$. (As in § 2.)

These postulates, with I and II, make the class an *abelian group* with respect to the operation $+$.

POSTULATE $RA1$. If there is a uniquely determined element 0 such that $0 + 0 = 0$; and if $a$, $b$, and $a + b$ are elements of the class; and if $a > 0$ and $b > 0$; then $a + b > a$.

POSTULATE $RA2$. If there is a uniquely determined element 0 such that $0 + 0 = 0$; and if $a$, $b$, and $a + b$ are elements of the class; and if $a < 0$ and $b < 0$; then $a + b < a$.

These last two postulates serve to connect the symbols $<$ and $+$. In view of postulates I, II, and $A1$–$A5$, they give us at once:

*Theorem* 10. *If $a$ and $b$ are both above* 0, *then $a + b$ is above either of them; and if $a$ and $b$ are both below* 0, *then $a + b$ is below either of them.*

These sixteen postulates: I–II, $R1 - R6$, $A1 - A6$, $RA1$ and $RA2$, are *independent*, as will be shown in § 4, so that no one of them can be deduced from the remaining fifteen. On the other hand, these sixteen postulates, or the theorems 1–10 which follow from them,* are *sufficient to define completely the algebra of a real variable.* The remainder of this section will be devoted to the proof of this latter statement, the necessary preliminary theorems, besides the theorems 1–10, grouping themselves under the following heads: 1) positive and negative elements; 2) multiples, with remarks on BURALI-FORTI's definition of number; 3) ARCHIMEDES' principle; and 4) submultiples, with the theorem of infinite divisibility.

*Positive and negative elements.*

The following theorems 11–17 are proved without the use of postulate $R5$ (on density) or postulate $R6$ (on continuity).

*Theorem* 11. *If $a$ and $b$ are both $> 0$, then $a + b > 0$; and if $a$ and $b$ are both $< 0$, then $a + b < 0$.* (By 10 and 3.)

*Theorem* 12. *If $a > 0$, then $-a < 0$; and if $a < 0$, then $-a > 0$.* (By 11 and 9.)

Thus the class $K$ is composed of three mutually exclusive, non-empty subclasses: 1) the element 0; 2) the elements above 0, called the *positive* elements; 3) the elements below 0, called the *negative* elements. (By I, II, 12 and 3.)

*Theorem* 13. *If $x > 0$, then $a + x > a$; if $x < 0$, then $a + x < a$.*

The first part is clearly true when $a = 0$, and (by 10) when $a > 0$; the proof for the remaining case, $a < 0$, is as follows: Suppose $a + x \leqq a$, while

* It should be noted that the sixteen postulates are clearly deducible from the theorems 1–10.

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$x > 0$ and $a < 0$; then $a + x < 0$, by 3, and $-x < 0$, by 12; hence by 10, $(a + x) + (-x) < a + x$, or, by 6 and 9, $a < a + x$, which contradicts the supposition. The second part may be proved in a similar way.

*Theorem* 14. *If $a < b$ there is a positive element $x$ such that $a + x = b$ and a negative element $y$ such that $a = b + y$; and conversely, if $x > 0$ in $a + x = b$, or if $y < 0$ in $a = b + y$, then $a < b$.* (By 9 and 13.)

*Theorem* 15. *If $x < y$, then $a + x < a + y$, and conversely.*

For, by 14, take $w < 0$ so that $x = y + w$. Then, by 13,

$$(a + y) + w < (a + y),$$

whence $a + (y + w) < a + y$, or $a + x < a + y$. The converse is proved indirectly, as usual.

*Theorem* 16. *If $a < b$ and $x < y$, then $a + x < b + y$.*

For, $a + x < a + y$, and $a + y < b + y$, by 15 and the commutative law; hence the theorem, by 3.

*Theorem* 17. *There is no highest element, and no lowest element, in the class.* (By 13.)

If we admit also the postulate of density ($R5$), we have further:

*Theorem* 18. *There is no highest or lowest element in the sub-class of positive elements, or in the sub-class of negative elements.*

## *Multiples.*

The theory of the multiples of an element $a$ is closely connected with the theory of the (finite) ordinal numbers (or positive integers). In fact C. BURALI-FORTI, in a recent memoir,* has shown that the theory of multiples can be developed without presupposing any knowledge of these numbers, and that the class of numbers can then be defined by means of the theory of multiples. For the present purpose, however, it seems best to assume the ordinal numbers as known, especially in view of a valuable discussion of BURALI-FORTI'S work by L. COUTURAT, in a memoir † which came into my hands while writing this paper.

In order to state precisely what is involved in thus assuming the ordinal numbers, it may be well to recall the following facts.

The class of ordinal numbers, however it may be defined, is a class $N$ which possesses the following characteristic properties, due to PEANO: ‡

* C. BURALI-FORTI, *Sulla teoria generale delle grandezze e dei numeri*, Atti della R. Accademia delle Scienze di Torino, vol. 39 (1903–4), pp. 256–272. On BURALI-FORTI'S proof of the commutative law, see footnote below, in § 6.

† L. COUTURAT, *Les principes des mathématiques; V: L' idée de grandeur*, Revue de Métaphysique et de Morale, vol. 12 (1904), pp. 675–698.

‡ For a brief account of these postulates, with bibliographical references, see Bulletin of the American Mathematical Society, ser. 2, vol. 9 (1902–3), pp. 41–46. An extended discussion is found in B. RUSSELL'S *Principles of Mathematics*, vol. 1 (1903), chap. 14. A revised list, in which the number of postulates is reduced to four, has been given by A. PADOA, *Théorie des nombres entiers absolus*, Revue de Mathématiques, vol. 8 (1902), p. 48.

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1. The class $N$ contains a peculiar element, called the "first" ordinal number, and denoted by 1.

2. Every element $m$ of the class $N$ determines uniquely another element of $N$ called the number "next following" the number $m$, and denoted by $m^+$, or seq $m$. $(1^+ = 2;\ 2^+ = 3;$ etc.)

3. For any number $m$, $m^+$ is different from 1.

4. If $m^+ = n^+$, then $m = n$.

5. If $S$ is any class such that: $(a)$ the number 1 belongs to $S$, and $(b)$ the number $m^+$ belongs to $S$ whenever the number $m$ belongs to $S$; then every element of $N$ belongs to $S$.

This list of five postulates comprises as much of the theory of the ordinal numbers as we need to assume in the present paper.

On the basis of these fundamental propositions, the *sum*, $p + q$, of two numbers, $p$ and $q$, is the number defined by the following recurrent formulæ:

$$p + 1 = p^+,\ p + 2 = (p + 1) + 1,\ p + 3 = (p + 2) + 1, \cdots,$$
$$p + (k + 1) = (p + k) + 1;$$

whence, by mathematical induction, $p + q = q + p$.

Their *product*, $pq$, is defined by the following recurrent formulae:

$$1p = p,\ 2p = 1p + p,\ 3p = 2p + p, \cdots, (k + 1)p = kp + p;$$

whence, by mathematical induction, $pq = qp$.

Finally, if $p \neq q$, then there is either a number $\lambda(= p - q)$ such that $p = q + \lambda$, or else a number $\mu(= q - p)$ such that $q = p + \mu$, and not both; in the first case we write $p > q$ ($p$ "greater than" $q$), in the second case, $p < q$ ($p$ "less than" $q$).

Making use of these properties of the ordinal numbers, we may now define the multiples of any element of our abstract class $K$. But we must be careful to notice that the symbols $+$ and $<$, as used between two ordinal numbers, have nothing to do with the general symbols $+$ and $\prec$ (or $\oplus$ and $\odot$) as used between the elements of $K$.

We define the $m$th *multiple*, $ma$, *of any element* $a$, $m$ being any ordinal number, by the following recurrent formulæ:

$$1a = a,\ 2a = 1a + a,\ 3a = 2a + a,\ \cdots,\ (k + 1)a = ka + a.$$

Hence, if $p$ and $q$ denote ordinal numbers, and $a$ an element of $K$, we have at once:

*Theorem* 19. 1°) $pa + qa = (p + q)a$;

2°) $q(pa) = p(qa) = (qp)a = (pq)a = pqa$.

This theorem depends only on postulates I, II, and $A1$; the following theorems

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20–22, however, require all the postulates except $R5$ (on density) and $R6$ (on continuity):

*Theorem* 20. *If $a$ is above* 0, *the successive multiples of $a$ form an ascending series:* $(m+1)a > ma$; *if $a$ is below* 0, *the successive multiples of $a$ form a descending series:* $(m+1)a < ma$; *and if* $a = 0$, *every multiple of $a$ is* 0: $ma = 0$.

*In particular, if* $a \neq 0$, $pa = qa$ *when and only when* $p = q$.

*Theorem* 21. *If* $a < b$, *then* $ma < mb$, *and conversely.*

For, if $a < b$, then $a = b + x$, where $x < 0$, by 14; hence $ma = mb + mx$, where $mx < 0$, by 20. Therefore $ma < mb$, by 14. Conversely, if $ma < mb$, then $a < b$, since $a \geqq b$ leads to contradictions.

*Theorem* 22. *If* $ma = mb$, *then* $a = b$. (From 21.)

*The principle of Archimedes.*

The following theorem requires all the postulates except the postulate of density ($R5$).

*Theorem* 23. *If $a$ and $b$ are both above* 0, *there are multiples of $a$ which are above $b$; and if $a$ and $b$ are both below* 0, *there are multiples of $a$ which are below $b$.* (Principle of ARCHIMEDES.)

The proof of the first part is as follows: Suppose every multiple of $a$ were $\leqq b$; then, by the postulate of continuity ($R6$), the class of multiples of $a$ would have an upper limit, $X$, having the following properties:

1°) $na \leqq X$, for every ordinal number $n$ (in particular, $a < X$); and

2°) if $x < X$, there is some multiple of $a$, say $n'a$, which is $> x$.

By theorem 14, take $x < X$ so that $x + a = X$; then by 2°), $n'a > x$ for at least one value of $n'$. Hence, by theorem 15 and the commutative law, $n'a + a > x + a$, whence $(n'+1)a > X$, which contradicts 1°).

The second part is proved in a similar way.

*Submultiples.*

The following lemma requires all the postulates except $R6$ (on continuity); while the theorem 24 to which it leads, requires the whole list.

*Lemma.* If $m$ is any ordinal number, then, if $a$ is positive, there is a positive element $x$ such that $mx < a$; and if $a$ is negative, there is a negative element $y$ such that $my > a$.

Proof. The lemma is clearly true when $m = 1$, by 18, and if it is true when $m = n$, it will be true when $m = n + 1$. For (considering the case $a > 0$), if $nx < a$, then there is a positive $\xi$ such that $nx + \xi = a$, by 14, and a positive $x'$ such that $x' < x$ and $x' < \xi$, by 18. Then $nx' < nx$ by 21, whence, $nx' + x' < nx + \xi$, by 16, or $(n+1)x' < a$. Therefore the lemma is true for every value of $m$.—Similarly for the case $a < 0$.

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*Theorem* 24. *If $m$ is any ordinal number, then every element $a$ determines uniquely an element denoted by $a/m$, called the $m$th submultiple of $a$, such that*

$$m(a/m) = a.$$

*In particular:*

$$a/1 = a \qquad \textit{and} \qquad 0/m = 0.$$

Proof. We notice in the first place, by 22, that this element $a/m$, if it exists at all, will be uniquely determined by $a$ and $m$. To prove the existence of any such element (for a positive $a$), we define two subclasses: $\Gamma$, containing all the positive elements $y$ for which $my < a$; and $\Gamma'$, containing all the positive elements $y'$ for which $a < my'$. Now $\Gamma$ and $\Gamma'$ are non-empty classes, by the lemma just proved, and 18; and every element in $\Gamma$ is $<$ every element in $\Gamma'$, by 21. Hence, by theorem 5, there is a positive element $X$ such that: 1°) every element in $\Gamma \leqq X$, and 2°) $X \leqq$ every element in $\Gamma'$. This element $X$ is the required element $a/m$. For if $mX$ were $< a$, there would be an element $y$ in $\Gamma$ which is $> X$, in contradiction with 1°)*; and if $mX$ were $> a$, there would be an element $y'$ in $\Gamma'$ which is $< X$, in contradiction with 2°).† Therefore $mx = a$. — The proof is similar for a negative $a$.

*Theorem* 25. *If $p$ and $q$ are ordinal numbers such that $p < q$, then if $a$ is above* 0, $a/p > a/q$, *and if $a$ is below* 0, $a/p < a/q$.

In other words, if $a \neq 0$, increasing the value of $m$ brings the element $a/m$ nearer to 0. In particular, if $a > 0$, $a > a/m > 0$, and if $a < 0$, $a < a/m < 0$; where $m > 1$. (Proof indirect, by 20, 21 and 24.)

*Theorem* 26. *If $a$ and $b$ are both above* 0, *there are submultiples of $a$ which are below $b$; and if $a$ and $b$ are both below* 0, *there are submultiples of $a$ which are above $b$.* (Proof indirect, by 23 and 24.)

*Theorem* 27. *If $p$ and $q$ denote ordinal numbers, and $a$ an element of $K$, then $p(a/q) = (pa)/q$.* (By 24.)

Hence we may denote either member of this equation by $pa/q$.

Any element of the form $pa/q$, where $p$ and $q$ are ordinal numbers, is called a *rational fraction* of $a$. In particular, $p0/q = 0$.

*Theorem* 28. *If $m, p, q, p', q'$, are ordinal numbers, then:*

(*a*) $mpa/mq = pa/q$.

(*b*) $pa/q + p'a/q' = (q'p + qp')a/qq'$.

(*c*) $pa/q = p'a/q'$ when and only when $q'p = qp'$.

*Theorem* 29. *If $a$ is above* 0 [*below* 0], *then $pa/q < p'a/q'$ when and only when $q'p < qp'$ [or $q'p > qp'$].* (Proof by 20, 21 and 24.)

---

* Take $\xi > 0$ so that $mX + \xi = a$, by 14, and $\eta > 0$ so that $m\eta < \xi$, by the lemma, and let $y = X + \eta$. Then $my = mX + m\eta < mX + \xi = a$.

† Take $\xi' < 0$ so that $mX + \xi' = a$, by 14, and $\eta' < 0$ so that $m\eta' > \xi'$, by the lemma, and let $y' = X + \eta'$. Then $my = mX + m\eta' > mX + \xi' = a$.

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The theorems 1–29 put us now in a position to establish the fact that the postulates of § 3 are sufficient to define completely the algebra of a real variable; this sufficiency is formulated in the following theorem:

*Theorem* 30. *Any two systems,* $(K, <, +)$ *and* $(K', <, +)$, *which satisfy the sixteen postulates of* § 3 *are* EQUIVALENT *with respect to* $<$ *and* $+$; *that is, they can be brought into one-to-one correspondence* in such a way that*

1°) *from* $a < b$ *follows* $a' < b'$, *and conversely; and*

2°) *from* $a + b = c$ *follows* $a' + b' = c'$, *and conversely;*

*where* $a'$, $b'$, $c'$ *are the elements in* $K'$ *which correspond to the elements* $a$, $b$, $c$ *in* $K$.

Proof. First make the zero-element in $K$ correspond to the zero-element in $K'$.

Next, to bring the positive elements of $K$ into one-to-one correspondence with the positive elements of $K'$, choose any positive element $E$ in $K$, and any positive element $E'$ in $K'$, and call $E$ and $E'$ corresponding elements; then if any positive element $a$ is given, the corresponding positive element $a'$ is determined as follows: consider in $K$ the subclass $\Gamma$ composed of all rational fractions of $E$ which are $< a$; form in $K'$ the subclass $\Gamma'$ composed of the *same* rational fractions of $E'$; then the upper limit of $\Gamma'$ is the required element $a'$. Similarly if $a'$ is given and $a$ required.

Finally, bring the negative element of $K$ into one-to-one correspondence with the negative elements of $K'$ by making $a$ correspond to $a'$ whenever $-a$ correspond to $-a'$.

The correspondence as thus established is readily shown to have the properties 1°) and 2°), by the aid of the theorems in regard to rational fractions.

*Note on multiplication.*

The deductions from the postulates have been carried, in this section, only so far as was necessary for the proof of theorem 30. The further development of the algebra depends on the definition of the *product* of two elements, with respect to an arbitrarily chosen (positive) "unit element."† [The arbitrary choice of this "unit element" is of great significance in the *theory of measurement*, into which I shall not enter here.]

Multiplication may also be regarded as a fundamental operation whose properties are determined by postulates, as is done below, in the appendix.

* And that in an infinite number of ways, since the "unit elements" $E$ and $E'$ in the proof may be chosen in an infinite number of ways.

† Cf. E. V. HUNTINGTON, Transactions of the American Mathematical Society, vol. 4 (1903), p. 365.

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## § 4. Summary of the postulates, and proof of their independence.

In the following summary of the postulates of § 3 for real algebra, each postulate is stated twice, first in the precise and convenient symbolism * of Peano's *Logica Mathematica*, and secondly in common language. To avoid confusion in the proofs of independence, we enclose the general symbols $<$ and $+$ in circles, and write $z$ in place of $0$.

Postulate I. $\qquad \exists K;$

that is, there is an entity which belongs to the class.

Postulate II. $\qquad a \epsilon K . \supset . \exists K \frown b \ni [a \neq b];$

that is, if $a$ is an element of the class, there is an element $b$ such that $a \neq b$.

Postulate $R1$. $\quad a, b \epsilon K . a \neq b . \supset : a \circledcirc b . \smile . a \circledcirc b;$

that is, if $a$ and $b$ are elements of the class, and $a \neq b$, then either $a \circledcirc b$ or $a \circledcirc b$.

Postulate $R2$. $\quad a, b \epsilon K . a \neq b . \supset : a \circledcirc b . a \circledcirc b . = . \Lambda;$

that is, if $a$ and $b$ are elements of the class, and $a \neq b$, then $a \circledcirc b$ and $a \circledcirc b$ cannot both be true.

Postulate $R3$. $\qquad a \epsilon K . \supset . a \circledcirc a . = . \Lambda;$

that is, if $a$ is an element of the class, then $a \circledcirc a$ cannot be true.

Postulate $R4$. $\quad a, b, c \epsilon K . a \neq c . a \circledcirc b . b \circledcirc c . \supset . a \circledcirc c;$

that is, if $a, b, c$ are elements of the class, and $a \neq c$, then from $a \circledcirc b$ and $b \circledcirc c$ follows $a \circledcirc c$.

Postulate $R5$. $\quad a, b \epsilon K : a \circledcirc b . \smile . a \circledcirc b : a \neq b : \supset :$

$$\exists K \frown x \ni [a \circledcirc x . x \circledcirc b] . \smile . \exists K \frown y \ni [a \circledcirc y . y \circledcirc b];$$

---

* See Burali-Forti. *Logica Matematica* (*Manuali Hoepli*, 1894), or the introductions to the several volumes of the *Formulaire de Mathématiques* (1895–1903). In reading statements written in these symbols, look first for the *dots*, which occur in groups of one, two, or more, and divide the sentence into its component parts; the larger the number of dots, the more important the point of division. Next find the symbol $\supset$, which is the sign of illation; thus, "$a . b . \supset . c$" means "*if* the propositions $a$ and $b$ are true, *then* the proposition $c$ is true;" or, "the truth of propositions $a$ and $b$ *implies* the truth of propositions $c$." For the other symbols used in this paper, the following glossary will be sufficient:

"$a . \smile . b$" means "either $a$ or $b$."

"$a \epsilon K$" means "$a$ belongs to the class $K$;" "$a, b \epsilon K$" means "$a$ and $b$ both belong to the class $K$. (The "$\epsilon$" is the initial letter of "ἐστί.")

"$\exists K$" means "there exists an element of the class $K$." "$\ni$" means "such that."

"$\exists K \frown a \ni$" means "there exists an element $a$, in the class $K$, such that."

"$\Gamma \epsilon$ Cls' $K$" means "$\Gamma$ is a sub-class in the class $K$."

"$a . = . \Lambda$" means "the proposition $a$ is false."

"Hp" means "hypothesis."

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that is, if $a$ and $b$ are elements of the class such that $a \circledcirc b$ or $a \circledcirc b$, then: if $a \neq b$, there is either an element $x$ such that $a \circledcirc x$ and $x \circledcirc b$, or an element $y$ such that $a \circledcirc y$ and $y \circledcirc b$.

POSTULATE $R6$. $\Gamma \epsilon \text{ Cls}' K . \exists \Gamma . \exists K \frown c \ni [\alpha \epsilon \Gamma . \supset . \alpha \circledcirc c] . \supset .$

$x \neq \bar{X}$.

$$\exists K \frown X \ni [\alpha \epsilon \Gamma . \supset . \alpha \underline{\circledcirc} X : x \epsilon K . x \circledcirc X . \supset . \exists \Gamma \frown \xi \ni (\xi \circledcirc x)] ;$$

that is, if $\Gamma$ is a non-empty sub-class in $K$, and if there is an element $c$ in $K$ such that every element of $\Gamma$ is $\circledcirc c$, then there is an element $X$ in $K$ having the two following properties with regard to the sub-class $\Gamma$:

1°) if $\alpha$ is an element of $\Gamma$, then $\alpha \underline{\circledcirc} X$; while

2°) if $x$ is any element of $K$ which is $\circledcirc X$, there is an element in $\Gamma$ which is $\circledcirc x$.

POSTULATE $A1$. $a, b \epsilon K . \supset . a \oplus b \epsilon K ;$

that is, if $a$ and $b$ are elements of the class, then $a \oplus b$ is an element of the class.

POSTULATE $A2$. $a, b, a \oplus b, b \oplus a \epsilon K . \supset . a \oplus b = b \oplus a ;$

that is, if $a$, $b$, $a \oplus b$, and $b \oplus a$ are elements of the class, then $a \oplus b = b \oplus a$.

POSTULATE $A3$. $a, b, c, a \oplus b, b \oplus c, (a \oplus b) \oplus c, a \oplus (b \oplus c) \epsilon K . \supset .$

$$(a \oplus b) \oplus c = a \oplus (b \oplus c) ;$$

that is, if $a, b, c, a \oplus b, b \oplus c, (a \oplus b) \oplus c$, and $a \oplus (b \oplus c)$ are elements of the class, then $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

POSTULATE $A4$. $a, x, y, a \oplus x, a \oplus y \epsilon K . a \oplus x = a \oplus y . \supset . x = y ;$

that is, if $a, x, y, a \oplus x$, and $a \oplus y$ are elements of the class, then from $a \oplus x = a \oplus y$ follows $x = y$.

POSTULATE $A5$. $\exists K . \supset . \exists K \frown z \ni (z \oplus z = z) ;$

that is, if there is any element in the class, then there is an element $z$ such that $z \oplus z = z$.

POSTULATE $A6$. $\exists K \frown z \ni (z \oplus z = z) : z_1 \oplus z_1 = z_1 . z_2 \oplus z_2 = z_2 . \supset . z_1 = z_2 : a \epsilon K : \supset :$

$$\exists K \frown \bar{a} \ni (a \oplus \bar{a} = z) ;$$

that is, if there is a uniquely determined element $z$ such that $z \oplus z = z$, then for every element $a$ there is an element $\bar{a}$ such that $a \oplus \bar{a} = z$.

POSTULATE $RA1$. Hp A6. $a, b, a \oplus b \epsilon K . a \circledcirc z . b \circledcirc z . \supset . a \oplus b \circledcirc a ;$

that is, if there is a uniquely determined element $z$ such that $z \oplus z = z$; and if $a$, $b$, and $a \oplus b$ are elements of the class; and if $a \circledcirc z$ and $b \circledcirc z$; then $a \oplus b \circledcirc a$.

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POSTULATE $RA2$. Hp A6. $a, b, a \oplus b \epsilon K. a \odot z. b \odot z. \supset . a \oplus b \odot a$;

that is, if there is a uniquely determined element $z$ such that $z \oplus z = z$; and if $a, b$, and $a \oplus b$ are elements of the class; and if $a \odot z$ and $b \odot z$; then $a \oplus b \odot a$.

This completes the list of postulates in § 3.

To show the mutual independence of these sixteen postulates—that is, to show that the list contains no redundancies—we exhibit, for each postulate, an interpretation of $K$, $\odot$, and $\oplus$ which satisfies all the other postulates, but not the one in question. Such postulate, then, cannot be a consequence of the other fifteen; for, if it were, every system $(K, \odot, \oplus)$ which had the other fifteen properties would have this property also, which is not the case.*

In constructing these systems, we assume the existence of the real numbers, as defined in the usual way by successive generalizations of the number-concept; and the symbols $<$, $+$, and $0$ are used, in this section, only in their ordinary arithmetic meaning. The proof-systems are the following:

*For* I. $K =$ an "empty" class.

*For* II. $K =$ a class of a single element $a$. $a \odot a$ false. $a \oplus a = a$.

*For* $R1$. $K =$ all real numbers. $(a \odot b) = (a < b)$ when $a$ and $b$ are both positive or both negative; otherwise $a \odot b$ is false. $\oplus = +$.

*For* $R2$. $K =$ all real numbers. $(a \odot b) = (a \neq b)$. $\oplus = +$.

*For* $R3$. $K =$ all real numbers. $(a \odot b) = (a \leqq b)$. $\oplus = +$.

*For* $R4$. $K =$ a class of three elements, $0, 1$, and $2$. $0 \odot 1, 1 \odot 2, 2 \odot 0$. $a \oplus b = a + b$ when $a + b \leqq 2$, and $= a + b - 3$ when $a + b > 2$; in other words, $a \oplus b \equiv a + b \pmod 3$.

*For* $R5$. $K =$ all integral numbers. $\odot = <$. $\oplus = +$.

*For* $R6$. $K =$ all rational numbers. $\odot = <$. $\oplus = +$.

*For* $A1$. $K =$ all real numbers. $\odot = <$. $a \oplus b = a + b$ when $a + b = 0$, otherwise $a \oplus b =$ an entity not in the class.

*For* $A2$. $K =$ all real numbers. $\odot = <$. $a \oplus b = b$.

*For* $A3$. $K =$ all real numbers. $\odot = <$. $a \oplus b = 2(a + b)$ when $a$ and $b$ are both positive or both negative; otherwise $a \oplus b = a + b$.

*For* $A4$. $K =$ all real numbers. $\odot = <$. $a \oplus b = 0$ when $a \neq b$; and $a \oplus a = a$.

*For* $A5$. $K =$ all positive real numbers. $\odot = <$. $\oplus = +$.

*For* $A6$. $K =$ all positive real numbers, with $0$. $\odot = <$. $\oplus = +$.

*For* $RA1$. $K =$ all real numbers. $(a \odot b) = (a > b)$ when $a$ and $b$ are both positive, otherwise $\odot = <$. $\oplus = +$.

*For* $RA2$. $K =$ all real numbers. $(a \odot b) = (a > b)$ when $a$ and $b$ are both negative, otherwise $\odot = <$. $\oplus = +$.

* This is the now familiar method of proving independence.



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Thus the independence of the sixteen postulates of § 3 is established.

The independence of the ten postulates of § 1 is proved as follows:

For postulates I, II, $R1$–$R6$, use the systems just given, omitting the specifications in regard to $\oplus$.

For postulate $R7$, namely, $a, b \epsilon K.\ a \neq b.\ \supset.\ {}_{\exists}K \frown x \ni (x \odot a)$, use the system: $K =$ all real numbers between 0 and 1, with 0; $\odot = <$.

For postulate $R8$, namely, $a, b \epsilon K.\ a \neq b.\ \supset.\ {}_{\exists}K \frown y \ni (y \odot a)$, use the system: $K =$ all real numbers between $-1$ and 0, with 0; $\odot = <$.

To prove the independence of the eight postulates of § 2, use the systems above, omitting the specifications in regard to $\odot$.

## § 5. Further analysis of the postulates for group-theory.*

The set of eight postulates given in § 2 can be used only for *abelian* groups. In the present section I give a set of twelve postulates for groups in general, with a thirteenth postulate which makes the group abelian; and moreover (except in the case of postulate $G9$, which can probably be further sub-divided), each postulate has been made as nearly a *simple statement* as seems possible.†

Since the theory of groups has many applications not connected with real algebra, I represent the fundamental operation of the group, in the present set of postulates, by the symbol $\circ$, or by simple juxtaposition, instead of by the symbol $+$ or $\oplus$. Thus, $a \circ b$ (read: "$a$ with $b$"), or $ab$, may be interpreted at pleasure as the "sum" of $a$ and $b$, or the "product" of $a$ and $b$, or the "resultant" of $a$ and $b$, etc. I also represent the "zero-element," 0 or $z$, of § 2, by the symbol $i$ ("identity"), which has a less special connotation.‡

(The letter $G$ is prefixed to the numbers of the postulates to suggest the word Group.)

Postulates I–II. The same as in § 2. Hence:

*Theorem* I. *There are at least two distinct elements in the class.*

Postulate $G1$. If $a$ is an element of the class, then $aa$ is an element of the class.

Postulate $G2$. If $a \neq b$, then $ab$ is an element of the class.—Hence:

*Theorem* II. *If $a$ and $b$ are elements of the class, then $ab$ is an element of the class.*

---

* Another set of postulates for groups will be presented in a subsequent paper.

† Most of the postulates embody, to be sure, a multitude of statements, corresponding to the multitude of elements in the group; but (except in case of $G9$) there seems to be no ground for distinguishing any of these statements from the rest; that is, there seems to be no ground for further subdivision of any of the postulates except $G9$. Cf. E. H. Moore, loc. cit.

‡ Compare footnote under postulate $A1$ in § 2.

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POSTULATE $G3$. If there is any element in the class, there is an element $i$ such that $ii = i$.

POSTULATE $G4$. If $x$ and $y$ are elements such that $xx = x$ and $yy = y$, then $x = y$. Hence:

*Theorem* III. *There is a uniquely determined element $i$ such that $ii = i$.*

POSTULATE $G5$. If $ii = i$ and $aa \neq a$, then $ai = a$.

POSTULATE $G6$. If $ii = i$ and $aa \neq a$, then $ia = a$. Hence:

*Theorem* IV. *For every element $a$ we have $ai = ia = a$.*

POSTULATE $G7$. If $ax = a$ and $aa \neq a$, then $xx = x$.

POSTULATE $G8$. If $xa = a$ and $aa \neq a$, then $xx = x$. Hence:

*Theorem* V. *If $ax = a$, or $xa = a$, then $x = i$.*

POSTULATE $G9$. If $aa \neq a$, $bb \neq b$, $cc \neq c$, and $ab \neq a$, $ab \neq b$, $bc \neq b$, $bc \neq c$; then

$$(ab)c = a(bc).$$

Hence:

*Theorem* VI. *For all elements* $(ab)c = a(bc)$. (Associative law.)

POSTULATE $G10$. If $ii = i$ and $aa \neq a$, then there is either an element $a_r$ such that $aa_r = i$, or an element $a_l$ such that $a_l a = i$.*

Now from $ax = i$ follows $xa = i$, and conversely. Thus, if $ax = i$, then $a(xa) = (ax)a = ia = a$, whence $xa = i$, by theorem V.

Further, if $ax = i$ and $ay = i$, then $x = y$. For, from $ax = i$ follows $yax = yi = y$, and from $ay = i$ follows $ya = i$ and $yax = ix = x$.

Hence, from postulate $G10$,

*Theorem* VII. *Every element $a$ determines uniquely an element $\bar{a}$ such that $a\bar{a} = \bar{a}a = i$.*

Then if $ax = ay$, we have $\bar{a}ax = \bar{a}ay$, whence $x = y$; and similarly if $xa = ya$, we have $xa\bar{a} = ya\bar{a}$, whence $x = y$. Hence:

*Theorem* VIII. *A change in either $a$ or $b$ alone produces a change in $ab$.*

These twelve postulates, I–II, $G1$–$G10$, are thus sufficient to define a *group*. In order to make this group an *abelian* group, we must add another postulate, namely:

POSTULATE $G11$. If $aa \neq a$, $bb \neq b$, $ab \neq a$, $ab \neq b$, $ba \neq a$, and $ba \neq b$, then

$$ab = ba.$$

Hence:

*Theorem* IX. *For all elements, $ab = ba$.* (Commutative law.)

These thirteen postulates, I–II, $G1$–$G11$, are *independent* of each other, as is shown by the following systems, or interpretations of $K$ and $\oplus$:

*For* I *and* II. The same systems as in § 4.

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* This postulate was suggested by E. H. MOORE's six-postulate definition of a group; Transactions, vol. 3 (1902), p. 489; vol. 5 (1904), p. 549.

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*For G1.* $K =$ all real numbers. $a \oplus b = a + b$ when $a \neq b$, or $a = 0$, or $b = 0$, or $a + b = 0$; otherwise $a \oplus b$ not in the class.

*For G2.* $K =$ all real numbers. $a \oplus b = a + b$ when $a = b$, or $a = 0$, or $b = 0$, or $a + b = 0$; otherwise $a \oplus b$ not in the class.

*For G3.* $K =$ all positive real numbers. $\oplus = +$.

*For G4.* $K =$ all positive real numbers. When $a \neq b$, $a \oplus b =$ the greater of the two numbers $a$ and $b$; when $a = b$, $a \oplus a = a$.

*For G5.* $K =$ all positive real numbers with 0. $a \oplus b = a + b$, except that $a \oplus 0 = 0$.

*For G6.* $K =$ all positive real numbers with 0. $a \oplus b = a + b$, except that $0 \oplus a = 0$.

*For G7.* $K =$ all real numbers. $a \oplus a = 0$, $0 \oplus a = a \oplus 0 = a$; otherwise $a \oplus b = a$.

*For G8.* $K =$ all real numbers. $a \oplus b = 0$, $0 \oplus a = a \oplus 0 = a$; otherwise $a \oplus b = b$.

*For G9.* $K =$ all real numbers. $a \oplus b = 2(a + b)$, when $a$ and $b$ are both positive or both negative; otherwise $a \oplus b = a + b$.

*For G10.* $K =$ all positive real numbers with 0. $\oplus = +$.

*For G11.* Any non-abelian group with respect to $\oplus$. For example: $K =$ the class of all complex numbers $(\alpha, \beta)$ where $\alpha$ is positive and $\beta$ real, with $\oplus$ defined so that $(\alpha_1, \beta_1) \oplus (\alpha_2, \beta_2) = (\alpha_1\alpha_2, \alpha_2\beta_1 + \beta_2)$. The complex numbers $(\alpha, \beta)$ may be represented by the points of the $\alpha\beta$-plane which lie to the right of the $\beta$-axis, the element $i$ being the point $(1, 0)$.

## § 6. Another set of postulates for real algebra.

The set of postulates for groups given in § 5 suggests at once, (if we replace* $ab$ by $a + b$ and $i$ by 0), another set of postulates for real algebra, one which does not involve the commutative law,— namely the set consisting of the following twenty postulates:

Postulates I–II. (As in § 1 or § 2.)—These postulates give the class at least two distinct elements.

Postulates $R1$–$R6$. (As in § 1.)—These postulates, with I and II, make the class a *continuous scale*, or a *one-dimensional continuum*, with respect to the relation $<$.

Postulates $G1$–$G10$. (As in § 5.)—These postulates, with I and II, make the class a *group* (not necessarily an abelian group) with respect to the operation $+$.

Postulate $RG1$. If there is a uniquely determined element 0 such that $0 + 0 = 0$; and if $a$, $b$, $a + b$, and $b + a$ are elements of the class such that $a + b \neq a$, $a + b \neq b$, $b + a \neq a$, and $b + a \neq b$; then: if $a$ and $b$ are $> 0$, then $a + b$ is $> a$ and $> b$.

* Compare footnote under postulate $A1$ in § 2.

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POSTULATE $RG2$. If the hypothesis of $RG1$ holds true, then: if $a$ and $b$ are $< 0$, then $a + b$ is $< a$ and $< b$.

These last two postulates serve to connect the symbols $<$ and $+$. In view of Postulates I, II, and $G1$–$G10$ [see especially theorem V in § 5] they give at once theorem 10 of § 3, namely: *If $a$ and $b$ are both above* 0, *then $a + b$ is above either of them; and if $a$ and $b$ are both below* 0, *then $a + b$ is below either of them.*

It remains only to prove the commutative law. From theorem 10 we have, as in § 3, theorems 13 and 15, and also:

*Theorem* 13*a*. *If $x > 0$, then $x + a > a$; and if $x < 0$, then $x + a < a$;* and

*Theorem* 15*a*. *If $x < y$, then $x + a < y + a$, and conversely.*

From these theorems the principle of ARCHIMEDES can be deduced (as in theorem 23) without the aid of the commutative law, and hence that law itself can be proved by the method employed in my paper on the postulates of magnitude.*

Thus the equivalence of the sets of postulates in § 3 and § 6 is established.

The *independence* of the twenty postulates of this set (§ 6) is shown by the following systems:

*For* I–II, *and* $R1$–$R6$. The same systems as in § 4.

*For* $G1$–$G10$. The systems used in § 5, with $\circ$ defined as $<$.

*For* $RG1$ *and* $RG2$. The systems used for $RA1$ and $RA2$ in § 4.

*An unsolved problem.*†

I conclude this section with the statement of the following problem: Suppose that the postulates $RG1$ and $RG2$ in § 6 are "weakened" so as to read "$\cdots > a$ OR $> b$" and "$\cdots < a$ OR $< b$," instead of "$\cdots < a$ AND $< b$" and "$\cdots < a$ AND $< b$"; is the commutative law then still deducible, or is postulate $G11$ an independent postulate?

In attempting to prove the latter alternative, two "proof-systems" suggest

* Transactions, vol. 3 (1902), pp. 270–271. The method is due to O. HÖLDER.—The same method is used by BURALI-FORTI (*Teoria generale* ···, loc. cit.) in deducing the commutative law from a set of eight postulates for the class of positive real numbers with 0. In order, however, to make his proof of the principle of ARCHIMEDES convincing, certain modifications are necessary. In a letter of November 11, 1904, he proposes, therefore, the following corrected form of his definition 1·3, page 6, and his postulate VI, page 7:

$\cdot 3.\ a\varepsilon u.\ \supset.\ \theta(u,f)a = u\frown x\ni[a\varepsilon\{xf(u\text{-Nul}(u,f))\smile(u\text{-Nul}(u,f))fx\}]$ Df

$[\mathrm{VI}]\ x, y\varepsilon u.\ \supset_{x,y}: x = y.\smile.\ x\varepsilon\theta(u,f)y.\smile.\ y\varepsilon\theta(u,f)x.\therefore$

With these changes, his proofs of the principle of ARCHIMEDES and the commutative law present no further difficulty.

† This problem is closely connected with the unsolved problem No. 1 in my paper on magnitudes, Transactions, vol. 3 (1902), p. 279.

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themselves. First, we might try the system used for $G11$ in § 5, with $\odot$ defined as follows: $(\alpha_1, \beta_1) \odot (\alpha_2, \beta_2)$ when $\alpha_1 < \alpha_2$, or when $\alpha_1 = \alpha_2$ and $\beta_1 < \beta_2$. This system satisfies $RG1$ and $RG2$, but fails for $R6$ (on continuity). Again, we might try the same system as to $K$ and $\oplus$, with $\odot$ defined as follows: bring the points $(\alpha, \beta)$ of the half-plane into one-to-one correspondence with the points of a straight line as G. CANTOR has shown how to do;* and then interpret $\odot$ as "below" (or "on the left of"). This system satisfies all the postulates $R1$–$R6$, but it is not easy to see how to make it satisfy *both* of the postulates $RG1$ and $RG2$.—The independence of postulate $G11$, in the revised set of postulates proposed in the problem, is therefore an open question.

## § 7. A CONDENSED LIST OF POSTULATES.

In the previous sections I have attempted to analyze the fundamental propositions of real algebra into their simplest (independent) component statements, and the number of postulates in each list is therefore large. In the present section, on the other hand, I give a condensed list of postulates, in which the number is much smaller. The list comprises the following ten postulates:

1. There are at least two distinct elements in the class.
2. If $a < b$ and $b < c$, then $a < c$.
3. $a < a$ is false.
4. If $a \neq b$, then there is either an element $x$ such that $a < x$ and $x < b$, or an element $y$ such that $a > y$ and $y > b$.
5. If $\Gamma$ and $\Gamma'$ are two non-empty sub-classes in $K$, such that every element of $\Gamma$ is $<$ every element of $\Gamma'$, then there is at least one element, $M$, which is $\geqq$ every element of $\Gamma$ and $\leqq$ every element of $\Gamma'$.
6. If $a$ and $b$ are elements of the class, then $a + b$ is an element of the class.
7. $a + b = b + a$, whenever the sums involved are elements of the class.
8. $(a + b) + c = a + (b + c)$, whenever the sums involved are elements of the class.
9. If $a$, $b$, and $a + b$ are elements of the class, there is an element $x$ such that $a = b + x$.
10. If $x < y$ then $a + x < a + y$, whenever the sums involved are elements of the class.

The sufficiency of these ten postulates is shown as follows:

From postulates 2 and 3 we have: $a < b$ and $a > b$ cannot both be true. From postulates 2 and 4 we have: if $a \neq b$, then either $a < b$ or $a > b$. Postulates 4 and 5 give density and continuity. Hence by § 1, postulates 1–5

* G. CANTOR, *Ein Beitrag zur Mannigfaltigkeitslehre*, CRELLE'S Journal für die reine und angewandte Mathematik, vol. 84 (1877), pp. 242–258; translated under the title *Une contribution à la théorie des ensembles*, Acta Mathematica, vol. 2 (1883), pp. 311–328.

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make the class a *continuous scale*, or a *one-dimensional continuum*, with respect to $<$. Further, postulates 6–9 make the class an *abelian group* * with respect to $+$; and postulate 10 gives theorem 10 of § 3.

Hence the set of ten postulates in § 7 defines the same algebra as the set of sixteen postulates in § 3.

The *independence* of postulates 1–10 is shown as follows: For 1–5, and 7–10, use the systems given in § 4 for postulates II, $R4$, $R3$, $R5$, $R6$, and $A2$, $A3$, $A5$, $RA1$, respectively. For 6, use the system: $K =$ all real numbers; $\odot = <$; $a \oplus b = a + b$ when $a$ and $b$ are integers; otherwise $a \oplus b$ not in the class.

HARVARD UNIVERSITY, CAMBRIDGE, MASS.,
*September*, 1904.

## APPENDIX. A SET OF POSTULATES FOR REAL ALGEBRA, IN WHICH MULTIPLICATION IS INCLUDED AS A FUNDAMENTAL CONCEPT.

In the preceding sections of this paper, multiplication has been thought of as a derived concept, the product of two elements with respect to an arbitrarily chosen (positive) unit-element being definable in terms of the fundamental concepts $<$ and $+$. In this appendix, on the other hand, multiplication is regarded as itself a fundamental concept, whose properties are determined by postulates.

We consider, namely, a class, $K$, a relation, $<$ (read: "below" or "before") and *two* operations, $+$ and $\times$ (read: "plus" and "times"), and inquire what conditions must be imposed upon the system $(K, <, +, \times)$ to make it equivalent to the algebra of a real variable with respect to $<$, $+$, and $\times$. These conditions are expressed in the following twenty postulates (the letters $R$, $A$, and $M$ indicating the postulates which concern the Relation $<$, the operation of Addition, and the operation of Multiplication, respectively):

POSTULATES I, II, $R1$–$R6$, $A1$–$A6$, $RA1$–$RA2$. The same as in § 3.

POSTULATE $M1$. If $a$ and $b$ are elements of the class, then $a \times b$ (or $ab$) is an element of the class.

POSTULATE $M2$. If $a, b, ab$, and $ba$ are elements of the class, then $ab = ba$. (Commutative law for multiplication.)

POSTULATE $AM1$. If $a, b, c, b + c, ab, ac, a(b + c)$, and $(ab) + (ac)$ are elements of the class, then $a(b + c) = (ab) + (ac)$. (Distributive law for multiplication with respect to addition.)

POSTULATE $RAM1$. If 0 is a uniquely determined element such that

* E. V. HUNTINGTON, Transactions, vol. 4 (1903), p. 27. But note that E. H. MOORE's query as to "definition $W_1'$," in Transactions, vol. 3 (1902), p. 489, is not yet completely answered.

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$0 + 0 = 0$, and if $a$, $b$, and $ab$ are elements of the class, and $a > 0$ and $b > 0$, then $ab > 0$.

From these postulates we have:

*Theorem* 31. $c \times 0 = 0 \times c = 0$ *for every element* $c$.

*Theorem* 32. $a \times (-b) = (-a) \times b = -(a \times b)$.

*Theorem* 33. *If* $a$ *and* $b$ *are both positive or both negative,* $ab$ *is positive; if either is positive and the other negative,* $ab$ *is negative.* (See definitions under theorem 12.) Hence:

*Theorem* 34. *If* $a \neq 0$ *and* $b \neq 0$, *then* $ab \neq 0$.

If now we select at pleasure a positive element $E$, and notice that every positive element $a$ is the upper limit of the class of all the rational fractions of $E$ which are $< a$ (see definitions under postulate $R6$ and theorem 27), we can prove the associative law for multiplication (first for positive elements, and then, by the aid of theorem 32, for all elements), and also the possibility of "division" when the divisor is different from zero; that is:

*Theorem* 35. $(ab)c = a(bc)$; and

*Theorem* 36. *For every two elements* $a$ *and* $b$, *provided* $a$ *is not zero, there is an element* $x$ *such that* $ax = b$.

Thus we see that the elements of $K$ *excluding zero* from an *abelian group* with respect to $\times$. [Hence:* 1°) there is a uniquely determined "unit-element," 1, not zero, such that $1 \times 1 = 1$; 2°) for every element $a$, $1 \times a = a \times 1 = a$; and 3°) if $a \neq 0$, then from $ax = ay$ follows $x = y$.] Therefore the whole system is a *field*† with respect to $+$ and $\times$.

Making use again of classes of rational fractions, we can now prove:

*Theorem* 37. *Any two systems,* $(K, <, +, \times)$ *and* $(K', <, +, \times)$, *which satisfy the twenty postulates of this appendix are* EQUIVALENT *with respect to* $<$, $+$ *and* $\times$; *that is, they can be brought into one-to-one correspondence in such a way that*

1°) *from* $a < b$ *follows* $a' < b'$, *and conversely;*

2°) *from* $a + b = c$ *follows* $a' + b' = c'$, *and conversely; and*

3°) *from* $ab = c$ *follows* $a'b' = c'$, *and conversely;—*

*where* $a'$, $b'$, $c'$ *are the elements in* $K'$ *which correspond to the elements* $a$, $b$, $c$ *in* $K$. *Further, this correspondence can be established in only one way, since the "unit-element" is fixed in each system.*

The last clause of the theorem, namely that the correspondence can be established in only one way, marks the essential distinction between the postulates of the appendix (expressed in terms of $<$, $+$, and $\times$) and the postulates of

* Cf. Transactions, vol. 4 (1903), pp. 27-28.

† Cf. L. E. Dickson and E. V. Huntington, Transactions, vol. 4 (1903), pp. 13 and 31. Notice that postulates I, II, $A1$–$A6$, $M1$–$M2$, $AM1$ are not sufficient to define a field, without the aid of the postulates concerning $<$ (as witness the system of all integers with respect to $+$ and $\times$). A further discussion of the postulates for a field will be presented in a later paper.

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§ 3 (expressed in terms of $<$ and $+$). For in § 3 the correspondence could be established in an infinite number of ways, according to the infinite number of ways in which the "unit-elements" might be selected. The postulates of the appendix, therefore, determine the algebra *uniquely* in a stronger sense than do the postulates of the preceding sections.

As examples of systems $(K, \circledcirc, \oplus, \odot)$ which satisfy all the postulates, and to illustrate theorem 37, I mention the following "equivalent" systems (the symbols $<$, $+$, and $\times$, being now used in their ordinary arithmetical meanings):*

1) $K =$ all real numbers; $\circledcirc = <$; $\oplus = +$; $\odot = \times$. (Here $z = 0$, $u = 1$.)

2) $K =$ all real numbers $> 0$; $\circledcirc = <$; $a \oplus b = ab$; $a \odot b = \exp(\log a \cdot \log b)$. (Here $z = 1$, $u = e =$ base of the system of logarithms.)

3) $K =$ all real numbers $> 1$; $\circledcirc = <$; $a \oplus b = \exp(\log a \cdot \log b)$; $a \odot b = \exp\{\exp[\log(\log a) \cdot \log(\log b)]\}$. (Here $z = e$, $u = e^e$.)

4) $K =$ all real numbers; $\circledcirc = <$; $a \oplus b = (a^{\frac{1}{3}} + b^{\frac{1}{3}})^3$; $a \odot b = ab$. (This example was suggested to me by Professor BOUTON.)

The existence of any such system proves the *consistency* of the postulates.

It remains to show that the twenty postulates of the appendix are *independent*; that is, to exhibit twenty interpretations of $K$, $\circledcirc$, $\oplus$, and $\odot$, in which all the other postulates are satisfied, but not the one in question. These proof-systems are the following:

*For* I. $K =$ an "empty" class.

*For* II. $K =$ a class of a single element $a$. $a \circledcirc a$ false. $a \oplus a = a$. $a \odot a = a$.

*For* $R1$–$R6$, $A1$–$A6$, $RA1$–$RA2$. The systems used in § 4, with $\odot$ defined as ordinary multiplication, except that in the system for $R4$ the product is taken modulo 3.

For the remaining postulates, use the system of all real numbers, with $\circledcirc$ and $\oplus$ defined as the ordinary $<$ and $+$, and $\odot$ defined as follows:

*For* $M1$. $a \odot b = ab$ when $a$ and $b$ are both rational, otherwise $a \odot b$ not in the class.

*For* $M2$. $a \odot b = b$.

*For* $AM1$. $a \odot b = a + b$.

*For* $RAM1$. $a \odot b = -ab$.

Thus the independence of the twenty postulates is established.

We notice in conclusion that we may replace the first sixteen postulates of this list by the twenty postulates of § 6, or by the ten postulates of § 7, thus forming two new sets of twenty-four and fourteen postulates, respectively. The independence of the postulates of each of these sets is readily verified, by the aid of the proof-systems already employed.

* See the opening paragraph in § 1 and § 2. For convenience in printing, $\exp x$ is written for $e^x$.

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# ON THE PRIMITIVE GROUPS OF CLASS $3p$*

BY

W. A. MANNING

In this paper are considered only those groups which contain a substitution of order $p$ and degree $3p$, $p$ an odd prime. Two general theorems are first established and then class 9 is disposed of before the general problem is considered.

THEOREM I. *Let $A$ be a substitution of degree $pq$ and order $p$ in a group of class $pq$, $q \leqq p$. No substitution similar to and non-commutative with $A$ can be free from all the letters of any one cycle of $A$. An exception may occur when $q = p$ and the group contains a transitive subgroup of order $p^2$.*

Let $B$ be a substitution similar to $A$, non-commutative with $A$, and free from all the letters of $r$ cycles of $A$. If $q < p$, no two substitutions similar to $A$ can displace exactly the same letters unless one is a power of the other,† and we may assume this to be true in the groups of class $p^2$ here considered, since the knowledge that $G$ contains a transitive subgroup of degree $p^2$ makes its consideration and determination relatively simple.

If $B$ does not connect old and new letters transitively in its cycles, $A^{-1}B^{-1}AB$ is of degree not greater than $(q-r)p$, and is not the identity. We can now assume that $B$ and all its powers connect old and new letters transitively.

It will be shown that a substitution $F$ can always be found among the substitutions similar to $A$, which transforms into themselves the $r$ cycles of $A$ left fixed by $B$ and which displaces not more than $q-r$ letters new to $A$. The existence of $F$ depends only upon the existence of $B$ and leads to a substitution, not the identity, which displaces at most $(q-r)p+q-r$ letters. If $B$ displaces not more than $q$ new letters and $q \neq p$, we have at once a substitution $A^{-1}B^{-1}AB$ of degree less than $pq$. If $q = p$, an apparent exception arises when $r = 1$, and $B$ displaces just $p$ new letters. But here $A^{-1}BA$ is not a power of $B$ and displaces the same $p^2$ letters as $B$.

It is now assumed that $B$ displaces more than $q$ new letters, so that some cycle contains at least two new letters. In $B^{-\rho}AB^{\rho} = C$, suppose $\rho$ so chosen that two new letters which occur in the same cycle of $B$ are adjacent in $B^{\rho}$.

* Presented to the Society (San Francisco) April 25, 1903. Received for publication June 2, 1904.

† Transactions of the American Mathematical Society, vol. 4 (1903), p. 351.



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The substitution $C$ does not displace as many new letters as $B$ and in it $r$ cycles of $A$ occur unchanged. $C$ certainly contains one or more new letters. We now wish to show that the new letters which are in $C$ cannot merely fill up isolated cycles of $C$, but that $C$ also must connect old and new letters in its cycles. Let $C = C_1 SR$, where $C_1$ contains only old letters, $S$ only new ones, and $R$ is made up of the $r$ unchanged cycles of $A$. Let $S$ have $s$ cycles. Break $A$ up into two parts, $A = A_1 R$, where $A_1 = c_1 c_2 \cdots c_{q-r}$. The substitution $A^{-1}C = (A_1^{-1}C_1)S$ contains not more than $(q-r+s)p$ letters. Unless $s \geqq r$, $G$ is of class less than $pq$. Again $A^{-1}C^{-1}AC = A_1^{-1}C_1^{-1}A_1C_1$ lowers the class of $G$ to $(q-r)p$ or less unless we have $A_1^{-1}C_1A_1 = C_1$. This condition can be satisfied only if $C_1 = c_1^{x_1}c_2^{x_2}\cdots c_{q-r-s}^{x_{q-r-s}}$, since $C_1$ has at most $p-2$ cycles. From this form of $C$ it follows that if a letter of any cycle of $A$ is left fixed by $B^\rho$, no letter of that cycle occurs in $B$. But by hypothesis $B$ is free from just $r$ complete cycles of $A$. Then $B^\rho$ contains just $(q-r)p$ old letters. The number of new letters in $B^\rho$ is $sp = rp$, and since these $rp$ new letters are all found in $C$, each one of them is in $B^\rho$ preceded by an old letter. But $\rho$ was chosen so that two new letters would be adjacent in $B^\rho$. We conclude that $C$ connects old and new letters transitively.

Suppose that in some cycle of $C$ two or more new letters are found. Again we choose $\rho$ so that two new letters are adjacent in $C^\rho$. Then $D = C^{-\rho}AC^\rho$ displaces fewer new letters than does $C$, retains unchanged the $r$ cycles of $A$ left fixed by $B$, and furthermore connects old and new letters. The last statement requires proof.

In case $D$ does not connect old and new letters, $D = D_1 SR$, where $D_1$ contains old letters only; $S$, $sp$ new letters only; and $R$ repeats $r$ cycles of $A$ without change. The degree of $A^{-1}D = (A_1^{-1}D_1)S$ is not greater than $(q-r+s)p$; hence $s \geqq r$. Again $A^{-1}D^{-1}AD = A_1^{-1}D_1^{-1}A_1D_1 = 1$, since this substitution cannot displace more than $(q-r)p$ letters. Hence $D_1 = c_1^{x_1}\cdots c_{q-r-s}^{x_{q-r-s}}$. Now $C^{-\rho}AC^\rho = D = c_1^{x_1}\cdots c_{q-r-s}^{x_{q-r-s}}SR$. It follows that if a letter of any cycle of $A$ is missing from $C^\rho$, no letter of that cycle occurs in $C$. Therefore $C$ leaves fixed all the letters of at least $s$ cycles of $A$. But we have seen that $s \geqq r$. The same reasoning can now be applied to $C$ as was applied to $B$. Then $D$ has the properties stated. Applying the same method to $D$ we obtain another substitution $E$ similar to $A$, connecting old and new letters transitively, containing unchanged at least $r$ cycles of $A$, and displacing fewer new letters than $D$. This process can be continued until a substitution $F$ is reached which has at least $r$ cycles of $A$ unchanged, is similar to $A$, and introduces $k$ $(q-r \geqq k \geqq 1)$ new letters with no two new letters in the same cycle. The substitution $A^{-1}F$ displaces not more than $(q-r)p + q - r$ letters, which is contrary to the hypothesis that $r \geqq 1$.

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THEOREM II. *Among the substitutions similar to $A$ in a primitive group of class $pq$ $(1 < q \leqq p)$, $p$ odd, a substitution $B$ can be found connecting transitively two cycles of $A$ and having not more than one new letter in any cycle.*

Since $G$ is primitive the similar substitutions $A, \cdots$ generate a transitive group. If no one of the set replaces all the letters $a_1, a_2, \cdots, a_p$ by other letters, one of them connects two cycles of $A$ and has not more than one new letter $\alpha$ in any cycle.* But if $A_1$ replaces all the letters $a_1, \cdots$ by other letters, these $p$ letters $a$ are found in at least three of the $q$ cycles of $A_1$, so that by the theorem just proved some cycle of $A_1$ contains letters from different cycles of $A$. Therefore there always is in the set $A, \cdots$ a substitution $B = (a_1 b_1 \cdots) \cdots$.

Among all the substitutions $A, \cdots$ which connect cycles of $A$, there is one which displaces a minimum number $\lambda$ of the new letters $\alpha$. It is immaterial which two cycles of $A$ are connected. Let $B$ be a substitution of the form $(ab \cdots) \cdots$ displacing $\lambda$ new letters. Also let $B$ leave fixed one of the letters $a$. It cannot have two new letters $\alpha$ consecutive, for then $B^{-1}AB$ would connect letters $a$ and $b$ in one of its cycles and would displace fewer than $\lambda$ new letters. Suppose that $B$ has two or more new letters in its first cycle. A convenient power $B^\rho$ makes these two new letters consecutive. In $B^\rho$ letters $a$ can only be followed (or preceded) by other letters $a$ and new letters $\alpha$. Hence in the first cycle of $B^\rho$ there are the sequences $a'\alpha'$ and $b'\alpha''$, where $a'$ is one of the letters $a_1, \cdots, a_p$, and $b'$ is one of the remaining $(q-1)p$ letters of $A$. Now choose $\sigma$ so that $B^{\rho\sigma} = (a'b' \cdots \alpha'\alpha'' \cdots) \cdots$. Since by hypothesis $B$ leaves an $a$ fixed, $B^{-\rho\sigma}AB^{\rho\sigma}$ connects cycles of $A$ and has fewer than $\lambda$ letters $\alpha$. Then $B$ has just one $\alpha$ in its first cycle. It is clear that any power of $B$ has a letter $a$ followed (or preceded) in its first cycle by a letter from another cycle of $A$. Hence $B$ cannot have two new letters in any cycle.

We shall now show that it may always be assumed that $B$ leaves a letter $a$ fixed. Suppose that $B$ displaces all the $a_1, \cdots, a_p$. Evidently the same is true of $b_1, b_2, \cdots, b_p$. If the $2p$ letters $a_1, \cdots, b_1, \cdots$ occupy just two cycles of $B$, any power of $B$ replaces some $a$ by an $a$ and some other $a$ by a $b$, and, as before, $B$ has not more than one new letter $\alpha$ to a cycle. If the $2p$ letters $a_1, \cdots, b_1, \cdots$ are found in more than two cycles of $B$, two cases arise. First, let no letter $c, d, \cdots$ be in the same cycle with an $a$ or $b$. Some power $B^x$ of $B$ now connects $c$ and $d$, say, because of Theorem I. Then $B$ must displace the $2p$ letters $c_1, \cdots, d_1, \cdots$, and these letters again are found in at least two cycles of $B$. Now if no $e$ is in a cycle of $B$ with $c$ or $d$, we have a $B^y$ connecting $e$ and $f$, with the same conditions. Proceeding thus we finally find that either $B$ connects two cycles $k$ and $l$ of $A$, leaving fixed some of the letters $k_1, \cdots, k_p$, or else $B$ connects three or more cycles of $A$. So we have the second case,

* JORDAN, Journal für Mathematik, vol. 79 (1874), pp. 249-253.

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a letter $c$ is in a cycle with $a$ and $b$. Here $B$ displaces the $3p$ letters $a_1, \cdots, b_1, \cdots, c_1, \cdots$. These $3p$ letters cannot occupy just 3 cycles of $B$, for then any power of $B$ would transform $A$ into a new substitution connecting cycles of $A$. In fact $B$ cannot have $kp$ letters of $k$ cycles of $A$ in $k$ cycles by themselves for the same reason. Hence $a_1, \cdots, b_1, \cdots, c_1, \cdots$ are to be found in at least 4 cycles of $B$. Continuing thus it is evident that $B$ either displaces all the $qp$ letters of $A$ or connects two cycles of $A$ without displacing all the letters of one of the two cycles.

*Class* 9.

Let there be a transitive subgroup $(F)$ of degree 9 in $G$. This subgroup cannot be cyclic for it would then be contained in a doubly transitive $G^{10}$,* which does not exist. If $F$ is non-cyclic it leads to a doubly transitive $G^{13}_{13.12.9}$, also impossible.

We can now say that there is a substitution $B$ similar to $A$ which connects transitively two cycles of $A$ and displaces one, two, or three new letters.

Suppose that $I_1 = \{A, B\}$ is intransitive. It is a simple isomorphism between two transitive constituents, one of which is of degree 4 and order 12. Now the other constituent can only be of degree 6, and class 4, lowering the class of $G$ to 8.

Then $I_1$ is transitive. It is of degree 12 and order 36. The 4 systems of imprimitivity of three letters each can be chosen in only one way. Hence $I_1$ must be maximal in a doubly transitive $G^{13}_{13.12.3}$, an absurdity. No primitive group of class 9 exists.

*Class* $3p, p > 3$.

If a primitive group contains a cyclic subgroup $F$ on $3p$ letters, it also contains a doubly transitive $G^{3p+1}_{(3p+1)3p}$. Then $3p = 2^{2m} - 1$, and $p = 5$. We have here a $G^{16}_{16.15}$ † which is maximal in turn in a $G^{17}$, but is not contained in a 4-ply transitive group of degree 18. ‡

In case $F$ is non-Abelian only the doubly transitive $G^{3p+4}$ need be examined. Here the subgroup transforming $F$ into itself has a tetrahedral subgroup in its quotient group. But such a subgroup is not to be found in the group of isomorphisms of $F$.

Let $I_1 = \{A, B\}$, of degree greater than $3p$, be intransitive, and let $I_1'$ and $I_1''$ be the two simply isomorphic transitive constituents of degrees $2p + k'$,

* JORDAN, Journal de Mathématiques, ser. 2, vol. 16 (1871), p. 383; MARGGRAFF, *Ueber primitive Gruppen mit transitiven Untergruppen geringeren Grades*, Dissertation, Giessen, 1889.

† MILLER, *The primitive groups of degree* 16, American Journal of Mathematics, vol. 20 (1898), p. 229; *The transitive groups of degree* 17, Quarterly Journal of Mathematics, vol. 31 (1899), p. 49.

‡ JORDAN, Journal de Mathématiques, ser. 2, vol. 17 (1872), p. 351.

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$p + k''$, respectively; where $k', k'' = 0, 1$; $k' = k'' \neq 0$. Suppose $I_1''$ of degree $p$. It is then of class $p - 2$, and hence* is the simple triply transitive $G_{p.p-1.p-2}$. To all the substitutions not of order $p$ in $I_1''$ must correspond substitutions of degree $2p + 2$ in $I_1$. Hence $(p-1)(p-2) = 2p + 2$, from which $p = 5$. The group $I_1$ is icosaedral of degree 17. Next suppose that $I_1''$ is of degree $p + 1$. It can only be of class $p - 1$ and hence is of order $(p+1)p \cdot 2$. Now $I_1''$ has $(p+1)p/2$ subgroups of order 2 on $p - 1$ letters, and each is invariant in a subgroup of order 4. But the substitutions of order 2 involve all possible transpositions of $p + 1$ letters, so that a given transposition is found in $(p-1)/2$ distinct substitutions. These $(p-1)/2$ substitutions generate an Abelian group since the product of any two of them is of order 2. Hence $(p-1)/2 = 2$, $p = 5$.†

Since the degree of $I_1'$ exceeds $(3p-1)/2$ a substitution $C$ similar to $A$ can be found in $G$ which connects $I_1'$ and $I_1''$, and introduces at most three letters new to $I_1$.

We take up $I_{60}^{17}$ first. A transitive group of degree 17 and class 15 is triply transitive and has already been considered. It may be remarked that $I_{60}^{17}$ cannot be included in a larger intransitive group of the same degree. Then $I_2 = \{A, B, C\}$, if of degree 18, is of order $18 \cdot 60$. This group cannot be primitive, as may be shown as follows. There are in $I_2$ 36 conjugate subgroups of order 5, each of which is invariant in a subgroup of order 30. By considering the transitive representation of $I_2$ on 36 letters it is seen that $I_2$ has one conjugate set of 6 subgroups of order 3, and since no operator of order 5 can be permutable with each of the 6 subgroups of order 3, $I_2$ is isomorphic to a multiply transitive group on 6 letters. Then $I_2$ has either an invariant intransitive subgroup or a regular invariant subgroup of order 18 containing negative substitutions. But $I_2$ is a positive primitive group by hypothesis. Since $I_2$ is generated by $I_1^{17}$ and $C$, it cannot be imprimitive. Continuing in much the same way the examination of the limited number of cases to which $I_{60}^{17}$ and $I_{60}^{18}$ lead, we reach the conclusion that the subgroup $I_1$ of $G$ is never intransitive.

If the transitive group $I_1$ is of degree $3p + 1$ it is primitive of order $(3p+1)p$. Here again $p = 5$, because of the condition $3p + 1 = 2^\alpha$. This well-known $G_{80}^{16}$ is not maximal in a group of degree 17. If $I_1$ is of degree $3p + 2$, the number of subgroups of order $p$ in it is $(3p+2)/2$, an absurdity. Let $I_1 = \{A, B\}$ be of degree $3p + 3$. Since any substitution of $I_1$ which replaces one new letter by another must merely permute the new letters among themselves, $I_1$ is imprimitive. There are $p + 1$ systems of 3 letters each. Since a system of three letters can be chosen in only one way, $I_1$ leads to a

* Cf. MAILLET, *Recherches sur les Substitutions*, etc., Dissertation, Paris, 1892, p. 78.

† Cf. DE SÉGUIER, Comptes Rendus de l'Académie des Sciences de Paris, vol. 137 (1903), p. 37.

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doubly transitive $G^{3p+4}$ of order $(3p+4)(3p+3)p$ or $(3p+4)(3p+3)2p$. In $G$ the Sylow subgroup of order $p$ is invariant in a group in which the quotient-group is tetrahedral or octahedral. This is impossible.

There exist then only three primitive groups of class $3p$, $p$ odd, containing a substitution of order $p$ and class $3p$. These groups are of class 15 and order 80, 240 and 4080.

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# THE MINIMUM DEGREE $\tau$ OF RESOLVENTS FOR THE $p$-SECTION OF THE PERIODS OF HYPERELLIPTIC FUNCTIONS OF FOUR PERIODS*

BY

LEONARD EUGENE DICKSON†

## *Introduction.*

The chief object of the investigation‡ is to prove that, if $p > 3$,

$$\tau = (p^4 - 1)/(p - 1).$$

The case $p = 3$ alone is exceptional, the problem then being equivalent to that of the 27 lines on a general cubic surface. On the final page of his *Traité*,§ JORDAN states that he had established the theorem for $p = 5$, by methods analogous to those used in his complicated discussion for $p = 3$, and says "mais ici la complication est beaucoup plus grande."

It is rather remarkable that the minimum $\tau$ should be so large as $p^3 + p^2 + p + 1$, since the fractional form of the general quaternary linear group modulo $p$ can be represented as a substitution group of this degree (and of no lower in view of the present theorem).

The paper makes considerable headway in the problem of all the subgroups of the quaternary abelian group modulo $p$, which plays the same rôle in the hyperelliptic modular theory (as yet but little developed) as the binary congruence group plays in the classic elliptic modular theory.

## *Properties of three maximal subgroups of $SA(4, p^n)$.*

1. We consider special abelian || transformations of the three types

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† Of the Carnegie Institution of Washington.

‡ A sequel to my series of articles in these Transactions. Occasional reference to them is made by Roman numerals as in the list in the paper, *Determination of all the subgroups of the known simple group of order* 25920, in vol. 5 (1904), p. 127.

§ Note *E*, p. 667. Also in Comptes Rendus (1870), p. 1028.

|| The abelian conditions on $(1)_1$ and $(1)_2$ are given by (7) of $\mathrm{II}_{374}$ and (19) of $\mathrm{II}_{380}$, respectively. The sign $\pm$ preceding the matrices in II is now to be omitted.



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$$(1)\quad \begin{bmatrix} \alpha_{11} & \gamma_{11} & \alpha_{12} & \gamma_{12} \\ 0 & \alpha_{11}^{-1} & 0 & 0 \\ 0 & \gamma_{21} & \alpha_{22} & \gamma_{22} \\ 0 & \delta_{21} & \beta_{22} & \delta_{22} \end{bmatrix}, \quad \begin{bmatrix} \alpha_{11} & \gamma_{11} & \alpha_{12} & \gamma_{12} \\ 0 & \delta_{11} & 0 & \delta_{12} \\ \alpha_{21} & \gamma_{21} & \alpha_{22} & \gamma_{22} \\ 0 & \delta_{21} & 0 & \delta_{22} \end{bmatrix}, \quad \begin{bmatrix} \alpha_{11} & \gamma_{11} & 0 & 0 \\ \beta_{11} & \delta_{11} & 0 & 0 \\ 0 & 0 & \alpha_{22} & \gamma_{22} \\ 0 & 0 & \beta_{22} & \delta_{22} \end{bmatrix}.$$

Those of type $(1)_1$ form a group $G_\omega$ of order $\omega = (p^{2n}-1)(p^n-1)p^{4n}$. The operators with $\alpha_{11} = 1$ form a subgroup $G_{\omega'}$ of order $\omega' = (p^{2n}-1)p^{4n}$. Next, the operators of type $(1)_2$ form a group $H_\omega$; those with $\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} = 1$ form a subgroup $H_{\omega'}$. Finally, there are $\{(p^{2n}-1)p^n\}^2$ operators $(1)_3$ with

$$(2)\qquad \alpha_{11}\delta_{11} - \beta_{11}\gamma_{11} = 1, \qquad \alpha_{22}\delta_{22} - \beta_{22}\gamma_{22} = 1.$$

These and their products by $P_{12} = (\xi_1\xi_2)(\eta_1\eta_2)$ form a group $K_\pi$, $\pi = 2(p^{2n}-1)^2p^{2n}$. A subgroup $K_{\pi'}$, $\pi' = (p^{2n}-1)p^{2n}$ is formed of the operators

$$(3)\quad \xi_1' = \xi_1 + \gamma_{11}\eta_1, \qquad \eta_1' = \eta_1, \qquad \xi_2' = \alpha_{22}\xi_2 + \gamma_{22}\eta_2, \qquad \eta_2' = \beta_{22}\xi_2 + \delta_{22}\eta_2.$$

2. THEOREM. *If a subgroup of $SA(4, p^n)$ contains $G_{\omega'}$, it lies in $G_\omega$. In particular, $G_\omega$ is a maximal subgroup.*

We extend $G_{\omega'}$ by a transformation $S$ of $SA(4, p^n)$, not in $G_\omega$, and prove that the group obtained is $SA(4, p^n)$. Give to $S$ the notation (1) of $\mathrm{II}_{372}$. By hypothesis, $\beta_{11}, \beta_{12}, \delta_{12}$ are not all zero. We may assume that $\beta_{11}$ and $\beta_{12}$ are not both zero, otherwise $M_2 S$ has $\beta_{12}' = -\delta_{12} \neq 0$, while $M_2$ lies in $G_\omega$.

Let first $\beta_{11} = 0$, $\beta_{12} \neq 0$. Employing $S^{-1}$ if necessary, we may take $\beta_{11} = 0$, $\beta_{21} \neq 0$. Then $S' \equiv SN_{1,2,\rho}$, where $\alpha_{11} + \rho\beta_{21} = 1$, is of the form $S$ with $\alpha_{11} = 1$, $\beta_{11} = 0$, $\beta_{21} \neq 0$. Now $G_{\omega'}$ contains a transformation $T$ which leaves $\eta_1$ fixed and replaces $\xi_1$ by the same function $\xi_1 + \cdots$ that $S'$ does. Then $S_1 \equiv T^{-1}S'$ leaves $\xi_1$ fixed and has $\beta_{11} = 0$. The abelian conditions give

$$\delta_{11} = 1, \qquad \gamma_{21} = \delta_{21} = 0, \qquad \alpha_{22}\delta_{22} - \beta_{22}\gamma_{22} = 1.$$

The product $S_2 \equiv S_1U^{-1}$, where $U = \left(\begin{smallmatrix} \alpha_{22} & \gamma_{22} \\ \beta_{22} & \delta_{22} \end{smallmatrix}\right)$ on $\xi_2$ and $\eta_2$, is

$$\xi_1' = \xi_1, \qquad \eta_1' = \eta_1 + \beta_{12}\xi_2 + \delta_{12}\eta_2, \qquad \xi_2' = \xi_2 + \alpha_{21}\xi_1, \qquad \eta_2' = \eta_2 + \beta_{21}\xi_1.$$

By the hypothesis on $S$, $S_2$ is not in $G_\omega$, so that $\beta_{12}$ and $\delta_{12}$ are not both zero. Now $G_{\omega'}$ contains an operator $V$ which leaves $\xi_1$ and $\eta_1$ unaltered and replaces $\xi_2$ by $-\kappa^{-1}(\beta_{12}\xi_2 + \delta_{12}\eta_2)$, where $\kappa$ is any mark $\neq 0$. Then $V^{-1}S_2V = R_{1,2,\kappa}$. Now $M_1^{-1}$ transforms the latter into $Q_{2,1,\kappa}$. But $G_{\omega'}$ contains $Q_{1,2,\kappa}$. Hence we reach

$$(4)\qquad P_{12} = Q_{2,1,1}^{-1}Q_{1,2,1}Q_{2,1,1}^{-1}T_{2,-1}, \qquad M_1 = P_{12}M_2P_{12},$$

and hence (*Linear Groups*, p. 92) all the generators of $SA(4, p^n)$.



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Let next $\beta_{11} \neq 0$. Then $S' \equiv N_{1,2,-\delta_{12}\beta_{11}^{-1}} Q_{1,2,-\beta_{12}\beta_{11}^{-1}} S$ replaces $\eta_1$ by $\beta_{11}\xi_1 + \delta\eta_1$. Then $S_1 \equiv L_{1,-\delta\beta_{11}^{-1}} S'$ replaces $\eta_1$ by $\beta_{11}\xi_1$. For $S_1$,

$$\gamma_{11} = -\beta_{11}^{-1}, \qquad \gamma_{21} = 0, \qquad \delta_{21} = 0, \qquad \alpha_{22}\delta_{22} - \beta_{22}\gamma_{22} = 1,$$

by certain abelian conditions. Then $S_2 \equiv S_1 U^{-1}$, $U$ as above, is

$$\xi_1' = \alpha_{11}\xi_1 - \beta_{11}^{-1}\eta_1 + \alpha_{12}\xi_2 + \gamma_{12}\eta_2, \qquad \eta_1' = \beta_{11}\xi_1, \qquad \xi_2' = \xi_2 + \alpha_{21}\xi_1,$$
$$\eta_2' = \eta_2 + \beta_{21}\xi_1.$$

Then $S_3 \equiv S_2 N_{1,2,-\gamma_{12}} Q_{1,2,-\alpha_{12}}$ is of the form

$$\xi_1' = \alpha\xi_1 - \beta_{11}^{-1}\eta_1, \qquad \eta_1' = \beta_{11}\xi_1, \qquad \xi_2' = \xi_2, \qquad \eta_2' = \eta_2.$$

Then $S_4 \equiv S_3 L_{1,-\alpha\beta_{11}^{-1}}$ is of the form $S_3$ with $\alpha = 0$. Now $S_4$ transforms $L_{1,\lambda}$ into $L'_{1,-\lambda\beta_{11}^{2}}$. Hence we reach every $L'_{1,\rho}$ and hence $M_1 = L'_{1,-1} L_{1,1} L'_{1,-1}$.

3. THEOREM. *If a subgroup of $SA(4, p^n)$ contains $H_{\omega'}$, it lies in $H_\omega$. In particular, $H_\omega$ is a maximal subgroup.*

I omit the proof, which is of the same character as that of § 2.

4. THEOREM. *If, for $p > 2$, a subgroup of $SA(4, p^n)$ contains $K_{\pi'}$, it lies in $K_\pi$ or $G_\omega$. In particular, $K_\pi$ is a maximal subgroup.**

We extend $K_{\pi'}$ by a transformation $S$ of $SA(4, p^n)$, lying in neither $K_\pi$ nor $G_\omega$, and prove that the group obtained is $SA(4, p^n)$. Giving $S$ the notation (1) of $\Pi_{372}$, we have $\alpha_{12}, \gamma_{12}, \beta_{12}, \delta_{12}$ not all zero, since $S$ does not lie in $K_\pi$.

Applying $L_{1,\lambda}$ on the right, we may suppose that $\alpha_{12}$ and $\gamma_{12}$ are not both zero. We may take $\alpha_{12} \neq 0$, applying $M_2$ on the left if necessary. Finally, applying $T_{2,\rho} L_{2,\sigma}$ on the left, we may take $\alpha_{12} = 1$, $\gamma_{12} = 0$.

Case ($a$). Let $\delta_{12} = 0$. Then $(2)_2$ holds. Then $SU^{-1}$, where $U = \left(\begin{smallmatrix} \alpha_{22} & \gamma_{22} \\ \beta_{22} & \delta_{22} \end{smallmatrix}\right)$ on $\xi_2$ and $\eta_2$, becomes $S_1$ in view of abelian conditions $C_{14}$, $C_{24}$:

$$(5) \qquad S_1 = \begin{bmatrix} \alpha_{11} & \gamma_{11} & 1 & 0 \\ \beta_{11} & \delta_{11} & \beta_{12} & 0 \\ 0 & 0 & 1 & 0 \\ \beta_{21} & \delta_{21} & 0 & 1 \end{bmatrix}, \qquad W = \begin{bmatrix} \alpha & \gamma & 0 & 0 \\ \beta & \delta & 1 & 0 \\ 0 & 0 & 1 & 0 \\ \alpha & \gamma & 0 & 1 \end{bmatrix}.$$

Since $S_1$ does not lie in $G_\omega$, $\beta_{11}$ and $\beta_{12}$ are not both zero.

Let first $\beta_{12} \neq 0$ in $S_1$. Then $T_{2,\beta_{12}}^{-1} S_1 L_{1,-\beta_{12}^{-1}} T_{2,\beta_{12}}$ is the form $W$. Then for any $\lambda \neq 0$, $W^{-1} T_{2,\lambda} W T_{2,\lambda}^{-1} = R_{1,2,1-\lambda}$. Transforming by $T_{2,\rho}$ and $M_2^{-1}$, we reach every $R_{1,2,\mu}$ and $Q_{2,1,\mu}$, if $p^n > 2$. Then† we reach $N_{1,2,\mu}$ and its

* The latter is true also for $p = 2$.

†*Linear Groups*, p. 97, formula (83) for $i = 2, j = 1$.

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transform $Q_{1,2,-\mu}$ by $M_2$. Applying (4), we obtain the remaining generators of $SA(4, p^n)$.

Let next $\beta_{12} = 0$, $\beta_{11} \neq 0$, in $S_1$. Then $\beta_{21} = -\beta_{11}$, $\delta_{21} = -\delta_{11}$ by abelian conditions $C_{13}$ and $C_{23}$. Then $L_{1,-\delta_{11}\beta_{11}^{-1}} S_1 L_{1,-\alpha_{11}\beta_{11}^{-1}}$ is $Z$:

$$(6) \qquad Z = \begin{bmatrix} 0 & -\beta_{11}^{-1} & 1 & 0 \\ \beta_{11} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta_{11} & 0 & 0 & 1 \end{bmatrix}, \qquad V = \begin{bmatrix} \alpha_{11} & \gamma_{11} & 1 & 0 \\ \beta_{11} & \delta_{11} & 0 & \delta_{12} \\ \alpha_{21} & \gamma_{21} & 1 & 0 \\ -\beta_{11} & -\delta_{11} & 0 & \delta_{22} \end{bmatrix}$$

$$(\delta_{12} \neq 0,\ \delta_{12} + \delta_{22} = 1.)$$

Then if $\gamma \neq 0$, $Z^{-1} T_{2,\lambda} Z T_{2,\lambda}^{-1} = Q_{1,2,\lambda-1}$. If $p > 2$, we reach every $Q_{1,2,\tau}$. Then

$$ZQ_{1,2,-1}:\ \xi_1' = -\beta_{11}^{-1}\eta_1,\ \eta_1' = \beta_{11}\xi_1,\ \xi_2' = \xi_2,\ \eta_2' = \eta_2$$

transforms $L_{1,\rho}$ into $L'_{1,-\rho\beta_{11}^2}$. We thus reach every transformation of determinant unity on $\xi_1$ and $\eta_1$, and then $SA(4, p^n)$.

Case ($b$). Let next $\delta_{12} \neq 0$. Applying $L'_{2,\rho}$ on the left we may make $\beta_{12} = 0$. We thus have a transformation $S'$ with $\alpha_{12} = 1$, $\gamma_{12} = 0$, $\beta_{12} = 0$, $\delta_{12} \neq 0$. Since $S'$ is not in $K_\pi$, $\alpha_{22}$, $\gamma_{22}$, $\beta_{22}$, $\delta_{22}$ are not all zero. Applying $M_2$ on the right, we may assume that $\alpha_{22}$ and $\gamma_{22}$ are not both zero in $S'$.

($b_1$). Let first $\alpha_{22} \neq 0$. We may set $\alpha_{22} = 1$ by multiplying by $T_{2,\alpha_{22}}^{-1}$ on the right. Then $V = L_{2,-\gamma_{22}} S' L_{1,\gamma_{22}\delta_{12}^{-1}} L'_{2,-\beta_{22}}$ is of the form $(6)_2$.

If $\alpha_{21} \neq 0$ in $V$, then $V_1 = L_{1,-\gamma_{11}\alpha_{21}^{-1}} V$ has $\gamma_{21} = 0$, $\gamma_{11} = 0$. Then $V_1^{-1} L'_{2,-\rho} V_1 L'_{2,\rho}$ is $Z_t$, with $t = \rho - \rho\delta_{22}^2$:

$$(7) \qquad Z_t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\rho\delta_{12}^2 & 1 & -\rho\delta_{12}\delta_{22} & 0 \\ 0 & 0 & 1 & 0 \\ -\rho\delta_{12}\delta_{22} & 0 & t & 1 \end{bmatrix}, \qquad X = \begin{bmatrix} \alpha & \gamma & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & \gamma \\ 0 & 1 & 0 & -\alpha \end{bmatrix}.$$

Then $Z_t L'_{2,-t} = Z_0$. For $\delta_{22} \neq 0$, $Z_0$ with $\rho = -\delta_{12}^{-1}\delta_{22}^{-1}$ is of the earlier type $(5)_2$. For $\delta_{22} = 0$, $Z_0 = L'_{1,-\rho}$ and $\delta_{12} = 1$, $\alpha_{11} = 0$, $\delta_{11} = -\alpha_{21}^{-1}$ in $V_1$. Then

$$V_2 \equiv L'_{1,\beta_{11}\alpha} V_1 T_{2,\alpha^{-1}}:\ \xi_1' = \xi_2,\ \eta_1' = \eta_2 - \alpha^{-1}\eta_1,\ \xi_2' = \xi_1 + \alpha^{-1}\xi_2,\ \eta_2' = \eta_1.$$

Then $(V_2^{-1} T_{2,-1} V_2 T_{2,-1})^2 = Q_{2,1,-4\alpha^{-1}}$. We thus reach $SA(4, p^n)$.

If $\alpha_{21} = 0$ in $V$, then $\alpha_{11} = 0$ by abelian condition $C_{14}$. Applying $L_{1,\rho}$ on the left, we may set $\delta_{11} = 0$. Then $L_{2,-\delta_{11}} V L_{1,\delta_{22}\delta_{21}^{-1}} L_{2,1}$ is of the form $(6)_2$ with $\alpha_{21}' = -\beta_{11} \neq 0$.

($b_2$). Let finally $\alpha_{22} = 0$, $\gamma_{22} \neq 0$ in $S'$. We may set $\beta_{22} = 0$, since otherwise $S' L_{2,1}$ is of the form $S'$ with $\alpha_{22} \neq 0$. Applying on the right the inverse

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of $\left(\begin{smallmatrix}\alpha_{21} & \gamma_{21}\\ \beta_{21} & \delta_{21}\end{smallmatrix}\right)$ on $\xi_2$ and $\eta_2$, we reach $(7)_2$, in view of the abelian conditions. If $\alpha=\gamma=0$, $X=P_{12}$ would belong to $K_\pi$. If $\alpha\neq 0$, $X^{-1}$ is of the form $S'$ with $\alpha_{22}\neq 0$. Hence we may set $\alpha=0$, $\gamma\neq 0$. Then $X^2=N_{1,2,2\gamma}$. If $p>2$, we reach every $N_{1,2,\rho}$. Now $XN_{1,2,-\gamma}=P_{12}$. Hence we reach the generators of $SA(4,p^n)$.

*The subgroups of order a power of $p$.*

5. Consider the subgroup $G_{p^{4n}}$ of the operators $S=[k,a,c,d]$ defined by (3) of $\mathrm{II}_{372}$. Let $\Sigma=[\kappa,\alpha,\gamma,\delta]$. Then the commutator $S^{-1}\Sigma^{-1}S\Sigma$ is

$$(8)\qquad [k',0,c',0]\quad (k'=2\alpha c-2\gamma a+\alpha^2 d-\alpha^2\delta,\ c'=\alpha d-a\delta).$$

If $p>2$, we may make $k'$ and $c'$ assume arbitrary values in the field by taking $a=-1$, $\alpha=0$, $\delta=c'$, $2\gamma=k'+c'$. If $p=2$, we may make $k'$ and $c'$ assume arbitrary values each $\neq 0$ by taking $\delta=0$, $\alpha=k'/c'$, $d=c'^2/k'$; also we may make $k'=c'=0$. The number of operators thus reached is $(2^n-1)^2+1$, which exceeds $\frac{1}{2}2^{2n}$ if $n>1$, so that they generate $K_{2^{2n}}$ below. If $p=2$, $n=1$, then $k'=c'=ad-a\delta$.

THEOREM. *For $p^n>2$, the commutator subgroup of $G_{p^{4n}}$ is*

$$(9)\qquad K_{p^{2n}}=\{[k,0,c,0],\ k,c\ \textit{arbitrary}\};$$

*for $p^n=2$ it is the group of the two operators $[k,0,k,0]$, $k=0,1$.*

6. It is easily shown that if the $p$th power of every operator of a group $G_{p^a}$ belongs to its commutator subgroup $G_{p^b}$, there are exactly $(p^{a-b}-1)/(p-1)$ subgroups of order $p^{a-1}$ in $G_{p^a}$.

For $G_{p^{4n}}$ the condition is satisfied. Hence the number of its subgroups of order $p^{4n-1}$ is $(p^{2n}-1)/(p-1)$ if $p^n>2$, and 7 if $p^n=2$.

When a subgroup $G_{p^{4n}}$ can be defined by certain independent relations $f_1=0,\cdots,f_r=0$ between the coefficients $k,a,c,d$ of $S$, we denote it $\{f_1=0,\cdots,f_r=0\}$. Thus (9) is denoted $\{a=0,d=0\}$.

*For $p>2$, $n=1$, the $p+1$ subgroups of order $p^3$ of $G_{p^4}$ are*

$$(10)\qquad \{d=0\},\qquad \{a=td\}\qquad (t=0,1,\cdots,p-1).$$

7. The group $\{a=0\}$ is commutative of type $(1,1,1)$ and hence has $p^2+p+1$ subgroups of order $p^2$. As in §5, it follows that, for $p>2$, the commutator group of either $\{d=0\}$ or $\{a=td\}$, $t\neq 0$, is formed of the operators $[k,0,0,0]$, and contains the $p$th power of every $S$. Hence either group has exactly $p+1$ subgroups of order $p^2$. They are seen to be the ones given in the following table:

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| Order $p^3$. | Subgroups of order $p^2$. |
|---|---|
| $\{d=0\}$ | $\{d=0,\, a=0\}$, $\{d=0,\, c=sa\}$, |
| $\{a=td\},\, t\neq 0$ | $\{a=0,\, d=0\}$, $\{a=td,\, c=sd+\frac{1}{2}td^2\}$, |
| $\{a=0\}$ | $\{a=0,\, d=0\}$, $\{a=0,\, c=sd\}$, $\{a=0,\, k=rc+sd\}$ |

where $r$ and $s$ take independently the values $0, 1, \cdots, p-1$.

Now $T_{1,t^{-1}}$ transforms $\{a=td\}$, $t\neq 0$, into $\{a=d\}$. The $2p^2+p+1$ distinct subgroups of order $p^2$ of $G_{p^4}$ are found to be conjugate within $SA(4,p)$ with the four types* in the 5th–8th rows of the table of § 8.

8. In the following table is given in the first column a representative of each set of subgroups of $G_{p^4}$ conjugate within $SA(4,p)$, $p>2$; in the second column the largest subgroup of $SA(4,p)$ transforming into itself the representative.

| $G_{p^4}$ | $G_{p^4}$, $T_{1,\alpha_{11}}T_{,\alpha_{22}}$ |
|---|---|
| $\{a=0\}$ | $H_\omega$ of operators $(1)_2$ |
| $\{d=0\}$ | $G_\omega$ of operators $(1)_1$ |
| $\{a=d\}$ | $G_{p^4}$, $T_{1,\alpha^3}T_{2,\alpha}$ |
| $\{a=d=0\}$ | $G_{p^4}$, $T_{1,\alpha_{11}}T_{2,\alpha_{22}}$ |
| $\{a=c=0\}$ | $\{a=0\}$, $T_{1,\alpha_{11}}T_{2,\alpha_{22}}$, $P_{12}$ |
| $\{a=0,\, k=\nu d\}$ | $2p^3(p^2-1)$ operators $(11)_1$ |
| $\{a=d,\, c=\frac{1}{2}d^2\}$ | $\{a=d\}$, $T_{1,\alpha^3}T_{2,\alpha}$ |
| $(L_{11})$ | $G_\omega$ of operators $(1)_1$ |
| $(L_{1\mu}L_{21})$ | $(p\mp\epsilon)(p-1)p^3$ operators $(11)_2$ |
| $(B)$ if $p>3$ | $\left[\gamma,\, \alpha,\, \dfrac{\alpha+\alpha^2}{2},\, \alpha\right]$, $T_{1,s^3}T_{2,s}\left[0,\, 0,\, \dfrac{s^4-1}{12},\, \dfrac{s^2-1}{2}\right]$ |

where $\nu$ is a particular not-square, $\mu=1$ or $\nu$, $B=[0,\,-1,\,0,\,-1]$, and $\epsilon=\pm 1$ according as $p=4l\pm 1$, and where

$$(11)\quad \begin{bmatrix} \pm\alpha_{22} & \gamma_{11} & \pm\nu\alpha_{21} & \gamma_{12} \\ 0 & \alpha_{22}/\Delta & 0 & -\alpha_{21}/\Delta \\ \alpha_{21} & \gamma_{21} & \alpha_{22} & \gamma_{22} \\ 0 & \mp\nu\alpha_{21}/\Delta & 0 & \pm\alpha_{22}/\Delta \end{bmatrix},\quad \begin{bmatrix} \alpha_{11} & \gamma_{11} & \alpha_{12} & \gamma_{12} \\ 0 & t^{-1}\alpha_{11} & 0 & t^{-1}\mu^{-1}\alpha_{12} \\ \mp\mu^{-1}\alpha_{12} & \gamma_{21} & \pm\alpha_{11} & \gamma_{22} \\ 0 & \mp t^{-1}\alpha_{12} & 0 & \pm t^{-1}\alpha_{11} \end{bmatrix}$$

$$\Delta=\pm(\alpha_{22}^2-\nu\alpha_{21}^2)\neq 0,\quad t=\alpha_{11}^2+\mu^{-1}\alpha_{12}^2\neq 0.$$

* These correspond to $K_{p^2}$, $K^*_{p^2}$, $K^{**}_{p^2}$, and (16′), respectively of II. The types of period $p$ are taken from I; the transform of $A_1$ of $\mathrm{I}_{112}$ by $M_1M_2T_{1,-1}$ gives $B$.

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*General plan of the subsequent investigation.*

9. Let $H$ be a subgroup of order $p^iN$ of $SA(4, p)$, $i > 0$, $N$ prime to $p$, $p > 2$. Applying a suitable transformation within $SA(4, p)$, we may assume that $H$ contains a subgroup $G_{p^i}$ lying in the $G_{p^4}$ of §§ 5–7. If $H$ contains $G_{p^i}$ self-conjugately, $H$ lies in one of the groups in the second column of the table in § 8, and hence lies in $G_\omega$, $H_\omega$, or $(G_{p^4}, T_{1,a_{11}}T_{2,a_{22}})$. In this case the determination of $H$ depends upon the determination* of all subgroups of the binary linear-homogeneous group of determinant unity. Suppose next that $G_{p^i}$ is not self-conjugate in $H$. Let $p^m$ be the maximal order of a subgroup common to $G_{p^i}$ and any of its conjugates under $H$; let $G_{p^m}$ be such a subgroup. By a theorem† discovered independently by BURNSIDE and FROBENIUS, $H$ must contain an operator $S$, of period prime to $p$, commutative with $G_{p^m}$ but not with $G_{p^i}$.

Now, if $p > 2$, $(p^4 - 1)(p^2 - 1)$ has no factor of the form $1 + p^3x$, $x > 0$. For if so, call the quotient $q$. Then $0 < q < p^3$, $-p^2 + 1 \equiv q \pmod{p^3}$. Hence $q = p^3 - p^2 + 1$. But the latter is relatively prime to $(p^2 - 1)^2$, and exceeds $p^2 + 1$ if $p > 2$. Hence the number of conjugates to $G_{p^i}$ in $H$ is not $\equiv 1 \pmod{p^3}$, so that‡ $m \geqq i - 2$. *We may set* $m = i - 1$ *or* $i - 2$.

10. LEMMA. *Any binary transformation* $B = \left(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right)$, $\beta \neq 0$, *together with all the* $S_\lambda = \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right)$ *generate every binary transformation of determinant unity.*

Indeed, $S_{-\delta\beta^{-1}}BS_{-\alpha\beta^{-1}} = \left(\begin{smallmatrix} 0 & \tau \\ \beta & 0 \end{smallmatrix}\right)$, where $\tau = -\beta^{-1}(\alpha\delta - \beta\gamma) \neq 0$. The latter operator transforms $S_\lambda$ into $\left(\begin{smallmatrix} 1 & 0 \\ \sigma & 1 \end{smallmatrix}\right)$, where $\sigma = \lambda\beta\tau^{-1}$ may be made arbitrary.

*The subgroups $H$ of order $p^4N$, $p > 2$.*

11. Now $i = 4$, $m = 3$ or $2$ in § 9. For $m = 3$, we may take $G_{p^m}$ to be $\{a = 0\}$ or $\{d = 0\}$, since every operator commutative with $\{a = d\}$ lies in $(G_{p^4}, T_{1a^2}T_{2,a})$ and hence is commutative with $G_{p^4}$ (§ 8).

For $\{d = 0\}$, $S$ is of the form $(1)_1$. Then $\beta_{22} \neq 0$ since $S$ is not commutative with $G_{p^4}$. The quotient-group§ $G_\omega/\{d = 0\}$ may be taken concretely as the group of the products $T_{1,a_{11}}U$, $U$ a binary transformation of determinant unity on $\xi_2$ and $\eta_2$. Also, $G_{p^4}/\{d = 0\}$ is $(L_{2,\gamma})$. Then, by § 10, we reach every $U$. These, with $G_{p^4}$, generate $G_{\omega'}$. Hence, by § 2, $H$ is a subgroup of $G_\omega$.

For $\{a = 0\}$, $S$ is of the form $(1)_2$, with $\alpha_{21} \neq 0$. The quotient-groups

* This has been done by the writer for any Galois field.

† References in BURNSIDE's *Theory of Groups*, p. 97.

‡ Compare, for example, BURNSIDE's *Theory of Groups*, p. 94, Cor. II.

§ Bulletin of the American Mathematical Society, vol. 10 (1904), pp. 178–184.

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$$\frac{H_\omega}{\{a=0\}} = \begin{bmatrix} \alpha_{11} & 0 & \alpha_{12} & 0 \\ 0 & \alpha_{22}/\Delta & 0 & -\alpha_{21}/\Delta \\ \alpha_{21} & 0 & \alpha_{22} & 0 \\ 0 & -\alpha_{12}/\Delta & 0 & \alpha_{11}/\Delta \end{bmatrix}, \qquad \frac{G_{p^4}}{\{a=0\}} = \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -a & 0 & 1 \end{bmatrix},$$

where $\Delta = \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}$, are simply isomorphic with the binary groups

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}, \qquad \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$

Hence, in view of § 10 and § 3, $H$ is a subgroup of $H_\omega$.

12. Let next $i = 4$, $m = 2$. Then, by § 8, $G_{p^3}$ is neither $\{a = d = 0\}$ nor $\{a = d,\ c = \frac{1}{2}d^2\}$. If $G_{p^3}$ is $\{a = c = 0\}$, $S = S_1 P_{12}$, where $S_1$ lies in $\{a = 0\}$ extended by $T_{1,\alpha_1}$, $T_{2,\alpha_2}$, so that $S_1$ transforms $G_{p^4}$ into itself. But $G_{p^4}$ and its transform by $P_{12}$ have $\{a = 0\}$ in common, in contradiction with $m = 2$. Finally, let $G_{p^3}$ be $\{a = 0,\ k = vd\}$, so that $S$ is of the form $(11)_1$. Now the general quaternary abelian operator with every $\beta_{ij} = 0$ transforms $[k, a, c, d]$ into an operator of the form $(1)_2$, written in capital letters, with

$$A_{ij} = \tau_{ij} + a\alpha_{i1}\delta_{j2}, \qquad D_{ij} = \tau_{ij} - a\delta_{i2}\alpha_{j1},$$

$$C_{ij} = k\alpha_{i1}\alpha_{j1} + c\alpha_{i1}\alpha_{j2} + (c - ad)\alpha_{i2}\alpha_{j1} + d\alpha_{i2}\alpha_{j2} - a\alpha_{i1}\gamma_{j2} - a\gamma_{i2}\alpha_{j1},$$

where $\tau_{ij} = 1$ $(i = j)$, $\tau_{ij} = 0$ $(i \neq j)$. Hence $G_{p^4}$ and its transform by $S$ would have $\{a = 0\}$ in common, in contradiction with $m = 2$.

*The subgroups $H$ of order $p^3 N$, $p > 2$.*

13. Let first $i = 3$, $m = 2$ in § 9. In view of § 8 the only case not immediately excluded is $G_{p^2} = \{a = d = 0\}$, $G_{p^3} = \{a = d\}$. Then $S$ lies in $(G_{p^4}, T_{1,\alpha_{11}} T_{2,\alpha_{22}})$ and hence transforms $\{a = d\}$ into $\{a = \alpha_{11}\alpha_{22}^{-3} d\}$; the latter two generate $G_{p^4}$ in contradiction with $i = 3$.

14. Let $i = 3$, $m = 1$. If $G_p = (L_{1,\mu} L_{2,1})$, then $G_{p^3} = \{a = 0\}$. Since $(11)_2$ is of the form $(1)_2$, this case is excluded by § 8. If $G_p = (B)$, $p$ being $> 3$, then $G_{p^3} = \{a = d\}$; so that (§ 8) any operator commutative with $G_p$ is commutative with $G_{p^3}$. Let finally $G_p = (L_{1,1})$. Let first $G_{p^3}$ be $\{a = 0\}$, so that $S$ is of the form $(1)_1$ with $\beta_{22} \neq 0$. Then $S$ transforms $[0, 0, 1, 0]$ into $[2\alpha_{11}\alpha_{12}, -\alpha_{11}\beta_{22}, \alpha_{11}\alpha_{22}, 0]$, which extends $G_{p^3}$ to $G_{p^4}$, in contradiction with $i = 3$. The same argument excludes $G_{p^3} = \{a = d\}$. Finally, $G_{p^3} = \{d = 0\}$ is excluded by § 8.

THEOREM. *Every subgroup of order $p^3 N$ has a self-conjugate $G_{p^3}$ and hence lies in either $G_\omega$ or $H_\omega$.*

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*The subgroups $H$ of order $p^2N$, $p > 2$.*

15. Let $i = 2$, $m = 1$. Let first $G_p = (L_{1,1})$, whence $S = (1)_1$. Then $G_{p^2}$ is

$$\{a = d = 0\}, \qquad \{a = c = 0\}, \qquad \text{or} \qquad \{a = d,\ c = sd + \tfrac{1}{2}d^2\}.$$

We may set $s = 0$ by transforming by $[0, 0, -s, 0]$, which is commutative with $(L_{1,1})$. The argument at the end of § 14 excludes $\{a = d = 0\}$.

Let $G_{p^2} = \{a = c = 0\}$. Now $S$ transforms $[k, 0, 0, \gamma]$ into $\Sigma$, given by the second matrix of $\mathrm{II}_{374}$ when $a = c = 0$. The supposition $\beta_{22} = 0$ contradicts $i = 2$. If $\alpha_{22} \neq 0$, we employ $\Sigma$ for $\gamma = \alpha_{22}^{-1}\beta_{22}^{-1}$. Hence we may set $\beta_{22} \neq 0$, $\alpha_{22} = 0$ in $S$. Then, for $\gamma = -\beta_{22}^{-2}$, $\Sigma L_{1,-k'}$ has the form $L'_{2,1}Q_{1,2,\delta}$. This is transformed into $L'_{2,1}L_{1,\delta^2}$ by $N_{1,2,-\delta}$, which is commutative with $\{a = c = 0\}$. Hence $H$ contains $L'_{2,1}$ and $\{a = c = 0\}$, which generate $K_{\pi'}$ of § 1. By § 4, $H$ lies in $K_\pi$ or $G_\omega$.

For $G_{p^2} = \{a = d,\ c = \frac{1}{2}d^2\}$, a similar argument shows $H$ contains an operator $U_{r,s}$, with $s \neq 0$:

$$U_{r,s} = \begin{bmatrix} 1 & 0 & r & s \\ 0 & 1 & 0 & 0 \\ 0 & s & 1 & 0 \\ 0 & s-r & 1 & 1 \end{bmatrix}, \qquad X = \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 1 \\ 0 & -\frac{3}{2} & -1 & 2 \end{bmatrix}.$$

Now $N_{1,2,-r}$, which is commutative with $G_{p^2}$, transforms $U_{r,s}$ into $L_{1,r^2-rs}U_{0,s}$. Hence, $H$ contains $G_{p^2}$ and $U_{0,s}$. Set $E_d = [0, d, d^2/2, d]$. Then $H$ contains $X \equiv L_{1,2s}U_{0,s}^{-1}E_1U_{0,s}$ and $Y$, the transform of $E_{-1}$ by $[2, 2, 2, 2]X$. Now $N_{1,2,2}$ transforms $G_{p^2}$ into itself, and $Y$ into $L_{1,27/4}U_{0,1}$. We may thus assume that $H$ contains $G_{p^2}$ and $U_{0,1}$. Then $H$ contains

$$X^{-1}L_{1,\frac{3}{2}}E_{-1}U_{0,1}^{-1}E_1 = [-1, -2, 0, 0],$$

which belongs to $G_{p^4}$, but not to $G_{p^2}$, contrary to $i = 2$.

16. Let next $G_p = (L_{1,\mu}L_{2,1})$. Then $S$ is of the form $(11)_2$, a special case of $(1)_2$. Hence $S$ transforms $\{a = 0\}$ into itself. But the only $G_{p^2}$ containing $G_p$ are $\{a = c = 0\}$ and $\{a = 0,\ k = rc + \mu d\}$. Hence $S$ transforms either of these into a subgroup of $G_{p^4}$, in contradiction with $i = 2$.

17. Let finally $G_p = (B)$, $p$ being $> 3$. Then $G_{p^2}$ must be either

$$\{a = d = 0\}, \qquad \text{or} \qquad \{a = d,\ c = \sigma d + \tfrac{1}{2}d^2\}.$$

The first is excluded by § 8. The second is transformed into itself by any

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operator of $\{a = d\}$, and into $\{a = d, c = s^2\sigma d + \frac{1}{2}d^2\}$ by $T_{1,s^2}T_{2,s}$. Transforming the result by $[0, 0, (s^4 - 1)/12, (s^2 - 1)/2]$, we obtain

$$\{a = d, c = [s^2\sigma - \tfrac{1}{2}(s^2 - 1)]d + \tfrac{1}{2}d^2\}.$$

This lies in $G_{p^4}$, in contradiction with $i = 2$.

18. It remains to consider the $H_{p^2N}$ no two of whose subgroups $G_{p^2}$ have in common an operator $\neq I$. Hence the number of conjugate $G_{p^2}$ is $M$, where $M \equiv 1 \pmod{p^2}$. It is readily shown that the only factors of the form $1 + p^2x$ of $\omega = (p^4 - 1)(p^2 - 1)$ are $1, p^2 + 1, (p^2 - 1)^2$ and $\omega$. Hence if $H$ is of index $< \tau$, where $\tau = (p^4 - 1)/(p - 1)$, we may set $M = p^2 + 1$ or $(p^2 - 1)^2$. The latter case is immediately excluded in view of the orders of the largest subgroups containing a $G_{p^2}$ self-conjugately (§ 8). For $M = p^2 + 1$, $G_{p^2}$ is self-conjugate in a $G_{p^2t}$ within $H$, where $t$ divides $(p^2 - 1)^2$. In fact, $t \leqq 2(p^2 - 1)$ by § 8. Hence, for $p > 3$, $H$ is of index $> \tau$.

19. If a subgroup $H$ of order $pN$ is of index $< \tau$, then $N = \omega$. The details of the exclusion (for $p > 3$) of this isolated case will be omitted, in view of an anticipated treatment of all orders $pN$. In this direction I have shown that any $H_{pN}$ with more than one $C_p$ conjugate with $(L_{1,1})$ may be transformed into the group $\Gamma$ of the binary transformations of determinant unity on $\xi_1$ and $\eta_1$, or else into a direct product of $\Gamma$ and a binary group on $\xi_2$ and $\eta_2$ with no operator of period $p$ (and hence of order $2, 4d, 24, 48$ or $120$).

THE UNIVERSITY OF CHICAGO,
*October* 1, 1904.

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# DETERMINATION OF ALL THE GROUPS OF ORDER $2^m$ WHICH CONTAIN AN ODD NUMBER OF CYCLIC SUBGROUPS OF COMPOSITE ORDER*

BY

G. A. MILLER

It has recently been proved that in every non-cyclic group of order $p^m$, $p$ being an odd prime, the number of cyclic subgroups of order $p^\beta$, $\beta > 1$, is always a multiple of $p$, and that this number is of the form $1 + p + kp^2$ when $\beta = 1$.† From this it follows almost directly that the number of cyclic subgroups of order $p^\beta$ in any group $(G)$ is always of the form $kp$ whenever the Sylow subgroups of order $p^m$ in $G$ are non-cyclic, and that the number of subgroups of order $p$ in such a $G$ is always of the form $1 + p + kp^2$.‡ When $p = 2$, both of these theorems have exceptions. The present paper is devoted to an exhaustive study of the exceptions of the former theorem. Since the cyclic groups are so elementary we shall confine our attention to the non-cyclic groups of order $2^m$. Moreover, every group of even order contains an odd

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† Proceedings of the London Mathematical Society, ser. 2, vol. 2 (1904), p. 142.

‡ The former of these facts was noticed in the article just mentioned. The second does not follow so easily, but it may be proved as follows: Since the number of subgroups of order $p$ in such a Sylow subgroup $(S)$ of order $p^m$ is $\equiv 1 + p \pmod{p^2}$, it is only necessary to observe that the number of those which are found in $G$ but not in $S$ is of the form $kp^2$. If $S$ transforms one of these subgroups $(P)$ into only $p$ conjugates, the operators of $P$ are commutative with every operator of a subgroup of order $p^{m-1}$ contained in $S$. In particular, $P$ and an invariant (under $S$) operator of order $p$ in this subgroup of order $p^{m-1}$ generate a group of order $p^2$ and of type $(1, 1)$ which has just $p$ conjugates under $S$. It could not be transformed into itself by $S$ since a Sylow subgroup cannot transform a group of order $p^\gamma$ into itself unless this group is contained in the Sylow subgroup. The given $p$ subgroups of order $p^2$ contain $p^2$ distinct subgroups of order $p$ which are not found in $S$ and whose generators are commutative with each operator of the given subgroup of order $p^{m-1}$. If there is any other operator $(P)$ of order $p$ in $G$ which is not found in $S$ and is commutative with each operator of the given subgroup of order $p^{m-1}$, we use the same invariant operator under $S$ and construct other $p$ conjugate subgroups under $S$ which have $p^2$ additional subgroups of order $p$, etc. Proceeding in the same manner with each of the other subgroups of order $p^{m-1}$ in $S$, the theorem clearly follows since we do not need to consider those subgroups of order $p$ which have more than $p$ conjugates under $S$. When the Sylow subgroups of order $p^m$ $(p > 2, m > 1)$ in $G$ are cyclic, the preceding argument proves that the number of subgroups of order $p$ is of the form $1 + kp^2$.



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number of subgroups of order 2 since the operators of order 2 and the identity are the only ones which are self inverse.

## § 1. *Cyclic subgroups whose order exceeds four.*

We shall first prove that a group of order $2^m$ cannot contain just $2n+1$, $n>0$, cyclic subgroups of order $2^\alpha$, $\alpha>2$. Suppose that such a group $(G)$ contains $2n+1$ cyclic subgroups of order $2^\alpha$. At least one of them $(H_1)$ is invariant under $G$. Let $H_2$ represent any other and let $H_2'$ be the subgroup of $H_2$ such that the group $\{H_1, H_2'\}$ generated by $H_1$, $H_2'$ is of order $2^{\alpha+1}$. When $H_2'$ coincides with $H_2$ then $\{H_1, H_2'\}$ contains just 2 cyclic subgroups of order $2^\alpha$.* If this condition is not satisfied $H_2'$ must transform $H_1$ according to the square of an operator in its group of isomorphism. Hence $\{H_1, H_2'\}$ has still just 2 cyclic subgroups of order $2^\alpha$ and at least $2^{\alpha-1}$ invariant operators.† These two cyclic subgroups, which we may call $H_1$, $H_2$, contain just $2^{\alpha-1}$ common operators. It will next be proved that $G$ contains an even number of cyclic subgroups of order $2^\alpha$ which involve these $2^{\alpha-1}$ common operators. The main result thus far is that any group which contains an odd number (greater than one) of cyclic subgroups of order $2^\alpha$ must contain two such subgroups which have $2^{\alpha-1}$ common operators.

Suppose that $G$ contains an odd number of cyclic subgroups of order $2^\alpha$ which have $2^{\alpha-1}$ operators in common with $H_1$. One of them $(H_3)$ must be transformed into itself by $\{H_1, H_2\}$. The group $\{H_1, H_2, H_3\}$ is clearly conformal to an abelian group with respect to its operators whose orders exceed 4. This follows almost directly from the fact that the order of the commutator subgroup of this group cannot exceed 2. Hence $\{H_1, H_2, H_3\}$ contains four cyclic subgroups of order $2^\alpha$. At least one of the remaining cyclic subgroups of order $2^\alpha$ which have $2^{\alpha-1}$ operators in common with $H_1$ is transformed into itself by $\{H_1, H_2, H_3\}$. Calling this $H_5$ we have again that the group $\{H_1, H_2, H_3, H_5\}$ is conformal with an abelian group with respect to its operators whose order exceed four. This group contains 8 cyclic subgroups of order $2^\alpha$. As this process could be continued indefinitely if there were an odd number of cyclic subgroups of order $2^\alpha$ having $2^{\alpha-1}$ operators in common with $H_1$, this hypothesis is disproved.

The theorem in question is now practically proved, for the cyclic subgroups of order $2^\alpha$ which have $2^{\alpha-1}$ common operators must occur in even sets and a cyclic group of order $2^\alpha$ which is found in one set cannot also occur in another set. Hence *$G$ cannot contain just $2n+1$, $n>0$, cyclic subgroups of order $2^\alpha$, $\alpha>2$.* We shall now consider the case when $G$ contains only one cyclic

* BURNSIDE, *Theory of Groups of Finite Order*, 1897, p. 77.

† Bulletin of the American Mathematical Society, vol. 7 (1901), p. 352.

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subgroup of order $2^\alpha$. It will result that in this case $G$ must be one of these three well known groups of order $2^\alpha + 1$ having this property; viz., the dihedral rotation group, the one obtained from this dihedral rotation group by replacing its non-invariant operators of order 2 by operators of order 4, and the one which transforms the operators of the cyclic subgroup of order $2^{\alpha_1}$ into their $2^{\alpha_1} - 1$ powers, containing $2^{\alpha_1} - 1$ operators of each of the orders 2 and 4 in addition to the given cyclic subgroup.

If $G$ contains only one cyclic subgroup of order $2^\alpha$ it cannot contain more than one cyclic subgroup of any higher order $(2^{\alpha_1})$, since some two subgroups of this order would generate a group which is conformal with the abelian group of the type $(\alpha_1, 1)$.* Let $2^{\alpha_1}$ be the order of the largest cyclic subgroup $H_1$ of $G$. The operators of $G$ must transform the operators of $H_1$ according to a group of order $2^{m-\alpha_1}$. Every subgroup of order 4 in the group of isomorphisms of $H_1$ contains the operator of order 2 which transforms just half of the operators of $H_1$ into themselves.† Hence $m = \alpha_1 + 1$, otherwise $G$ would contain more than one cyclic subgroup of order $2^{\alpha_1}$ as it would have to transform the operators of $H_1$ according to the given operator of order 2. As the groups of order $2^{\alpha_1} + 1$ which contain a cyclic subgroup of order $2^{\alpha_1}$ are well known, the results stated at the end of the preceding paragraph are established. That is, *if a group of order $2^m$ contains an odd number of cyclic subgroups of order $2^\alpha$, $\alpha > 2$, this number must be unity, and the group must be one of three containing a cyclic subgroup of order $2^{m-1}$.* As the properties of these three groups are very elementary and their forms are so dissimilar (one contains $2^{m-1} + 1$, the other $2^{m-2} + 1$, and the third only one operator of order 2) it is easy to distinguish them. When $\alpha$ is given, $m$ may have any arbitrary value which exceeds $\alpha$.

## § 2. *Cyclic subgroups of order four.*

The three groups mentioned in the last paragraph contain respectively one, $2^{m-3} + 1$, and $2^{m-2} + 1$ cyclic subgroups of order 4. We proceed to prove that these are the only non-cyclic groups of order $2^m$ which contain an odd number of cyclic subgroups of order 4. The method of proof is very similar to the one employed in the preceding section.

Let $G$ be any group of order $2^m$ which contains an odd number of cyclic subgroups of order 4. At least one of them $(K_1)$ is invariant under $G$. At least half the operators of $G$ are commutative with a generator $(s)$ of $K_1$. It will be proved that the subgroup $(G_1)$ formed by these $2^{m-1}$ operators must be cyclic. If $G_1$ were not cyclic $K_1$ would be contained in an abelian subgroup of

* American Journal of Mathematics, vol. 23 (1901), p. 173.

† Bulletin of the American Mathematical Society, vol. 7 (1901), p. 351.

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type (2, 1).* This subgroup would contain two cyclic subgroups of order 4, $(K_1, K_2)$ having a common square. It will now be proved that $G$ would then contain an even number of cyclic subgroups of order 4 including $s^2$.

If this number were odd, $\{K_1, K_2\}$ would transform at least one of the remaining ones $(K_3)$ into itself. The commutator subgroup of $\{K_1, K_2, K_3]$ is generated by $s^2$ and hence this group contains an even number of cyclic subgroups of order 4 and all of these subgroups contain $s^2$.† Hence there would be another invariant cyclic subgroup $K_4$ involving $s^2$. The number of cyclic subgroups of order 4 in $\{K_1, K_2, K_3, K_4\}$ is again even since the commutator subgroup is still generated by $s^2$, and all of these subgroups include $s^2$. This follows almost directly from the fact that the product of an operator of order 4 in $\{K_1, K_2, K_3, K_4\}$ into an operator of order 2 is of order 4 when the two factors are commutative and of order 2 when they are not commutative, while the converse is true when the second factor is of order 4.

As this process could be continued indefinitely if there were an odd number of cyclic subgroups of order 4 which contained $s^2$, it results that the number of these subgroups is even. If there were an odd numbers of cyclic subgroups of order 4 in $G$ which did not contain $s^2$, one of these and $s^2$ would again generate the abelian group of type (2, 1) and the number of those involving the same subgroup of order 2 would again be even. As similar remarks would apply to all the possible other cyclic subgroups of order 4 we have proved that $G_1$ *is cyclic whenever $G$ contains an odd number of cyclic subgroups of order* 4. This proves the statement in the first paragraph of this section, since the other non-cyclic group of order $2^m$ which contains a cyclic subgroup of order $2^{m-1}$ contains an even number of cyclic subgroups of order 4.

Combining the results of the two sections we have that every group of order $2^m$, $m > 3$, which contains an odd number of cyclic subgroups of order 4 contains just one cyclic subgroup of order $2^\alpha$, where $\alpha$ can have all the values from 3 to $m - 1$; and every group which contains only one cyclic subgroup of order $2^\alpha$ contains an odd number of cyclic subgroups of order 4. For each value of $\alpha$ and $m$ there are three such groups, hence there is a doubly infinite system of groups of order $2^m$ which contain an odd number of cyclic subgroups of composite order. When $m = 3$ there are only two groups having this property, viz. the quaternion group and the group of movements of the square. If all the non-cyclic groups of order $p^m$ ($m > 3$, $p$ an arbitrary prime) were determined there would be just three among them in which the number of cyclic subgroups of composite order would not always be a multiple of $p$. In these three special cases the number of cyclic subgroups of every composite order is not divisible by $p$.

* BURNSIDE, loc. cit., p. 75.

† Quarterly Journal of Mathematics, vol. 28 (1896), p. 269.

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In the exceptional groups noted above the number of the subgroups of order 2 is $\equiv 1 \pmod 4$. That this number is $\equiv 3 \pmod 4$ in every other non-cyclic group of order $2^m$ is a direct consequence of the fact that the number of cyclic subgroups of order 4 in all of these groups is even. From this fact it results that the number of operators whose order exceeds 2 is divisible by 4, since every cyclic subgroup of order $2^\sigma$ contains $2^{\sigma-1}$ operators which are not found in any other subgroup whose order $\leqq 2^\sigma$. Hence the given system of groups is composed of all the groups of order $p^m$ in which the number of cyclic subgroups of order $p$ is not $\equiv 1 + p \pmod{p^2}$. That is, the groups of order $p^m$ in which the number of cyclic subgroups of composite order is not divisible by $p$ coincide with those in which the number of subgroups of order $p$ is not of the form $1 + p + kp^2$.

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# ON THE COEFFICIENTS

# IN THE QUOTIENT OF TWO ALTERNANTS*

BY

E. D. ROE, JR.

## § 1. *Introduction.*

This paper is complementary to one published in an earlier number of the Transactions.† The notations used there are retained with the following additions:

$$(1)\qquad \begin{aligned} &\left\|\begin{matrix} p_1 p_2 \cdots p_m \\ \kappa_1 \kappa_2 \cdots \kappa_m \end{matrix}\right\| = \left|\begin{bmatrix} p_1 p_2 \cdots p_m \\ q_1 q_2 \cdots q_m \end{bmatrix}\right| = \left|\begin{bmatrix} p \\ q \end{bmatrix}\right|, \\ &\left|\begin{matrix} \kappa_1 \kappa_2 \cdots \kappa_m \\ (1)^{j_1}(1^2)^{j_2}\cdots(1^n)^{j_n} \end{matrix}\right| = \begin{bmatrix} q \\ (1^i)^j \end{bmatrix}, \end{aligned}$$

where $q_1 = \kappa_1, q_2 = \kappa_2 - 1, \cdots, q_m = \kappa_m - m + 1$, so that $p$ and $q$ are both partitions of weight $w$;

$$(2)\quad \delta_r f(p_1, p_2, \cdots, p_m) = \sum f(p_1 - 1, p_2 - 1, \cdots, p_r - 1, p_{r+1} \cdots p_m),$$

$$(3)\quad \bar{\delta}_r f(p_1, p_2, \cdots, p_m) = \sum f(p_1 - r_1, p_2 - r_2, \cdots, p_\lambda - r_\lambda, p_{\lambda+1} \cdots p_m),$$

where

$$r_1 + r_2 + \cdots + r_\lambda = r, \qquad 1 \leqq \lambda \leqq r, \qquad 0 \leqq p_1 - r_1 \leqq p_2 - r_2 \leqq \cdots,$$

and where such terms in $\sum$ are excepted as would present the subtraction of the same number from two or more equal $p$'s;

$$(4)\quad \partial_r f(p_1, p_2, \cdots, p_m) = \partial_r f(1^{x_1} 2^{x_2} \cdots r^{x_r} \cdots m^{x_m}) = f(1^{x_1} 2^{x_2} \cdots r^{x_r - 1} \cdots m^{x_m});$$

$$(5)\quad \partial_r f[(1)^{x_1}(1^2)^{x_2} \cdots (1^m)^{x_m}] = f[(1)^{x_1}(1^2)^{x_2} \cdots (1^r)^{x_r - 1} \cdots (1^m)^{x_m}].$$

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* Presented to the Society December 29, 1902. Received for publication May 9, 1904.

† *On the Coefficients in the Product of an Alternant and a Symmetric Function* vol. 5 (1904), no. 2, pp. 193–213. This paper will be referred to as I.



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The subject of investigation is the coefficients $\left|\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]\right|$ of the symmetric functions $(p)$ in the quotient

$$(6) \qquad \frac{|\kappa_1 \kappa_2 \cdots \kappa_m|}{|01 \cdots m|} = \sum \left|\begin{matrix} p \\ \kappa \end{matrix}\right| (p) = \sum \left|\begin{bmatrix} p \\ q \end{bmatrix}\right| (p).$$

A table of such coefficients for weight $w$ is called a quotient table for weight $w$.

The following results are obtained: The relations

$$\left|\begin{bmatrix} p \\ q \end{bmatrix}\right| = \frac{\partial_r}{\delta_r} \left|\begin{bmatrix} p \\ q \end{bmatrix}\right|, \qquad \text{and} \qquad \sum_\kappa \left|\begin{matrix} \kappa \\ p \end{matrix}\right| \cdot \left|\begin{matrix} q \\ \kappa \end{matrix}\right| = \begin{matrix} 0 \\ 1 \end{matrix}$$

according as $\begin{smallmatrix} p \neq q \\ p = q \end{smallmatrix}$, together with other recurrence formulas, are found in § 2. Generalizations for some of the columns of coefficients are brought out in § 3. In § 4, it is shown that: the table is contained in a triangle and fills up the remainder of the square with the product table; none of the coefficients is negative; the diagonal line next to the hypotenuse is skew-symmetrical with the corresponding diagonal of the product table; the hypotenuse consists of ones which the table has in common with the product table; the table possesses the invariant property; the coefficients are the conjugates of those of Dr. Taylor's product table; the last line of the table is proved to be symmetrical with respect to its middle element. In addition to the quotients aimed at, the table also gives the determinants $\{\kappa_1 \kappa_2 \cdots \kappa_m\}_a$, or $\{\kappa_1 \kappa_2 \cdots \kappa_m\}_t$ in terms of monomial symmetric functions, and by means of a two-fold application it gives the values of the elementary products of weight $w$ in terms of monomial symmetric functions (§ 5).

## § 2. *Recurrence Formulas for the* $\left|\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]\right|$.

1. Dr. Taylor has obtained the following formula,* which has been expressed in the writer's notation, viz.:

$$(7) \qquad \left|\begin{matrix} \kappa \\ (1^i)^j \end{matrix}\right| = \frac{\delta_r}{\partial_r} \left|\begin{matrix} \kappa \\ (1^i)^j \end{matrix}\right|,$$

where the upper operator $\delta_r$ applies to the upper complex only, and the lower one $\partial_r$ to the lower, and where such terms in the summation are excluded as would make two of the $\kappa$'s equal, since in this case the corresponding alternant would vanish. By using the partition $q$ of weight $w$, instead of the complex $\kappa$, obtained from it as explained in connection with (1), (7) becomes

$$(8) \qquad \begin{bmatrix} q \\ (1^i)^j \end{bmatrix} = \frac{\delta_r}{\partial_r} \begin{bmatrix} q \\ (1^i)^j \end{bmatrix},$$

* *On the Product of an Alternant by a Symmetric Function*, American Mathematical Monthly, vol. 10 (1903), pp. 119–130.

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where such terms in the sum vanish as cause a smaller number to follow a larger one in the series $q$.

Example: Using $\partial_3$, $\delta_3$ as operators, and having the table for weight 4,

$$\left[\begin{matrix} 12^3 \\ (1)(1^3)^2 \end{matrix}\right] = \left[\begin{matrix} 1^2 2 \\ (1)(1^3) \end{matrix}\right] + \left[\begin{matrix} 1^4 \\ (1)(1^3) \end{matrix}\right] = 1 + 1 = 2.$$

2. Expressing (7) in terms of conjugates by the relation (30) of I, and by considering the effect on its conjugate of an operator $\delta_r$ on a partition, we find that the operator $\delta_r$ goes over into its conjugate $\bar{\delta}_r$, whence,

$$\left|\left[\begin{matrix} p \\ q \end{matrix}\right]\right| = \begin{matrix} \partial_r \\ \bar{\delta}_r \end{matrix} \left|\left[\begin{matrix} p \\ q \end{matrix}\right]\right|. \tag{9}$$

Formula (9) reduces a coefficient in the quotient table of weight $w$, to a sum of coefficients in the table of weight $w - r$.

Example: Using $\partial_2$ and $\bar{\delta}_2$,

$$\left|\left[\begin{matrix} 1^3 2^2 \\ 2^2 3 \end{matrix}\right]\right| = \left|\left[\begin{matrix} 1^3 2 \\ 23 \end{matrix}\right]\right| + \left|\left[\begin{matrix} 1^3 2 \\ 12^2 \end{matrix}\right]\right| = 3 + 2 = 5,$$

by the table for weight 5.

Since the tables read horizontally give the values of the elementary products in terms of the determinants $\{\kappa_1 \kappa_2 \cdots \kappa_m\}_a = [q_1 q_2 \cdots q_m]_a$, we obtain from this and (9) the following proposition in determinants:*

$$a_r [q_1 q_2 \cdots q_r] = \sum [x_0, q_1 + x_1, q_2 + x_2, \cdots, q_r + x_r, \cdots, q_m], \tag{10}$$

where $x_0 + x_1 + x_2 + \cdots + x_r = r$, $0 \leqq x_0 \leqq q_1 + x_1, \leqq q_2 + x_2, \cdots$, and where the same number must not be added to two or more equal indices ($q$'s).

3. When the number of elements in $p$ and $q$ is the same, we may find the coefficient by a single reduction by subtracting unity from each element of $p$ and $q$, for the partitions of the reduced coefficient. We have

$$\left|\left[\begin{matrix} p_1 p_2 \cdots p_n \\ q_1 q_2 \cdots q_n \end{matrix}\right]\right| = \left|\left[\begin{matrix} p_1 - 1\, p_2 - 1 \cdots p_n - 1 \\ q_1 - 1\, q_2 - 1 \cdots q_n - 1 \end{matrix}\right]\right|. \tag{11}$$

If $n$ be the number of elements in each partition the coefficient of weight $w$ is reduced to one of weight $w - n$. Thus $\left|\left[\begin{smallmatrix} 2^2 3 \\ 124 \end{smallmatrix}\right]\right| = \left|\left[\begin{smallmatrix} 1^2 2 \\ 13 \end{smallmatrix}\right]\right| = 2$, by the table for weight 4. This depends upon the invariance of the table treated of in § 4, 8., and is there explained.

4. We have also the two following particular recurrence formulas:

$$\left|\left[\begin{matrix} p_1 p_2 p_3 \\ q_1 q_2 \end{matrix}\right]\right| = \begin{matrix} 1 \\ 0 \end{matrix} + \left|\left[\begin{matrix} p_1 - 1\, p_2 - 1\, p_3 - 1 \\ q_1 - 1\, q_2 - 2 \end{matrix}\right]\right|, \tag{12}$$

* Cf., MUIR, *Vanishing Aggregates of Secondary Minors of a Persymmetric Determinant*, Transactions of the Royal Society of Edinburgh, vol. 40 (1902), pp. 511–533.



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where $q_2 \geqq q_1 + 1$, and where in the first term 0 or 1 is to be taken according as any element of $p$ is or is not greater than $q_2$. Similarly

$$(13) \qquad \left|\begin{bmatrix} p_1 p_2 p_3 p_4 \\ q_1^2 \end{bmatrix}\right| = \begin{matrix} 1 \\ 0 \end{matrix} + \left|\begin{bmatrix} p_1 - 1\, p_2 - 1\, p_3 - 1\, p_4 - 1 \\ (q_1 - 2)^2 \end{bmatrix}\right|.$$

These formulas are derived from two relations given by W. W. JOHNSON,* taken in connection with the before-mentioned invariance. The relations, using the present notation, are

$$(14) \qquad \frac{|0\, q_1 + 1\, q_2 + 2|}{|\ 012\ |} = t_{q_2, q_1} + \alpha_1 \alpha_2 \alpha_3 \frac{|0\, q_1\, q_2|}{|\ 012\ |},$$

$$(15) \qquad \frac{|0\,1\, q_1 + 2\, q_1 + 3|}{|\ 0123\ |} = t_{q_1, q_1} + \alpha_1 \alpha_2 \alpha_3 \alpha_4 \frac{|0\,1\, q_1\, q_1 + 1|}{|\ 0123\ |}.$$

Examples:

$$\left|\begin{bmatrix} 2^2 3 \\ 3\ 4 \end{bmatrix}\right| = 1 + \left|\begin{bmatrix} 1^2 2 \\ 2^2 \end{bmatrix}\right| = 1 + 1 = 2,$$

$$\left|\begin{bmatrix} 1^2 5 \\ 3\ 4 \end{bmatrix}\right| = 0 + \left|\begin{bmatrix} 4 \\ 2^2 \end{bmatrix}\right| = 0 + 0 = 0,$$

$$\left|\begin{bmatrix} 1^2 2^2 \\ 3^2 \end{bmatrix}\right| = 1 + \left|\begin{bmatrix} 1^2 \\ 1^2 \end{bmatrix}\right| = 1 + 1 = 2,$$

$$\left|\begin{bmatrix} 1^3 3 \\ 3^2 \end{bmatrix}\right| = 1 + \left|\begin{bmatrix} 2 \\ 1^2 \end{bmatrix}\right| = 1 + 0 = 1.$$

5. From the product table, (I), we have

$$(16) \qquad (p) \equiv \sum_{\kappa} \left|\begin{matrix} \kappa \\ p \end{matrix}\right| \{\kappa\}_t;$$

from the quotient table,

$$(17) \qquad \{\kappa\}_t \equiv \sum_{q} \left\|\begin{matrix} q \\ \kappa \end{matrix}\right\| (q),$$

hence

$$(18) \qquad (p) \equiv \sum_{\kappa} \left|\begin{matrix} \kappa \\ p \end{matrix}\right| \cdot \left(\sum_{q} \left\|\begin{matrix} q \\ \kappa \end{matrix}\right\| (q)\right).$$

Again from the quotient table

$$(19) \qquad a_p \equiv \sum_{\kappa} \left\|\begin{matrix} p \\ \kappa \end{matrix}\right\| \{\kappa\}_a.$$

From the product table

$$(20) \qquad \{\kappa\}_a \equiv \sum_{q} \left|\begin{matrix} \kappa \\ q \end{matrix}\right| a_q,$$

* *On a Formula of Reduction for Alternants of the Third Order*, American Journal of Mathematics, vol. 7 (1885) pp. 347–352. See also the same author's paper *On the Calculation of the Co-Factors of Alternants of the Fourth Order*, l. c. pp. 380–388.

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hence

$$a_p \equiv \sum_\kappa \left\| \begin{matrix} p \\ \kappa \end{matrix} \right\| \cdot \left( \sum_q \left| \begin{matrix} \kappa \\ q \end{matrix} \right| a_q \right). \tag{21}$$

From either of the identities (18) or (21) follows the identity

$$\sum_\kappa \left| \begin{matrix} \kappa \\ p \end{matrix} \right| \cdot \left\| \begin{matrix} q \\ \kappa \end{matrix} \right\| \equiv \begin{matrix} 0 \\ 1 \end{matrix}, \tag{22}$$

according as $\begin{matrix} p \neq q \\ p = q \end{matrix}$. From (22) we derive the recurrence formula,

$$\left\| \begin{bmatrix} p \\ q \end{bmatrix} \right\| = -\sum_\kappa \left| \begin{matrix} \kappa \\ q \end{matrix} \right| \cdot \left\| \begin{matrix} p \\ \kappa \end{matrix} \right\|, \tag{23}$$

where $\kappa$ goes from $1\,2\,3 \cdots m$ to the complex next preceding

$$q_1\, q_2 + 1\, q_3 + 2 \cdots q_m + m - 1.$$

Thus

$$\left\| \begin{bmatrix} 1^4 2 \\ 1^3 3 \end{bmatrix} \right\| = -(-1-3) = 4.$$

By the help of the product table one can by means of this recurrence formula calculate any line of the quotient table beginning at the left of the line. If the quotient table were given the product table might similarly be found by the same relation.

## § 3. *General formulas for the* $\left\| \begin{matrix} p \\ \kappa \end{matrix} \right\|$.

As in the product table, so also in the quotient table it has been possible to generalize certain of the coefficients by columns. Since

$$\frac{|\kappa_1 \kappa_2 \cdots \kappa_m|}{|01 \cdots m-1|} = \{\kappa_1 \kappa_2 \cdots \kappa_m\}_t = \{\kappa\}_t$$

and since the determinants $\{\kappa\}_t$ are given in the product table, we may assume as known the function $\{\kappa\}_t$, which gives all the coefficients in the column headed by the complex $\kappa$ or, what is the same thing, by the partition $q$. Let $n_{-r}$ denote the number of different ways of subtracting $r$ units from the complex of exponents $p_1 p_2 \cdots p_m$ of the symmetric function $(p)$, and let $(n_{-r})_{r_1 r_1 \cdots r_\lambda}$ where $r_1 + r_2 + \cdots + r_\lambda = r$, denote the number of ways of subtracting $r$ units at a time from $p$, $r_1$ from one index, $r_2$ from another and so on. Then

$$n_{-r} = \sum (n_{-r})_{r_1 r_2 \cdots r_\lambda}. \tag{24}$$

Also it is evident that

$$\left\| \begin{matrix} p \\ \kappa \end{matrix} \right\| = f(n_0, n_{-1}, n_{-2}, \cdots) = f(n), \tag{25}$$

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and the problem of determining the $\|{}^{p}_{\kappa}\|$ is a purely combinatorial one. In particular with $p_1 p_2 \cdots p_m = 0^{x_0} 1^{x_1} 2^{x_2} \cdots m^{x_m}$,

$$(26) \qquad n_0 = \binom{x_1 + \cdots + x_m}{0} = 1, \qquad (n_{-2})_2 = \binom{x_2 + \cdots + x_m}{1},$$

$$(27) \qquad n_{-1} = \binom{x_1 + \cdots + x_m}{1}, \qquad (n_{-2})_{11} = \binom{x_1 + \cdots + x_m}{2},$$

$$(28) \qquad n_{-2} = (n_{-2})_2 + (n_{-2})_{11} = \binom{x_2 + \cdots + x_m}{1} + \binom{x_1 + \cdots + x_m}{2},$$

$$(29) \qquad \begin{gathered} (n_{-3})_3 = \binom{x_3 + \cdots + x_m}{1}, \quad (n_{-3})_{21} = 2\binom{x_2 + \cdots + x_m}{2} + \binom{x_2 + \cdots + x_m}{1}\binom{x_1}{1}, \\ (n_{-3})_{111} = \binom{x_1 + \cdots + x_m}{3}, \end{gathered}$$

$$(30) \qquad \begin{aligned} n_{-3} = (n_{-3})_3 + (n_{-3})_{21} + (n_{-3})_{111} &= \binom{x_3 + \cdots + x_m}{1} \\ &+ 2\binom{x_2 + \cdots + x_m}{2} + \binom{x_2 + \cdots + x_m}{1}\binom{x_1}{1} + \binom{x_1 + \cdots + x_m}{3}, \end{aligned}$$

We thus find for the following partitions $q$, $\{\kappa\}_t = F(t)$, and $f(n)$, together with the coefficient which holds for the entire column headed by the partition $q$, as shown by the table on page 007.

Example: By (35)

$$\left\| \begin{bmatrix} 1^3 2^2 \\ 1^2 5 \end{bmatrix} \right\| = \binom{4}{2} = 6.$$

## §4. *The construction and properties of the table.*

1. The partitions are written horizontally and vertically as explained for the product table, I, with complexes of alternants or determinants corresponding to the several partitions heading the columns. The quotient table is read vertically downwards and not horizontally as in the case of the product table. The argument is taken from the horizontal line of complexes at the top of the table in the column headed by the complex $\kappa$. The quotient $|\kappa_1 \kappa_2 \cdots \kappa_m| / |0\,1 \cdots m-1|$, is equal to the sum of all the products got by multiplying each coefficient of the column into the symmetric function signified by the partition at the left of the line in which the coefficient occurs.

2. From the nature of division* it is seen that the first term in the quotient

* See MUIR, *The Theory of Determinants*, § 130, p. 177.

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| Partition | $F(t)$ | $f(n)$ | Coefficient |
|---|---|---|---|
| $(31)(n)$ | $t_n$ | $n_0$ | $1$ |
| $(32)(n-11)$ | $t_1 t_{n-1} - t_n$ | $n_{-1} - n_0$ | $x_1 + \cdots + x_m - 1$ |
| $(33)(n-22)$ | $t_2 t_4 - t_1 t_5$ | $n_{-2} - n_{-1}$ | $\binom{x_1 + \cdots + x_m}{2} - \binom{x_1}{1}$ |
| $(34)(n-33)$ | $t_3 t_{n-3} - t_2 t_{n-2}$ | $n_{-3} - n_{-2}$ | $\binom{x_2 + \cdots + x_m}{2} + \binom{x_1 + \cdots + x_m}{3} - \binom{x_2}{1} - \binom{x_1}{2}$ |
| $(35)(n-21^2)$ | $t_1^2 t_{n-2} + t_n - t_1 t_{n-1} - t_2 t_{n-2}$ | $1 - n_{-1} - n_{-2} + (n_{-2})_2 + 2(n_{-2})_{11}$ | $\binom{x_1 + \cdots + x_m - 1}{2}$ |
| $(36)(n-321)$ | $t_1 t_{n-1} - t_3 t_{n-3} - t_1^2 t_{n-2} + t_1 t_2 t_3$ | $n_{-1} - n_{-3} - (n_{-2})_2 - 2(n_{-2})_{11} + (n_{-3})_3 + 2(n_{-3})_{21} + 3(n_{-3})_{111}$ | $\binom{x_1}{1} - \binom{x_2 + \cdots + x_m}{1}\binom{x_1}{1} - 2\binom{x_1}{2} + 2\binom{x_1 + \cdots + x_m}{3}$ |
| $(37)(n-431)$ | | | $\binom{x_2}{1} - \binom{2x_2 + x_3 + \cdots + x_m}{1}\binom{x_1 + \cdots + x_m - 1}{1} + \binom{x_2 + \cdots + x_m}{2} + 2\binom{x_2 + \cdots + x_m}{1}\binom{x_1 + \cdots + x_m - 1}{2} + \binom{x_1 + \cdots + x_m}{2} + \frac{3}{4}\binom{x_1 + \cdots + x_m}{3}\binom{x_1 + \cdots + x_m - 1}{1}$ |
| $(38)(n-44)$ | | | $\binom{x_2 - x_3}{1} - \binom{x_1 + \cdots + x_m}{1}\binom{x_2}{1} - \binom{x_1 + \cdots + x_m}{3} + \binom{x_2 + \cdots + x_m}{2} + \binom{x_2 + \cdots + x_m}{1}\binom{x_1 + \cdots + x_m - 1}{2} + \binom{x_1 + \cdots + x_m}{4}$ |

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is the symmetric function $(q)$, where $q = \kappa_1, \kappa_2 - 1, \cdots, \kappa_m - m + 1$; whence it follows that

$$\left|\begin{bmatrix} p \\ q \end{bmatrix}\right| = 0, \tag{39}$$

when $p$ precedes $q$, and

$$\left|\begin{bmatrix} p \\ p \end{bmatrix}\right| = 1. \tag{40}$$

From (39) and (40) it follows that the table is contained in a triangle which fills up the remainder of the square with the product table; also that the hypotenuse which belongs in common to both tables, consists of ones.

3. Of the coefficients none can be negative. This can be seen from (9) in which no minus sign occurs in the summation. It can also be seen from the standpoint of the coefficients as coefficients in the product of the elementary functions by $|01 \cdots m - 1|$. In the formation of this product, units must be laid on the complex in such a way that no inversions of order can arise among the elements of the resulting complexes.

4. The last column consists of ones; in the next to the last column the coefficient is one less than the number of elements in the partition at the left in the same horizontal line with it. In the third last column the coefficient is equal to the number of combinations of the number of elements taken two at a time, minus the number of elements equal to unity. In the column headed by the partition $(n - 21^2)$ the coefficient is equal to the number of combinations of one less than the number of elements in the partition at the left, taken two at a time. These and other like generalizations follow from formulas (31) to (38).

5. The diagonal line next to the hypotenuse is skew symmetrical with the corresponding diagonal of the product table. This follows at once from formula (23). It can also be seen by noticing that the first term of the first remainder in the process of division comes from changing the sign of the alternant next to the hypotenuse. As a consequence of this property no coefficient of the product table in the diagonal next to the hypotenuse can be positive.

6. It may be recalled that it was shown in I, § 4 (30), that the coefficients are those of Dr. Taylor's product table, when conjugate columns have been interchanged. This also assumes a rearrangement of Dr. Taylor's table according to the writer's method of ordering, as explained in I.

7. The last line of the table is symmetrical with respect to its middle element or elements, as the first line of the product table is symmetrical or skew symmetrical in the same way according as $w$ is odd or even.* In fact

* To prove this we note, by the method of § 3, 2. I, that all coefficients in the first line will be zero, except those whose column partition consists of $r$ units and the element $m - r$. The value of the coefficient is then, by § 3, 2. I,

$$\begin{bmatrix} m \\ 1^r m - r \end{bmatrix} = (-1)^r.$$

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$$(41)\quad \left|\begin{bmatrix} 1^m \\ p_m p_{m-1}\cdots p_2 p_1 \end{bmatrix}\right| = \left|\begin{bmatrix} 1^{m-1} \\ p_m p_{m-1}\cdots p_2 p_1 - 1 \end{bmatrix}\right| + \left|\begin{bmatrix} 1^{m-1} \\ p_m p_{m-1}\cdots p_2 - 1 p_1 \end{bmatrix}\right| + \cdots + \left|\begin{bmatrix} 1^{m-1} \\ p_m - 1 p_{m-1}\cdots p_1 \end{bmatrix}\right|,$$

$$(42)\quad \left|\begin{bmatrix} 1^m \\ 1^{p_1-p_2} 2^{p_2-p_3}\cdots m^{p_m} \end{bmatrix}\right| = \left|\begin{bmatrix} 1^{m-1} \\ 1^{p_1-1-p_2} 2^{p_2-p_3}\cdots m^{p_m} \end{bmatrix}\right| + \left|\begin{bmatrix} 1^{m-1} \\ 1^{p_1+1-p_2} 2^{p_2-1-p_3}\cdots m^{p_m} \end{bmatrix}\right| + \cdots + \left|\begin{bmatrix} 1^{m-1} \\ 1^{p_1-p_2} 2^{p_2-p_3}\cdots (m-1)^{p_{m-1}+1-p_m} m^{p_m-1} \end{bmatrix}\right|.$$

From (41) and (42), since the lower partitions of the left members and also of the right members term by term are conjugates, it follows, if the property is true for the table of weight $w-1$, it is true for the table of weight $w$. But it is seen to be true when $w=2$, and hence it is true for all values of $w$.*

8. Like the product table it possesses the invariant property. In fact since each is the unique inverse of the other table, the invariance of the quotient table follows from that of the product table. It also appears as follows. We have

$$(43)\qquad \frac{|\kappa_1 \kappa_2 \cdots \kappa_m|}{|01\cdots m-1|} = \{\kappa\}_t = [q]_t.$$

If $i_1 i_2 \cdots i_{m+r}$ are so taken that

$$i_1,\ i_2-1,\ i_3-2,\ \cdots,\ i_{m+r}-m-r-1 = 0^r q_1 q_2 \cdots q_m,$$

then as an explicit function of $t$,

$$(44)\qquad \frac{|i_1 i_2 \cdots i_{m+r}|}{|01\cdots m+r-1|} = \{\kappa\}_t = [q]_t.$$

Hence, in their development in terms of monomial symmetric functions, as far as these functions contain no more than $m$ elements, $|i_1 i_2 \cdots i_{m+r}|/|01\cdots m+r-1|$ and $|\kappa_1 \kappa_2 \cdots \kappa_m|/|01\cdots m-1|$ must have identical coefficients. In the former the symmetric functions are functions of $m+r$ quantities $\alpha_1, \alpha_2, \cdots, \alpha_{m+r}$, in the latter they are functions of the $m$ quantities $\alpha_1, \alpha_2, \cdots, \alpha_m$. The latter quotient can be obtained from the former by putting $\alpha_{m+1} = \alpha_{m+2} = \cdots = \alpha_{m+r} = 0$

* The value of the conjugate coefficient is

$$\begin{bmatrix} m \\ 1^{m-r-1}(r+1) \end{bmatrix} = (-1)^{m+r+1},$$

whence

$$\begin{bmatrix} m \\ 1^r m-r \end{bmatrix} = (-1)^{m+1} \begin{bmatrix} m \\ 1^{m-r-1}(r+1) \end{bmatrix},$$

which proves the proposition.

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in it. It is also evident that (44) and its development are invariant if all the indices of both numerator and denominator be increased by the same positive integer $\lambda$.

9. If $n_p$ denote the number of terms in the symmetric function $(p)$, then $\sum n_p \left\| {p \atop \kappa} \right\|$ is the whole number of terms in the quotient $|\kappa_1 \kappa_2 \cdots \kappa_m| / |01 \cdots m-1|$. But this number is equal to the difference product of the $\kappa$'s divided by the difference product of $0, 1, 2, \cdots, m-1$ * hence we have the formula

$$(45) \qquad \sum_p \frac{n!}{x_0!\, x_1!\, x_2! \cdots x_m!} \left\| {p \atop \kappa} \right\| = \frac{(\kappa_m - \kappa_1)(\kappa_m - \kappa_2) \cdots (\kappa_2 - \kappa_1)}{1!\, 2! \cdots (m-1)!},$$

as a farther relation and check for the coefficients of the quotient table.

## § 5. *The Quotient Table as a Table for Elementary Products (as a Product Table).*

A two-fold application of the table gives the elementary products in terms of monomial symmetric functions. Since $|\kappa_1 \kappa_2 \cdots \kappa_m| / |01 \cdots m-1| = \{\kappa\}_t$, the table gives the values of the determinants $\{\kappa\}_t$, in terms of monomial symmetric functions (formula (17)). Also since (I, (41)),

$$(46) \qquad \{\kappa\}_a = (-1)^w a_0^w \{\bar{\kappa}\}_t,$$

it gives the values of the determinants $\{\kappa\}_a$ in terms of monomial symmetric functions.

Thus

$$(47) \qquad \begin{aligned} [6]_a &= [1^6]_t = (1^6), \\ [15]_a &= [1^4 2]_t = 5(1^6) + (1^4 2), \\ [24]_a &= [1^2 2^2]_t = 9(1^6) + 3(1^4 2) + (1^2 2^2), \end{aligned}$$

with $a_0 = 1$.

Moreover, looked upon as a product table in Dr. Taylor's sense and read horizontally, it gives, by the application of the theorem of corresponding matrices, the elementary products directly in terms of the $\{\kappa\}_a$, (19), or, after multiplying the coefficients by $(-1)^w$ and using conjugate headings, in terms of the $\{\kappa\}_t$. In combination with the product table this gives the elementary products in terms of elementary products and thus leads to the identity (21), or it gives the elementary products developed in terms of the $t$'s:

$$(48) \qquad a_p \equiv \sum_\kappa \left\| {p \atop \kappa} \right\| \cdot \left( \sum_q \left| {\bar{\kappa} \atop q} \right| t_q \right).$$

* O. H. Mitchell, *Note on Determinants of Powers*, American Journal of Mathematics, vol. 4 (1881), p. 341.

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By the use of conjugate alternants as headings, it gives directly the product of $|01\cdots m-1|$ and the elementary symmetric functions.

Examples:

$$|012345|(1^4)(1^2) = |123456| + |023457| + |013467|. \tag{49}$$

$$\begin{aligned}(1^4)(1^2) &= [24]_a + [15]_a + [6]_a \\ &= [1^2 2^2]_t + [1^4 2]_t + [1^6]_t \qquad (50)\\ &= (1^2 2^2) + 4(1^4 2) + 15(1^6) \quad [\text{by } (47)].\end{aligned}$$

## § 6. *Tables.*

Quotient tables from weight one to weight seven inclusive are given below With each table is also given the product table. The latter consists of the diagonal of ones and the triangle of coefficients above this diagonal, while the quotient table consists of the same diagonal and triangle of coefficients below it. The quotient table is read as explained in § 4, 1, the product table as explained in I.

WEIGHT 1.

| | 1 (1) |
|---|---|
| (1) | **1** |

WEIGHT 2.

| | 12 ($1^2$) | 03 (2) |
|---|---|---|
| (2) | −1 | **1** |
| ($1^2$) | **1** | 1 |

WEIGHT 3.

| | 123 ($1^3$) | 024 (21) | 015 (3) |
|---|---|---|---|
| (3) | 1 | −1 | **1** |
| (21) | −2 | **1** | 1 |
| ($1^3$) | **1** | 2 | 1 |

WEIGHT 4.

| | 1234 ($1^4$) | 0235 ($21^2$) | 0145 ($2^2$) | 0136 (31) | 0127 (4) |
|---|---|---|---|---|---|
| (4) | −1 | 1 | 0 | −1 | **1** |
| (31) | 2 | −1 | −1 | **1** | 1 |
| ($2^2$) | 1 | −1 | **1** | 1 | 1 |
| ($21^2$) | −3 | **1** | 1 | 2 | 1 |
| ($1^4$) | **1** | 3 | 2 | 3 | 1 |

WEIGHT 5.

| | 12345 ($1^5$) | 02346 ($21^3$) | 01356 ($2^21$) | 01347 ($31^2$) | 01257 (32) | 01248 (41) | 01239 (5) |
|---|---|---|---|---|---|---|---|
| (5) | 1 | −1 | 0 | 1 | 0 | −1 | **1** |
| (41) | −2 | 1 | 1 | −1 | −1 | **1** | 1 |
| (32) | −2 | 2 | −1 | −1 | **1** | 1 | 1 |
| ($31^2$) | 3 | −1 | −1 | **1** | 1 | 2 | 1 |
| ($2^21$) | 3 | −2 | **1** | 1 | 2 | 2 | 1 |
| ($21^3$) | −4 | **1** | 2 | 3 | 3 | 3 | 1 |
| ($1^5$) | **1** | 4 | 5 | 6 | 5 | 4 | 1 |

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**Weight 6.**

| | 123456 $(1^6)$ | 023457 $(21^4)$ | 013467 $(2^21^2)$ | 012567 $(2^3)$ | 013458 $(31^3)$ | 012468 $(321)$ | 012459 $(41^2)$ | 012378 $(3^2)$ | 012369 $(42)$ | 01235 10 $(51)$ | 01234 11 $(6)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| (6) | −1 | 1 | 0 | 0 | −1 | 0 | 1 | 0 | 0 | −1 | 1 |
| (51) | 2 | −1 | −1 | 0 | 1 | 1 | −1 | 0 | −1 | 1 | 1 |
| (42) | 2 | −2 | 1 | −1 | 1 | 0 | −1 | −1 | 1 | 1 | 1 |
| $(3^2)$ | 1 | −1 | 0 | 1 | 1 | −1 | 0 | 1 | 1 | 1 | 1 |
| $(41^2)$ | −3 | 1 | 1 | 1 | −1 | −1 | 1 | 0 | 1 | 2 | 1 |
| $(321)$ | −6 | 4 | 0 | −2 | −2 | 1 | 1 | 1 | 2 | 2 | 1 |
| $(31^3)$ | 4 | −1 | −1 | 0 | 1 | 2 | 3 | 1 | 3 | 3 | 1 |
| $(2^3)$ | −1 | 1 | −1 | 1 | 0 | 2 | 1 | 1 | 3 | 2 | 1 |
| $(2^21^2)$ | 6 | −3 | 1 | 1 | 1 | 4 | 3 | 2 | 4 | 3 | 1 |
| $(21^4)$ | −5 | 1 | 3 | 2 | 4 | 8 | 6 | 3 | 6 | 4 | 1 |
| $(1^6)$ | 1 | 5 | 9 | 5 | 10 | 16 | 10 | 5 | 9 | 5 | 1 |

**Weight 7.**

| | 1234567 $(1^7)$ | 0234568 $(21^5)$ | 0134578 $(2^21^3)$ | 0124678 $(2^31)$ | 0134569 $(31^4)$ | 0124579 $(321^2)$ | 0123679 $(32^2)$ | 012456 10 $(41^3)$ | 0123589 $(3^21)$ | 012357 10 $(421)$ | 012356 11 $(51^2)$ | 012348 10 $(43)$ | 012347 11 $(52)$ | 012346 12 $(61)$ | 012345 13 $(7)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (7) | 1 | −1 | 0 | 0 | 1 | 0 | 0 | −1 | 0 | 0 | 1 | 0 | 0 | −1 | 1 |
| (61) | −2 | 1 | 1 | 0 | −1 | −1 | 0 | 1 | 0 | 1 | −1 | 0 | −1 | 1 | 1 |
| (52) | −2 | 2 | −1 | 1 | −1 | 0 | −1 | 1 | 1 | 0 | −1 | −1 | 1 | 1 | 1 |
| (43) | −2 | 2 | 0 | −1 | −2 | 1 | 1 | 1 | −1 | −1 | 0 | 1 | 1 | 1 | 1 |
| $(51^2)$ | 3 | −1 | −1 | −1 | 1 | 1 | 1 | −1 | 0 | −1 | 1 | 0 | 1 | 2 | 1 |
| $(421)$ | 6 | −4 | 0 | 0 | 2 | 1 | −1 | −2 | −1 | 1 | 1 | 1 | 2 | 2 | 1 |
| $(3^21)$ | 3 | −2 | −1 | 2 | 2 | −1 | −1 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 1 |
| $(41^3)$ | −4 | 1 | 1 | 1 | −1 | −1 | 0 | 1 | 0 | 2 | 3 | 1 | 3 | 3 | 1 |
| $(32^2)$ | 3 | −3 | 2 | −1 | 1 | −1 | 1 | 0 | 1 | 2 | 1 | 2 | 3 | 2 | 1 |
| $(321^2)$ | −12 | 6 | 1 | −2 | −3 | 1 | 1 | 1 | 2 | 4 | 3 | 3 | 4 | 3 | 1 |
| $(31^4)$ | 5 | −1 | −1 | 0 | 1 | 3 | 2 | 4 | 3 | 8 | 6 | 4 | 6 | 4 | 1 |
| $(2^31)$ | −4 | 3 | −2 | 1 | 0 | 2 | 3 | 1 | 3 | 6 | 3 | 4 | 5 | 3 | 1 |
| $(2^21^3)$ | 10 | −4 | 1 | 2 | 1 | 6 | 5 | 4 | 6 | 11 | 6 | 6 | 7 | 4 | 1 |
| $(21^5)$ | −6 | 1 | 4 | 5 | 5 | 15 | 10 | 10 | 11 | 20 | 10 | 9 | 10 | 5 | 1 |
| $(1^7)$ | 1 | 6 | 14 | 14 | 15 | 35 | 21 | 20 | 21 | 35 | 15 | 14 | 14 | 6 | 1 |

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# GENERAL THEORY OF CURVES ON RULED SURFACES*

BY

E. J. WILCZYNSKI†

## § 1. *Relation between the differential equations of the surface and of the curves situated upon it.*

Let a ruled surface be given by means of the system of differential equations

$$(1)\qquad \begin{aligned} y'' + p_{11}y' + p_{12}z' + q_{11}y + q_{12}z &= 0, \\ z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z &= 0, \end{aligned}$$

so that the curves $C_y$ and $C_z$ will be two curves upon it, the lines joining corresponding points of these two curves being generators of the surface. We shall eliminate once $z$ and once $y$, so as to obtain the linear differential equations of the fourth order which each of these functions must satisfy.

We have from (1), by differentiation,

$$(2)\qquad \begin{aligned} y^{(3)} &= r_{11}y' + r_{12}z' + s_{11}y + s_{12}z, \\ z^{(3)} &= r_{21}y' + r_{22}z' + s_{21}y + s_{22}z, \end{aligned}$$

where

$$(3)\qquad \begin{aligned} r_{11} &= p_{11}^2 + p_{12}p_{21} - p_{11}' - q_{11}, & s_{11} &= p_{11}q_{11} + p_{12}q_{21} - q_{11}', \\ r_{12} &= p_{12}(p_{11} + p_{22}) - p_{12}' - q_{12}, & s_{12} &= p_{11}q_{12} + p_{12}q_{22} - q_{12}', \\ r_{21} &= p_{21}(p_{11} + p_{22}) - p_{21}' - q_{21}, & s_{21} &= p_{21}q_{11} + p_{22}q_{21} - q_{21}', \\ r_{22} &= p_{22}^2 + p_{12}p_{21} - p_{22}' - q_{22}, & s_{22} &= p_{21}q_{12} + p_{22}q_{22} - q_{22}'. \end{aligned}$$

We find, by another differentiation,

$$(4)\qquad \begin{aligned} y^{(4)} &= l_{11}y' + l_{12}z' + m_{11}y + m_{12}z, \\ z^{(4)} &= l_{21}y' + l_{22}z' + m_{21}y + m_{22}z, \end{aligned}$$

where

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$$
(5)\qquad
\begin{aligned}
l_{11} &= -p_{11}r_{11} - p_{21}r_{12} + r'_{11} + s_{11}, & m_{11} &= -r_{11}q_{11} - r_{12}q_{21} + s'_{11},\\
l_{12} &= -p_{12}r_{11} - p_{22}r_{12} + r'_{12} + s_{12}, & m_{12} &= -r_{11}q_{12} - r_{12}q_{22} + s'_{12},\\
l_{21} &= -p_{11}r_{21} - p_{21}r_{22} + r'_{21} + s_{21}, & m_{21} &= -r_{21}q_{11} - r_{22}q_{21} + s'_{21},\\
l_{22} &= -p_{12}r_{21} - p_{22}r_{22} + r'_{22} + s_{22}, & m_{22} &= -r_{21}q_{12} - r_{22}q_{22} + s'_{22}.
\end{aligned}
$$

If we put further

$$
(6)\qquad \Delta_1 = p_{12}s_{12} - q_{12}r_{12}, \qquad \Delta_2 = p_{21}s_{21} - q_{21}r_{21},
$$

we can find, from the above equations,

$$
(7)\qquad
\begin{aligned}
\Delta_1 z &= p_{12}y^{(3)} + r_{12}y'' + (p_{11}r_{12} - p_{12}r_{11})y' + (q_{11}r_{12} - p_{12}s_{11})y,\\
-\Delta_1 z' &= q_{12}y^{(3)} + s_{12}y'' + (p_{11}s_{12} - q_{12}r_{11})y' + (q_{11}s_{12} - q_{12}s_{11})y,
\end{aligned}
$$

and similarly,

$$
(8)\qquad
\begin{aligned}
\Delta_2 y &= p_{21}z^{(3)} + r_{21}z'' + (p_{22}r_{21} - p_{21}r_{22})z' + (q_{22}r_{21} - p_{21}s_{22})z,\\
-\Delta_2 y' &= q_{21}z^{(3)} + s_{21}z'' + (p_{22}s_{21} - q_{21}r_{22})z' + (q_{22}s_{21} - q_{21}s_{22})z.
\end{aligned}
$$

Finally, we obtain the required differential equations for $y$ and $z$, viz:

$$
(9)\qquad
\begin{aligned}
\Delta_1 y^{(4)} = {} & (p_{12}m_{12} - q_{12}l_{12})y^{(3)} + (r_{12}m_{12} - s_{12}l_{12})y''\\
& + [(p_{11}r_{12} - p_{12}r_{11})m_{12} - (p_{11}s_{12} - q_{12}r_{11})l_{12} + \Delta_1 l_{11}]y'\\
& + [(q_{11}r_{12} - p_{12}s_{11})m_{12} - (q_{11}s_{12} - q_{12}s_{11})l_{12} + \Delta_1 m_{11}]y,
\end{aligned}
$$

and

$$
(10)\qquad
\begin{aligned}
\Delta_2 z^{(4)} = {} & (p_{21}m_{21} - q_{21}l_{21})z^{(3)} + (r_{21}m_{21} - s_{21}l_{21})z''\\
& + [(p_{22}r_{21} - p_{21}r_{22})m_{21} - (p_{22}s_{21} - q_{21}r_{22})l_{21} + \Delta_2 l_{22}]z'\\
& + [(q_{22}r_{21} - p_{21}s_{22})m_{21} - (q_{22}s_{21} - q_{21}s_{22})l_{21} + \Delta_2 m_{22}]z.
\end{aligned}
$$

These equations are capable of a vast number of applications. Any question, in fact, in regard to the existence of curves of a specified character on a ruled surface must make use of them.

We notice that the conditions $\Delta_1 = 0$ or $\Delta_2 = 0$ will be necessary and sufficient to make $C_y$ or $C_z$ plane curves; the differential equations (of the third order) of these plane curves are found by putting $\Delta_1 = 0$ or $\Delta_2 = 0$ in (7) or (8) respectively. We will merely indicate a few other applications of these formulas. Let us write (9), more briefly,

$$
[(9')]\qquad y^{(4)} + 4p_1y^{(3)} + 6p_2y'' + 4p_3y' + p_4y = 0.
$$

It is easy to write down the conditions that the integral curve of (9′) shall belong to a linear complex, or that it shall be a twisted cubic. In one case its invariant of weight 3, and in the other both of its invariants, must vanish. But

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these conditions, which we now find expressed in terms of the coefficients of (1), become conditions for a particular kind of ruled surface, which contains such curves. One can impose other conditions, for example, that these curves shall be flecnode curves or asymptotic curves on the surface, and then proceed to study the particular class of surface characterized.

It is not our intention to follow up any of these *special* problems, interesting as they are. We shall, however, apply our equations for the purpose of answering some questions of a fundamental nature in the *general* theory of ruled surfaces.

§ 2. *On ruled surfaces, one of the branches of whose flecnode curve is given.*

The flecnode curve is so important in the general theory of ruled surfaces, that it seems essential to investigate to what extent it may be arbitrarily assigned.

If one of the sheets of the flecnode surface, $F'$, of $S$ is given, there remain only two possibilities for $S$, namely, one or the other of the two sheets of the flecnode surface of $F'$. But let us suppose that we merely know that a certain curve $C$ is one of the branches of the flecnode curve on $S$. Then there are two questions to answer. Can this curve be chosen arbitrarily? And how far does it determine the surface $S$?

Let the curve $C$ be given by means of its differential equation

$$\frac{d^4\bar{y}}{d\bar{x}^4} + 4\bar{p}_1\frac{d^3\bar{y}}{d\bar{x}^3} + 6\bar{p}_2\frac{d^2\bar{y}}{d\bar{x}^2} + 4\bar{p}_3\frac{d\bar{y}}{d\bar{x}} + \bar{p}_4\bar{y} = 0, \tag{11}$$

where $\bar{p}_1, \cdots, \bar{p}_4$ are given functions of $\bar{x}$. In the system of differential equations (1) defining our surface $S$, we must regard the coefficients $p_{ik}$ and $q_{ik}$ as unknown functions. We may, however, assume without exception that $u_{12} = 0$, so that $C_y$ is one of the branches of the flecnode curve on $S$, that $p_{21} = 0$, so that $C_x$ is an asymptotic curve on $S$, and that $p_{11} = p_{22} = 0$. Under these assumptions we form the differential equation (9) of the curve $C_y$. Since $C_y$ is to be identical with $C$ it must be possible to convert equation (9) into (11) by a transformation of the form

$$y = \phi(x)\bar{y}, \qquad \bar{x} = f(x). \tag{12}$$

The functions $\phi$ and $f$ are not independent however. For, while the equations $u_{12} = 0$ and $p_{21} = 0$ are not disturbed by any transformation of this form, the conditions $p_{11} = p_{22} = 0$ are. In fact a transformation of the form (12) converts (1) into another system of the same form whose corresponding coefficients $\bar{p}_{11}$ and $\bar{p}_{22}$ will be

$$\bar{p}_{11} = \frac{p_{11} + 2\dfrac{\phi'}{\phi} + \dfrac{f''}{f'}}{f'}, \qquad \bar{p}_{22} = \frac{p_{22} + 2\dfrac{\phi'}{\phi} + \dfrac{f''}{f'}}{f'}.$$

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In order, therefore, that after this transformation $\bar{p}_{11}$ and $\bar{p}_{22}$ may again vanish, we must have

$$\phi = \frac{C}{\sqrt{f''}}, \tag{13}$$

where $C$ is an arbitrary constant, which may be put equal to unity.

If then we apply the transformation (12) to (11), we shall get an equation

$$\frac{d^4y}{dx^4} + 4p_1\frac{d^3y}{dx^3} + 6p_2\frac{d^2y}{dx^2} + 4p_3\frac{dy}{dx} + p_4y = 0,$$

which we must identify with (9). Equating coefficients gives us a system of four equations with five unknown functions of $x$, viz.: $f$, $p_{12}$, $q_{11}$, $q_{21}$, $q_{22}$.

We find, therefore, the following theorem: *An arbitrary space curve being given, it can be considered as one branch of the flecnode curve of an infinity of ruled surfaces, into whose general expression there enters an arbitrary function.* One may, therefore, impose another condition and still obtain an infinity of ruled surfaces.

The most general curve $C_z$ which is capable of being the second branch of the flecnode curve on a ruled surface for which $C_y$ is the first branch, involves, therefore, in its expression one arbitrary function. It cannot, therefore, be an arbitrary curve, as that would involve three arbitrary functions.

Therefore, *two curves taken at random cannot be connected point to point in such a way as to constitute the complete flecnode curve upon the ruled surface thus generated.*

We can also prove our theorem by purely synthetic considerations. Let us take points $P_1, P_2, P_3, P_4, \cdots$ on an arbitrary curve, corresponding for example to equal increments $\Delta x$ of the parameter. Through $P_1, P_2, P_3$ draw three arbitrary lines $g_1, g_2, g_3$. We can draw a line $f_1$ through $P_1$ intersecting $g_2$ and $g_3$, say in $Q_2$ and $Q_3$. Take an arbitrary point $Q_4$ on $f_1$, and join it to $P_4$ by a line $g_4$. Then $f_1$ intersects $g_1, g_2, g_3, g_4$. Through $P_2$ we draw a line $f_2$ intersecting $g_3$ and $g_4$ in points $Q_3'$, $Q_4'$ and of course $g_2$ in $Q_2' = P_2$. Take an arbitrary point $Q_5'$ on $f_2$ and join it to $P_5$ by a line $g_5$. Continue this process. Clearly, we shall get two assemblages of lines $g_1, g_2, \cdots$ and $f_1, f_2, \cdots$, which when $P_1, P_2, \cdots$ are taken closer and closer together, approach as a limit two ruled surfaces having the given curve as flecnode curve, and which are flecnode surfaces of each other. The first three lines $g_1, g_2, g_3$ are arbitrary, and thus give rise to six constants of integration. Further, the double ratios $(P_1, Q_2, Q_3, Q_4)$, $(Q_2', Q_3', Q_4', Q_5')$, etc., may be chosen arbitrarily, which brings into evidence the arbitrary function involved in the construction of these surfaces.

The construction which has just been described becomes indeterminate if the given curve $C$ is a straight line. For then $Q_4$ coincides with $P_4$, etc. In fact,

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the most general ruled surface with a given straight line directrix depends on two arbitrary functions.

If the given curve $C$ is to be at the same time the second branch of the flecnode curve, i. e., if both of the branches of the flecnode curve of $S$ coincide with $C$, $g_4$ must be tangent to the hyperboloid determined by $g_1, g_2, g_3$; $g_5$ must be tangent to the hyperboloid determined by $g_2, g_3, g_4$; etc. This condition, therefore, clearly fixes the double ratios $(Q_1 Q_2 Q_3 Q_4)$, etc., i. e., the arbitrary function. Therefore this problem has in general $\infty^6$ solutions.

Let us assume that $C_y$ is not a straight line. Let us call the developable surface formed by the tangents of $C_y$ its *primary* developable. There exists another important developable surface containing $C_y$, which we shall speak of as its *secondary* developable, as indicated in the following theorem.

1. *If at every point of the flecnode curve of $S$ there be drawn the generator of the surface, the flecnode tangent, the tangent of the flecnode curve, and finally the line which is the harmonic conjugate of the latter with respect to the other two, the locus of these last lines is a developable surface, the secondary developable of the flecnode curve.*

2. *We can find a single infinity of ruled surfaces, each having one branch of its flecnode curve in common with that of $S$. This family of $\infty^1$ surfaces can be described as an involution, of which any surface of the family and its flecnode surface form a pair. The primary and secondary developables of the branch of the flecnode surface considered, are the double surfaces of this involution. In fact, the generators of these surfaces, at every point of their common flecnode curve, form an involution in the usual sense.*

We proceed to prove these theorems. Since $C_y$ is a branch of the flecnode curve, we may assume $u_{12} = p_{11} = p_{22} = 0$. System (1) assumes the form

$$(14) \qquad y'' + p_{12}z' + q_{11}y + \tfrac{1}{2}p'_{12}z = 0, \qquad z'' + p_{21}y' + q_{21}y + q_{22}z = 0.$$

The flecnode tangent at $P_y$ is the line joining $P_y$ to $P_\rho$, where

$$\rho = 2y' + p_{12}z,$$

while the tangent of the flecnode curve joins $P_y$ to $P_{y'}$. In the plane pencil formed by these lines, the harmonic conjugate of $P_y P_{y'}$ with respect to $P_y P_z$ and $P_y P_\rho$ will be the line $P_y P_\tau$, where

$$\tau = y' + p_{12}z.$$

But from the first equation of (14) we find

$$(15) \qquad \tau' + q_{11}y - \tfrac{1}{2}\frac{p'_{12}}{p_{12}}(\tau - y') = 0,$$

i. e., $P_y P_\tau$ generates a *developable* surface as asserted in the first theorem.

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Put

$$(16)\qquad \begin{aligned} e &= \tau + ky' = (1+k)y' + p_{12}z, \\ f &= \tau - ky' = (1-k)y' + p_{12}z, \end{aligned}$$

where $k$ is a constant. Clearly the lines $P_yP_e$ and $P_yP_f$ form a pair of the involution whose double lines are $P_yP_{y'}$ and $P_yP_\tau$.

One finds that $y$ and $e$ satisfy the following system of differential equations:

$$(17)\qquad \begin{aligned} y'' + P_{11}y' + P_{12}e' + Q_{11}y + Q_{12}e &= 0, \\ e'' + P_{21}y' + P_{22}e' + Q_{21}y + Q_{22}e &= 0, \end{aligned}$$

where

$$(18)\qquad \begin{aligned} P_{11} &= -\frac{1+k}{2k}\frac{p'_{12}}{p_{12}}, \quad P_{12} = -\frac{1}{k}, \quad Q_{11} = -\frac{q_{11}}{k}, \quad Q_{12} = \frac{1}{2k}\frac{p'_{12}}{p_{12}}, \\ P_{21} &= (1+k)q_{11} + k(1+k)q_{22} - kp_{12}p_{21} + \frac{(1-3k)(1-k^2)}{4k}\left(\frac{p'_{12}}{p_{12}}\right)^2 + \frac{1-k^2}{2}\frac{p''_{12}}{p_{12}}, \\ P_{22} &= \frac{1-3k}{2k}\frac{p'_{12}}{p_{12}}, \\ Q_{21} &= (1+k)q'_{11} - kp_{12}q_{21} + \frac{(1-3k)(1+k)}{2k}\frac{p'_{12}}{p_{12}}q_{11}, \\ Q_{22} &= -\left[kq_{22} + \frac{(1-3k)(1-k)}{4k}\left(\frac{p'_{12}}{p_{12}}\right)^2 + \frac{1-k}{2}\frac{p''_{12}}{p_{12}}\right]. \end{aligned}$$

We find

$$U_{12} = 2P'_{12} - 4Q_{12} + P_{12}(P_{11} + P_{22}) = 0,$$

i. e., the curve $C_y$ is flecnode curve on the ruled surface $S_k$ generated by $P_yP_e$. The flecnode surface of $S_k$ is obtained by joining $P_y$ to the point

$$2y' + P_{11}y + P_{12}e = -\frac{1}{k}f - \frac{1+k}{2k}\frac{p'_{12}}{p_{12}}y,$$

a point on the line $P_yP_f$. We see therefore that the ruled surfaces $S_k$ and and $S_{-k}$ are flecnode surfaces of each other. We have now proved our second theorem, and we may speak of an involution of ruled surfaces having one branch of their flecnode curve in common. The double surfaces of the involution are developables, while the members of each pair of the involution are flecnode surfaces of each other.

We have seen that $P_yP_\tau$ generates a developable. If

$$g = \alpha y + \beta\tau$$

represents its edge of regression, it must be possible to represent $g'$ in the form

$$g' = \gamma y + \delta\tau,$$

since the line $P_yP_\tau$ must then be tangent to the curve $C_g$.

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We find, by differentiation, making use of (15),

$$g' = \alpha y' + \beta\left(\tfrac{1}{2}\frac{p'_{12}}{p_{12}}\tau - q_{11}y - \tfrac{1}{2}\frac{p'_{12}}{p_{12}}y'\right) + \alpha' y + \beta'\tau,$$

so that $g'$ will be of the required form if, and only if,

$$\alpha : \beta = p'_{12} : 2p_{12}.$$

Therefore

$$g = p'_{12}y + 2p_{12}\tau.$$

If we express $\tau$ in terms of $y$, $z$ and $\rho$, we shall find

$$g = p_{12}\rho + p'_{12}y + p^2_{12}z \tag{19a}$$

*as the expression for the edge of regression of the secondary developable of the branch $C_y$ of the flecnode curve.* Similarly, if $\theta_4 \neq 0$,

$$h = p_{21}\sigma + p'_{21}z + p^2_{21}y \tag{19b}$$

will represent the cuspidal edge of the secondary developable of the branch $C_z$ of the flecnode curve, assuming, of course, $u_{21} = 0$.

One easily finds

$$-\tfrac{3}{2}p'_{12}g + p_{12}g' = \lambda y, \qquad -\tfrac{3}{2}p'_{21}h + p_{21}h' = \mu z, \tag{20}$$

where

$$\begin{aligned} \lambda &= p_{12}p''_{12} + \tfrac{1}{2}p^2_{12}u_{11} - p^3_{12}p_{21} - \tfrac{3}{2}(p'_{12})^2, \\ \mu &= p_{21}p''_{21} + \tfrac{1}{2}p^2_{21}u_{22} - p^3_{21}p_{12} - \tfrac{3}{2}(p'_{21})^2. \end{aligned} \tag{21}$$

The system of differential equations of which $g$ and $h$ are the solutions has the coefficients

$$\begin{aligned} &P_{11} = -\frac{1}{\lambda}\left[\lambda' + \tfrac{3}{4}p^2_{12}p'_{12}p_{21}\right], \qquad P_{12} = \frac{p_{21}\lambda}{\mu}, \qquad Q_{12} = -\frac{3p'_{21}\lambda}{2\mu}, \\ &Q_{11} = -\frac{1}{\lambda p_{12}}\left[\lambda\left(2p''_{12} + \tfrac{1}{4}p_{12}u_{11} - \tfrac{1}{2}p^2_{12}p_{21}\right) - \tfrac{3}{2}p'_{12}\left(\lambda' + \tfrac{3}{4}p^2_{12}p'_{12}p_{21}\right)\right], \end{aligned} \tag{22}$$

while $P_{21}$, $P_{22}$, $Q_{22}$, $Q_{21}$ are obtained from these same equations by permuting the indices 1 and 2, and consequently also the letters $\lambda$ and $\mu$.

*We see that we thus obtain, corresponding uniquely to any ruled surface whose flecnode curve intersects every generator in two distinct points, another ruled surface which is generated by the lines joining corresponding points of the edges of regression of the secondary developables of the two branches of the flecnode curve.*

Equations (20) show that one of the secondary developables of $C_y$ and $C_z$ degenerates into a cone if $\lambda$ or $\mu$ vanishes. In that case our new ruled surface also becomes a cone. If both of the secondary developables are cones, this ruled



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surface degenerates into the straight line joining their vertices. If $\lambda = \Delta_1 = 0$, the secondary developable of $C_y$ is a plane pencil.

Equations (22) show that this new ruled surface cannot be developable except if $\lambda$ or $\mu$ is zero, i. e., unless it is a cone. For the possibility $p_{12} = 0$ or $p_{21} = 0$ is to be excluded, since we should then have a ruled surface $S$ with a straight line directrix.

§ 3. *On ruled surfaces one of the branches of whose complex curve is given.*

There exists an infinity of ruled surfaces each of which contains an arbitrarily given curve as one branch of its complex curve. Into the general analytical expression of these surfaces there enters an arbitrary function. The analytical proof of this statement is precisely similar to that of the corresponding theorem of § 2. We shall give at once a geometrical construction for these surfaces.

Let us consider five straight lines $g_1, \cdots, g_5$. Let $f_1', f_1''$ be the two transversals of $g_1, \cdots, g_4$, and $f_2', f_2''$ those of $g_2, \cdots, g_5$. Clearly $g_1, \cdots, g_5$ determine a linear complex, with respect to which $f_1', f_1''$ and $f_2', f_2''$ are two pairs of reciprocal polars. Take a point $P$ on $g_1$. The plane, which corresponds to it in the linear complex, passes through $g_1$ and the line $h$, which passes through $P$ and intersects both $f_2'$ and $f_2''$. If $g_2, \cdots, g_5$ are made to approach each other, we shall have in the limit five consecutive generators of a ruled surface and its osculating linear complex. The plane tangent to this ruled surface at $P$ is the limit of the plane containing $g_1$ and the line through $P$ which intersect $g_2$ and $g_3$, i. e., the asymptotic tangent of the surface at $P$. If $P$ is a point on the complex curve, $h_1$ must be in the plane tangent to the ruled surface at $P$.

Now let an arbitrary curve be given, and let us choose points upon it, $P_1, P_2, P_3, \cdots$, according to any law. Through $P_1, \cdots, P_4$ draw four arbitrary lines $g_1, \cdots, g_4$. Through $P_1$ draw $h_1$, any line which intersects $g_2$. Let $Q$ be this point of intersection. The line $g_5$, through $P_5$, is to be constructed in such a way that the two transversals of $g_2, \cdots, g_5$ shall both meet $h_1$. Now these transversals must be generators of the second set on the hyperboloid determined by $g_2, g_3, g_4$. They must, therefore, be those two generators of the second set, $f_2'$ and $f_2''$, which pass through the two points in which $h_1$ intersects the hyperboloid. One of these points is $Q$. There exists just one line through $P_5$ intersecting both $f_2'$ and $f_2''$. It is the line $g_5$. In the same way, starting with $g_2, \cdots, g_5$, we can construct $g_6$, etc. Finally we pass to the limit. There enters an arbitrary function, fixing the position of the successive lines $h_1, h_2, \cdots$ in the planes in which they must lie.

That a corresponding theorem is true for asymptotic curves, is obvious.

NICE, *February* 25th, 1904.

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# THEORY ON PLANE CURVES IN NON-METRICAL ANALYSIS SITUS*

BY

OSWALD VEBLEN

## § 1. *Introduction.*

JORDAN'S † explicit formulation of the fundamental theorem that a simple closed curve lying wholly in a plane decomposes the plane into an inside and an outside region is justly regarded as a most important step in the direction of a perfectly rigorous mathematics. This may be confidently asserted whether we believe that perfect rigor is attainable or not. His proof, however, is unsatisfactory to many mathematicians. It assumes the theorem without proof in the important special case of a simple polygon ‡ and of the argument from that point on, one must admit at least that all details are not given.

The work of SCHOENFLIES,§ especially in formulating a converse theorem has thrown much light on its relation to the theory of point sets and Analysis Situs in general, and elegant proofs under restrictive hypotheses have been given by AMES ‖ and BLISS.** All these discussions make more or less use of the ideas of analysis, thus implying either an axiom to the effect that a plane is a doubly extended number-manifold or a set of congruence axioms. Either of these hypotheses imposes a restriction upon the formal generality of Analysis Situs as a science independent of the magnitude of the figures treated.

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* Presented to the Society at the St. Louis meeting, September 17, 1904, under the title, *The fundamental theorem of Analysis Situs*. Received for publication August 22, 1904.

† C. JORDAN, *Cours d'Analyse*, Paris, 1893, 2d ed., p. 92.

‡ This case was under discussion at the University of Chicago in 1901–02 in connection with Professor MOORE's seminar on Foundations of Geometry. Mr. N. J. LENNES gave a proof in his master's thesis (1903), *Theorems on the simple polygon and polyhedron*. Another proof appears as theorem 28 in the writer's dissertation (for reference, see footnote below). The present paper owes much to the discussions of the subject that have taken place under the leadership of Professor MOORE.

§ A. SCHOENFLIES, *Ueber einen grundlegenden Satz der Analysis Situs*, Nachrichten der Göttinger Gesellschaft der Wissenschaften, 1902, p. 185; *Beiträge zur Theorie der Punktmengen*, Mathematische Annalen, Vol. 58 (1903), p. 195.

‖ L. D. AMES, *On the theorem of Analysis Situs relating to the division of the plane or of space by a closed curve or surface*, Bulletin of the American Mathematical Society (2), vol. 10 (1904), p. 301.

** G. A. BLISS, *The exterior and interior of a plane curve*, Bulletin of the American Mathematical Society (2), vol. 10 (1904), p. 398.



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In the following pages an attempt is made to discuss the theorem of JORDAN and a number of related questions under considerably more general hypotheses than are employed in any of the works referred to above. This undertaking practically results in a statement in logical terms of a body of information that formerly was used without explicit formulation and recently has been cramped by unnecessary restrictions. In other words we are inquiring how wide may be the application of our intuitive notion of a plane curve.*

The arguments and definitions are based upon axioms I–VIII, XI of the system adopted in the writer's dissertation † to which refer all the citations not otherwise indicated. These axioms are sufficient to determine the intersectional properties of straight lines, the ordinal relations ‡ of points on a straight line, and continuity. We accordingly assume nothing about analytic geometry, the parallel axiom, congruence relations, nor the existence of points outside a plane. For example, the theory is as valid in the non-desarguesian geometries of HILBERT and MOULTON § as in the geometries of EUCLID and LOBATCHEWSKY, and is of course as applicable in pure analysis as in geometry. ‖

The reader who prefers the JORDAN definition of a simple curve (which according to § 4 is equivalent to ours for purposes of analysis) and does not care about the question of non-metrical hypotheses, may conveniently begin with § 5. A relatively simple proof of the theorem of JORDAN about the decomposition of the plane which applies to any simple closed curve having a straight line inter-

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*The general problem of the "mathematics of precision" may be stated in similar terms.

† O. VEBLEN, *A System of Axioms for Geometry*, Transactions of the American Mathematical Society, vol. 5 (1904), pp. 343–384.

‡ The line is open, i. e., between every two points there is a third, and the order $ABC$ excludes $BAC$ and $ACB$. Single elliptic geometry and projective geometry are therefore excluded unless a properly chosen cut is introduced.

§ F. R. MOULTON, *A Simple non-desarguesian Geometry*, these Transactions, vol. 3 (1902), p. 192.

‖ As to the applicability of our results in analysis, it seems desirable to add a remark which, though obvious from the point of view of "foundations of mathematics," may be of service to some readers who are not directly interested in this point of view. Numerical analysis is ordinarily thought of as founded on the concept of the positive integers. In terms of these a proof of existence can be given of a set of elements, or quantities, satisfying the postulates of the system of rational numbers, positive and negative. In terms of the rational numbers, in turn, can be given a proof of the existence of elements satisfying the postulates of the continuous real number system. Finally, the processes of analysis have to do with pairs of real numbers $(x, y)$. The set of all such number-pairs is a set of objects about which (with proper definition of the term "order") our axioms I–VIII, XI are true theorems. From the axioms of analysis, the line of deduction of our theorems is therefore clear and simple. Not only that, but we may add that any theorem or any definition rigorously based on the assumptions of geometry is *ipso facto* a theorem or definition of analysis. Such considerations as these justify the assertion that while much may be lost in elegance and simplicity, nothing is gained in rigor by the banishment of geometrical language and geometrical styles of exact reasoning from pure analysis. (Of course, under sufficiently strong geometrical axioms, these remarks may be reversed and applied to the rôle of analysis in geometry.)

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val is to be obtained by considering theorem 8 applied to a triangular region, theorem 9 applied to a simple polygon, corollary 2 of theorem 9, lemma $B$, theorem 10 and lemma $C$. This "reduced proof" could in turn be slightly modified so as to apply to any curve having at least one non-cuspidal tangent.

## § 2. *Non-metrical definition of limit point.*

For the definition of the terms, triangle, polygon, broken line, triangular region, separate, decompose, the reader is referred to § 4, chapter II. Of the theorems there proved we assume for the present purpose only that a triangle decomposes a plane in which it lies into two regions, an interior and an exterior.

DEFINITION 1. A *triangular region* is the interior of a triangle. A *geometrical limit point* of a set of points, $[X]$,* in a plane is a point $P$ such that every triangular region including $P$ includes a point $X$, distinct from $P$. A *triangular region* including a point is called a *neighborhood* of the point.

The continuity axiom was assumed for only one segment of a straight line and proved by projection for all lines. In like manner by projection it can be proved that for every point, $P$, of any line there exists a numerably infinite set of segments $[\sigma_\nu]$ $(\nu = 1, 2, \cdots)$ such that $\sigma_\nu$ contains $\sigma_{\nu+1}$ and such that $P$ is the only point that lies on every $\sigma_\nu$. It is an easy consequence of this that for every point in a plane there exists a set of triangular regions $[t_\nu]$ with a similar property. We also prove without difficulty the theorem that a limit point of a set of limit points of a set of points, $[X]$, is itself a limit point of $[X]$.

DEFINITION 2. A *region* is a set of points, any two of which are points of at least one broken line composed entirely of points of the set. An *interior point* of a region, $R$, is one that can be surrounded by a triangle containing only points of $R$. Consequently, an interior point of $R$ is a geometrical limit point of no set of points that does not contain points of $R$. A *frontier point* of a region $R$ is a point or geometrical limit point of $R$ not an interior point, i. e., it is a limit point both of $R$ points and of not $R$ points. An *exterior point* of $R$ or a point exterior to $R$ is any point neither an interior nor a frontier point of $R$. The *frontier* or *boundary* of a region is a set of all frontier points. An *open region* contains no frontier points. A *closed region* contains all its frontier points.

One of the most familiar examples of an open region is obtained by letting $[C]$ stand for a closed set of points and $[P]$ for the set of all points that can be joined with a point $P_0$ not of $[C]$ by broken lines not meeting $[C]$; $[P]$ is an open region.—It is to be noted that the points exterior to a region $[P]$, if such exist, need not constitute only a single region.

* The notation $[X]$ denotes a set of elements any one of which is denoted by $X$ alone or with suffixes. If we wish to indicate that the set is *ordered* we use $\{X\}$ instead of $[X]$.

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§ 3. *Definition of simple curve.*

Simple curves, closed and unclosed, are composed of sets of points subject to certain conditions which we arrange in the following groups:

*A.* LINEAR ORDER. Among the points of a set of points $\{P\}$ there exists a relation, $\odot$, which we may read *precedes*, such that:

1. $\{P\}$ *contains at least two points.*

2. *If* $P_1$ *and* $P_2$ *are any two distinct points of* $\{P\}$, *then either* $P_1 \odot P_2$ *or* $P_2 \odot P_1$.

3.* *If* $P_1 \odot P_2$, *then not* $P_2 \odot P_1$.

4. *If* $P_1 \odot P_2$ *and* $P_2 \odot P_3$, *then* $P_1 \odot P_3$.

*B.* ORDINAL CONTINUITY.

1. *If* $P_1$ *and* $P_3$ *are any two points of* $\{P\}$, *such that* $P_1 \odot P_3$, *then there is a point* $P_2$ *of* $\{P\}$ *such that* $P_1 \odot P_2$ *and* $P_2 \odot P_3$.

2. *If every point of* $\{P\}$ *belongs to* $[P_1]$ *or* $[P_2]$, *two infinite subsets of* $\{P\}$ *such that for every* $P_1$ *and* $P_2$, $P_1 \odot P_2$, *then there is a point* $P'$ *such that for every* $P_1$ *and* $P_2$ *distinct from* $P'$, $P_1 \odot P'$ *and* $P' \odot P_2$.

*C.* GEOMETRICAL CONTINUITY.

1. *Let* $P_0$ *be any point of* $\{P\}$ *for which there is an infinity of points* $P'$ *such that* $P' \odot P_0$. *Denote the set of all such points by* $[P']$; *then for every triangular region,* $t$, *including* $P_0$, *there is a point of* $[P']$, $P'_t$ *such that* $t$ *includes all points of* $[P']$ *for which* $P'_t \odot P'$.

2. *Let* $P_0$ *be any point of* $\{P\}$ *for which there is an infinity of points* $P''$ *such that* $P_0 \odot P''$. *Denote the set of all such points by* $[P'']$; *then for every triangular region,* $t$, *including* $P_0$ *there is a point of* $[P'']$, $P''_t$ *such that* $t$ *includes all points of* $[P'']$ *for which* $P'' \odot P''_t$.

DEFINITION 3. By the term *arc* or *arc of curve* is meant a set of points $\{P\}$ satisfying conditions $A$, $B$, $C$ and including two points $P_1$, $P_2$ such that every point $P$, distinct from $P_1$ and $P_2$, satisfies the further conditions that $P_1 \odot P$ and $P \odot P_2$. The arc is said to *join* $P_1$ and $P_2$ which are called its *end-points.*

DEFINITION 4. A *simple closed curve*, $j$, is a set of points, $\{J\}$, consisting of two arcs joining two points $J_1$ and $J_2$ but having in common no points other than $J_1$, $J_2$.

THEOREM 1. *Any two points of* $j$ *may be taken as the points,* $J_1$, $J_2$ *in the above definition.*

The proof of this theorem is here omitted as it involves no difficulty. The existence of sets of points satisfying the conditions of our definitionis proved by the examples of an interval of a straight line, which is an arc, and the boundary of a simple polygon, which is a simple closed curve. We shall use the letter $j$,

* From this it follows that if $P_1 \odot P_2$, then $P_1 \neq P_2$.

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to denote a simple closed curve, in honor of CAMILLE JORDAN. The term *arc* of course does not cover the most general case of an unclosed curve. On the other hand the conditions $A$, $B$, $C$ are too general to define an unclosed curve since they are satisfied, for example, by the boundary of a triangle exclusive of one vertex. We therefore set down the following condition which is evidently satisfied by an arc.

DEFINITION 5. A *simple unclosed curve* is a set of points $\{C\} = \mathfrak{c}$ that satisfies conditions $A$, $B$, $C$ and also the following:

*D. If $C$ is any point of the curve, no point except $C$ is a limit point (in the geometrical sense of definition* 1) *both of the set of all points $C'$ such that $C' \circledcirc C$ and of the set of all points $C''$ such that $C \circledcirc C''$.*

Any simple closed or simple unclosed curve is called a *simple curve.* For a set of points satisfying conditions $A$, $B$ and $C$, it is evident that there hold all the propositions usually proved in the theory of linear point-sets with the exception of those that involve the length of intervals. We may mention particularly the propositions of section 5, chapter II, including the HEINE-BOREL theorem and the definition of ordinal limit point, the properties of point-free intervals in connection with closed sets, and the proposition that a line cannot be separated into two subsets each of which includes all its limit points. We do not stop here to prove these propositions though we make use of the last one in the following theorem.

DEFINITION 6. A relation satisfying conditions $A$, $B$, $C$ is called a *sense.* A sense in which $P_1 \circledcirc P_2$ is said to be *from* $P_1$ *to* $P_2$.

THEOREM 2. *From one point to another upon a simple unclosed curve there is one and but one sense, while upon a simple closed curve there are two and but two senses.*

*Proof.* We have to show that if $\rho$ is any relation satisfying the conditions $A$, $B$, $C$ and $D$ imposed on $\circledcirc$, then if $P_1 \rho P_2$ implies for one pair $P_1 P_2$ of $\{P\}$ that $P_1 \circledcirc P_2$, $P_1 \rho P_2$ implies that $P_1 \circledcirc P_2$ for every pair $P_1 \rho P_2$ of $\{P\}$.

If $P_1$ is any point of $\{P\}$, let $[P']$ be the set of all points such that simultaneously $P_1 \rho P'$ and $P_1 \circledcirc P'$. Every limit point $\bar{P}$ $(\bar{P} \neq P_1)$ of $[P']$ with respect to the sense $\rho$ must by conditions $A$, $B$ and $C$ be such that $P_1 \rho P'$. Moreover $\bar{P}$, by condition $C$, is a geometrical limit point of $[P']$. But in view of condition $D$, $\bar{P}$ being a point of $\{P\}$ and a geometrical limit point of points $P'$ such that $P_1 \circledcirc P'$ must be such that $P_1 \circledcirc \bar{P}$; otherwise $\bar{P}$ $(P \neq P_1)$ would be a geometrical limit point both of points $\bar{P}$ such that $\bar{P} \circledcirc P_1$ and of points $P'$ such that $P_1 \circledcirc P'$.

Therefore the set $[P']$, if existent, contains all its limit points with respect to the sense $\rho$, except the point $P_1$. Similarly the set of all points $P''$, such that simultaneously $P_1 \rho P''$ and $P'' \circledcirc P_1$ must, if existent, contain all its limit points with respect to the sense $\rho$. Therefore, since the set of points $\{P_2\}$

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such that $P_1 < P_2$ cannot consist of two subsets, each closed with the exception of $P_1$, every point $P_2$ must either be such that $P_1 \circledcirc P_2$ or every point $P_2$ must be such that $P_2 \circledcirc P_1$. From this result the conclusion of our theorem follows at once.

DEFINITION 7. If with respect to any sense on a curve, $P_1 \circledcirc P_2$ and $P_2 \circledcirc P_3$, $P_2$ is *between* $P_1$ and $P_3$ in that sense. The set of all points between $P_1$ and $P_3$ in the given sense is called a *segment* $P_1 P_2 P_3$ whose *end-points* are $P_1$, $P_3$. The segment and its end-points together constitute an *arc* or *interval* of the curve. On a simple unclosed curve, if $P_1 \circledcirc P_2 \circledcirc P_3$, $P_2$ is said to *separate* $P_1$ and $P_3$. On any simple curve if $P_1 \circledcirc P_2 \circledcirc P_3 \circledcirc P_4$, $P_1$ and $P_3$ are said to separate and be separated by $P_2$ and $P_4$. If a set $[P_\nu]$ $(\nu = 1, 2, 3, \cdots)$ is such that $P_\nu \circledcirc P_{\nu+1}$, the points $P_\nu$ are said to be in *the order along the curve*, $P_1 P_2 P_3 \cdots P_\nu P_{\nu+1} \cdots$. A point $P_0$ is the *first* of a set $[P]$ if $P_0 \circledcirc P$ for every $P \neq P_0$; $P_1$ is the *last* of the set $[P]$ if $P \circledcirc P_1$ for every $P \neq P_1$.

Either of the relations of "betweenness" or "separation" which are here defined in terms of "precedence" could have been used as fundamental* and a definition of a simple curve equivalent to the above would have resulted. The deduction of the properties of these relations will be omitted.

## § 4. *Remarks on the definition of a simple closed curve.*

While the definitions of the preceding section are stated so as to apply only to plane curves, it is obvious that if one replaces triangles by tetrahedrons or the corresponding figures in space of more dimensions, the conditions $A$, $B$, $C$, etc., give a definition of a simple curve in space of any number of dimensions.

It may be of interest to note that when one passes from the realm of plane geometry, the distinction between metric and non-metric theory loses much of its importance. For if we add to our assumptions (axioms I–VIII, XI) the assumption (axiom IX) that there exists a point outside a plane, then it is possible to define the ideal elements of projective geometry (cf. chapter III) and by choosing among these ideal elements an "absolute" plane† and polar system to establish a projective theory of congruence. We are thus enabled to operate in the most general case by ordinary analytic geometry as if dealing with the whole or a limited region of euclidean space.

As to the relation of the above definition to the current definition‡ in terms of a numerical parameter, it has not yet been determined whether, in the presence of axioms I–VIII, XI alone, the two definitions are or are not equivalent.

* Cf. B. RUSSELL, *The Principles of Mathematics*, Cambridge, 1903, chapters 24, 25. On the definition by postulate of "separation," see G. VAILATI, *Sulle relazioni di posizione tra punti d'una linea chiusa*, Rivista di Matematica, vol. 5 (1895), p. 75; and also ibid., p. 183.

† That there always is an ideal plane depends in particular on axioms III and XI which determine that straight lines shall be open.

‡ See JORDAN, loc. cit. p. 90.

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If however we introduce axiom IX as indicated above or bring in a set of congruence axioms like HILBERT's group IV, then the two definitions can be shown to be equivalent by reference to a theorem of CANTOR.*

Let $a$ be any arc of a simple closed curve $j$. The points of $a$, excluding the end points, evidently constitute what CANTOR calls a *perfect* set, i. e., with respect to the sense $\circledcirc$. Let $\{t_n\}$ denote a set of equilateral triangles concentric and similarly placed such that the lengths of the side of $t_n$ is $1/n$. For every point $J$ of $a$ there is such a set of triangles $\{t_n\}_J$, having $J$ as a common center. Each triangle $t_n$ determines an arc $i_n$ of $j$ which lies wholly within $t_n$ (cf. condition $C$) and includes the central point $J$ of $t_n$. Among the arcs $i_n$, there is by the HEINE-BOREL theorem applied to $a$, a finite subset such that every point of $a$ is interior to one of the arcs $i_n$. The end points of these arcs that lie on $a$, excluding the end points of $a$, we denote by

$$A_n^1, A_n^2, \cdots, A_n^{m_n}.$$

The set of points $\{A_n^k\}$ is evidently numerable, is ordered according to one of the senses of $a$, and moreover is *everywhere dense on* $a$. For if it were not everywhere dense on $a$ there would be some interval $i$ of $a$ which for *every* $n$ lies wholly within *some* $i_n$ and therefore within some $t_n$; whereas two of its points are a certain distance apart greater than $1/n$ for $n$ sufficiently great.

Now by the theorem of CANTOR cited above, any perfect set which possesses a numerable subset everywhere dense can be set in one-to-one reciprocal continuous correspondence, with the real numbers between 0 and 1. Thus we have a continuous one-to-one correspondence of the points of any arc, and hence of any simple closed curve, with a numerical parameter, $t$. If a system of coördinates $(x, y)$ has been introduced, the simple closed curve may be expressed in parameter form by defining $x(t)$ as the abscissa of the point of $j$ that corresponds to $t$ and $y(t)$ as the ordinate of the same point. The continuity of $x(t)$ and $y(t)$ is evident.

Regarding the conditions of definition 4 as a set of postulates for the determination of the notion, simple curve, the proposition just proved is in effect that in the presence of axioms I–VIII, XI, together with IX or a set of congruence axioms, the system of postulates is "categorical."† The conditions are also independent; i. e., each item of the definition is indispensable to the full definition. To prove this we give a list of point-sets each of which satisfies all the conditions except one. Our independence proofs apply to conditions $A$, $B$, $C$, $D$ since the closed simple curve is defined in terms of the unclosed arc.

$A_1$. $\{P\}$ consists of one point.

*G. CANTOR *Zur Begründung der transfiniten Mengenlehre* I, Mathematische Annalen, vol. 46 (1895), p. 510.

†See vol. 5, p. 346, of these Transactions.

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$A_2$. $\{P\}$ consists of a straight segment $P_1P_2$ and a point $P_3$ not on the straight line $P_1P_2$, the relation $\circledcirc$ referring to a fixed sense on the straight line $P_1P_2$.

$A_3$. $\{P\}$ consists of two points $P_1, P_2$ with the conventions $P_1 \circledcirc P_2$, $P_1 \circledcirc P_1$, $P_2 \circledcirc P_2$.

$A_4$. $\{P\}$ consists of seven points $P_1 \cdots P_7$, the relation $\circledcirc$ being defined by the following table

| $\circledcirc$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | | • | | • | | | • |
| 2 | | | • | | • | | • |
| 3 | • | | | | | • | • |
| 4 | | • | • | | | • | |
| 5 | • | | • | • | | | |
| 6 | • | • | | | • | | |
| 7 | | | | • | • | • | |

$B_1$. $\{P\}$ consists of two points $P_1$ and $P_2$ with the convention $P_1 \circledcirc P_2$.

$B_2$. $\{P\}$ consists of all the points of a straight line with one exception, $\circledcirc$ being one of the two senses along the line.

$C_1$. $\{P\}$ consists of all the points of a straight line $P_1P_2$ with the exception of the point $P_1$ and the segment $P_1P_2$, $\circledcirc$ being the sense from $P_1$ to $P_2$.

$C_2$. $\{P\}$ consists of all the points of a straight line $P_1P_2$ with the exception of the point $P_2$ and the segment $P_1P_2$, $\circledcirc$ being the sense from $P_1$ to $P_2$.

$D$. P consists of the points of a broken line $P_1P_2P_3P_4$, where $P_4$ is a point of the segment $P_1P_2$, $\circledcirc$ being the sense $P_1P_2$, and $P_4$ being counted as a point of $P_1P_2$. This case shows the necessity of condition D in theorem 2 since $\circledcirc$ may also be the sense along the broken line $P_1P_4P_3P_2$.

## § 5. *A simple curve as a planar point set.*

DEFINITION 8. A geometrically *closed* set of points is a set that includes all its geometrical limit points.

THEOREM 3. *If* [P] *is any geometrically closed set of points and a any arc that does not have any point in common with* [P], *then* (1) *there exists a finite set of triangles* $\{t_n\}$ *such that every point of a is interior to at least one* $t_n$ *and every point of* [P] *is exterior to every* $t_n$, *and* (2) *the two end points* $A_1A_2$ *of a can be joined by a broken line not meeting* [P].

*Proof.* (1) If $A$ is any point of $a$ there must be a triangle, $t$, including $A$ and not including any point of [P]; otherwise $A$ would be a limit point of [P]. By condition $c$, each of these triangles, $t$, determines an arc, $i$, of $a$ which lies entirely within $t$ and includes the point $A$ to which $t$ belongs. By

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the HEINE-BOREL theorem applied to the arc $a$, there is a finite subset $[i_n]$ of the arcs $i$ such that every point of $a$ belongs to one arc $i_n$. The finite set of triangles, $t_n$, that determined these arcs $i_n$ is the set required by conclusion (1) of the theorem.

(2) The end points of the arcs $i_n$ constitute a finite set of points which we take as ordered by the sense of $a$ from $A_1$ to $A_2$. The broken line* joining these points taken in order is such that each side lies within a triangle $t$ and therefore cannot meet $[P]$.

COROLLARY. If $[P]$ is any geometrically closed set of points and $Q_0$ a point not of $[P]$, then $Q_0$ and the set of points, $Q$, that can be joined to $Q_0$ by arcs not meeting $[P]$ constitute an open region.

The following theorem is a direct consequence of definitions 4 and 5 and its proof as well as that of theorem 5 is omitted.

THEOREM 4. *About any point of a segment of a simple curve there is a triangle which includes no points of the curve not on the segment.*

In the sense of definition 8, a straight line is a geometrically closed set. A straight line, however, lacks a property possessed by any one of its intervals, namely that every infinite subset has a limit point. For this kind of set we introduce the phrase "finitely closed" because any such set can be enclosed by a finite set of triangles. This property, however, is not used and not proved in the present paper.

DEFINITION 9. A *finitely closed* set of points is a geometrically closed set of which every infinite subset possesses a geometrical limit point. A finitely closed set, every point of which is a geometrical limit point, is a *finitely perfect* set. A finitely perfect set of points which cannot consist entirely of two closed subsets is called a *coherent* set of points.†

THEOREM 5. *A closed curve or an arc of curve is a finitely perfect set of points which cannot consist entirely of two subsets, each of which includes all its limit points. In other words a closed curve or an arc of curve is a coherent set of points.*

THEOREM 6. *If every point of a coherent set of points $[A]$ is on a simple curve $c$, closed or unclosed, then $[A]$ is an interval of $c$.*

*Proof.* If $[A]$ were not an arc of $c$ there must in case $c$ is unclosed be one, and in case $c$ is closed, two points, $C_1$, $C_2$, of $C$ not on $a$ which separate the points of $c$ into two sets, $c'$, $c''$ each containing points of $[A]$. Let $[A']$ denote the points common to $[A]$ and $c'$ and $[A'']$ denote the points common to $[A]$ and $c''$. Every geometrical limit point of $[A']$ would be a

* This broken line of course need not be simple. A broken line with multiple points has a sense independent of the definition of sense on a simple curve. See chapter II, § 4.

† This is the "Begriff des Zusammenhangs" of JORDAN and SCHOENFLIES. Cf. SCHOENFLIES, Mathematische Annalen, vol. 58 (1903), p. 208.

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geometrical limit point of $[A]$, therefore a point of $[A]$, and hence a point of $c$. Being a point of $c$ and a geometrical limit point of $c'$, by theorem 4, it would be a point of $c'$ and hence a point of $[A']$. $[A']$ would therefore be a closed set and by parity of reasoning $[A'']$ also would be closed and thus the definition of $[A]$ would be contradicted.

*Corollary.* If every point of an arc, $a$, is on a simple curve, $c$, then $a$ is an interval of $c$.

THEOREM 7. *If $c$ is any simple curve, any triangle, $t$, of the plane includes points not on $c$.*

*Proof.* Let $a$ be a straight line interval lying wholly within $t$. By theorem 6, $a$ either contains points not on $c$, in which case our conclusion holds, or $a$ is an arc of $c$. In the latter case, by theorem 4, a triangle, $t'$, exists about any interior point of $a$ including no points of $c$ not on $a$. Points of the boundary of this triangle within $t$ and not on $a$ are not on $c$.

## § 6. *The approach to and crossing of a boundary.*

DEFINITION 10. Let $P$ be an interior point of a region, $R$, and $B$ a point of the boundary $b$ of $R$. An arc of a curve, $a$, whose end points are $P$ and $B$ *approaches $B$ from $P$ through $R$* if every interval of $a$, one of whose end points is $B$, contains interior points of $R$. The approach is *one-sided* if, besides the above condition, the arc, $a$, contains no points exterior to $R$. The approach is *simple* if all the points of $a$, except $B$, are interior points of $R$.

An arc $a'$ *departs from* a point $B'$ of $b$ *to* a point $Q$ exterior to $R$ if every interval of $a'$ with $B'$ as an end point contains points exterior to $R$. The departure is *one-sided* if, besides the above condition, the arc $a'$ contains no points interior to $R$. The departure is *simple* if all the points of $a'$ except $B'$ are exterior to $R$.

A curve $c$ *crosses the boundary in a point $B$* if, with respect to a fixed sense, $B$ is between two points $C_1$, $C_2$ of $c$, $C_1$ interior and $C_2$ exterior to $R$, in such a way that the arc $C_1B$ approaches $B$ through $R$ and $BC_2$ departs from $B$ to $C_2$.

A curve $c$ *crosses the boundary $b$ in a pair of points $BB'$* if, with respect to a certain sense, one arc $BB'$ of $c$ is composed entirely of boundary points and if there are two points $C_1C_2$ of $c$ such that $C_1$ is interior to $R$ and an arc $C_1B$ of $c$ approaches $B$ from $C_1$ while $C_2$ is exterior to $R$ and an arc $B'C_2$ departs from $B'$ to $C_2$.

The crossing of a boundary is *simple* if both the approach and departure at the point $B$ or point pair $BB'$ are simple.

The crossing of a straight line by a curve is a special case of the definition just given. A curve is said to cross a segment $AB$ if the curve crosses the line $AB$ in a point or a pair of points.

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THEOREM 8. *Any simple curve joining an interior point of a region to an exterior point crosses the boundary in a point or a pair of points.*

*Proof.* Let $I$ be the interior point, $O$ the exterior point, and $a$ any arc of the curve from $I$ to $O$. Let $\{A\}$ be the set of all points, $A$, of the arc $a$ such that every point following $I$ and preceding $A$ is an interior or boundary point of the region. There are such points because of condition $C$ of the definition in § 3. By the ordinal continuity of $a$, the set $\{A\}$ has a first forward bound $B$, i. e., a first point in the sense from $I$ to $O$ that follows every point of $\{A\}$ except possibly $B$ itself.

The arc $BO$ of $a$ departs from $B$ to $O$ as otherwise every arc $BB''$ of $BO$ would contain only interior or boundary points of the region and thus $B$ would not be a bound of $\{A\}$. Two cases can now occur. Either $B$ is approached from $I$ by the arc $IB$ of $a$ in which case our conclusion follows, or there are points $A'$ of $\{A\}$ such that the arcs $A'B$ include only boundary points. In the last case the set of all points, $A'$, must have a first forward bound $B'$ in the sense from $B$ to $I$. The point $B'$ is evidently a boundary point and is approached from $I$ by the arc of $a$, $IB'$. Thus in the second case, the boundary is crossed in the pair of points $B'B$.

THEOREM 9. *If a simple closed curve crosses a side of a polygon (simple or not) in one point or point-pair, it must pass through a vertex or cross the same or another side in another point or point-pair.*

*Proof.* Let the polygon be $P_1P_2\cdots P_n$ and let the curve, $j$, cross it in a point of $P_1P_2$. If there is another crossing on the segment $P_1P_2$ or if $j$ passes through a vertex, $P_1\cdots P_n$, the theorem is verified. These cases disposed of, $P_1P_2P_3$ may either be collinear or non-collinear. In the first case the original crossing may have been on $P_2P_3$ in which case the theorem is verified or it may have been on $P_1P_3$ in which case we pass to the paragraph below. In case $P_1P_2P_3$ are non-collinear there must be a point $J_1$ of $j$ and a point $O_1$ common to $j$ and $P_1P_2$ such that in a certain sense on $j$ the arc $J_1O_1$ of $j$ approaches $O_1$ through the region on one side of $P_1P_2$; likewise there must be a point $J_2$ of $j$ on the opposite side of $P_1P_2$ from $J_1$ and a point $O_2$ common to $P_1J_1$ and $j$ such that in the same sense the arc $O_2J_2$ departs from $O_2$ to $J_2$. Moreover the points $J_1$ and $J_2$ may be so chosen that one and only one of them lies within the triangle $P_1P_2P_3$. Since $j$ crosses $P_1P_2$ only once, $O_1$ and $O_2$ are on the same arc of $j$ with end points $J_1J_2$. The other arc, $a$, of $j$ with end points $J_1J_2$ must, by theorem 8, cross the boundary of the triangle $P_1P_2P_3$ and since it does not pass through a vertex, must either cross $P_2P_3$ verifying the theorem or cross $P_1P_3$. In the latter case, let $O_1'$ be the first point in the sense from $J_1$ to $J_2$ in which $a$ meets $P_1P_3$ and $O_2'$ the last such point. Upon the arcs $J_1O_1'$ and $O_2'J_2$ there must be two points of $a$, $J_1'$ and $J_2'$ on opposite sides of $P_1P_3$ such that in opposite senses along $j$ the arcs $J_1'O_1'$ and $J_2'O_2'$ approach $O_1'$ and $O_2'$

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from opposite sides of $P_1P_3$. In case $P_1P_3P_4$ are non-collinear $J_1'$ and $J_2'$ may be so chosen that one is interior and the other exterior to the triangle $P_1P_3P_4$.

Thus whether $P_1P_3P_4$ are or are not collinear we proceed as with $P_1P_2P_3$, either verifying the theorem or arriving at the case $P_1P_4P_5$. Continuing this process, by a finite number of steps we come to $P_1P_{n-1}P_n$ and verify the theorem if it is not fulfilled at one of the intermediate steps.

COROLLARY 1. If $j_1$ is a simple closed curve having an arc which is a linear interval $J_1J_2$, and if the segment $J_1J_2$ is crossed by a simple closed curve $j_2$ in one point or point pair, then either $J_1J_2$ is crossed in another point or point pair or the non-linear arc $J_1J_2$ of $j_1$ has a point in common with $j_2$.

*Proof.* In case $J_1J_2$ were not crossed more than once and the other arc $J_1J_2$ of $j_1$ did not meet $j_2$, by theorem 3 $J_1$ and $J_2$ could be joined by a broken line not meeting $j_2$ and we should thus have a contradiction with theorem 9.

COROLLARY 2. Any simple closed curve $j_1$ having a linear arc $J_1J_2$ decomposes its plane into at least two regions.

*Proof.* Let $PQ$ be a linear segment crossing $J_1J_2$ in a point $O$. The region composed of all points that can be joined to P by broken lines not meeting $j_1$ is by theorem 9 separated from the region similarly connected with $Q$.

LEMMA *A*. *Any simple closed curve $j$ decomposes the plane in which it lies into at least two regions.*

*Proof.* Let $J_1$ and $J_2$ be two points of $j$ such that the linear segment $J_1J_2$ has no point in common with $j$. Such points $J_1J_2$ exist, for if $a$ is any line joining two points of $j$, it either has an interval free of $j$ points and whose end-points are the required points $J_1J_2$ or its points in common with $j$ constitute a single arc of $j$ (theorem 6, corollary). In the latter case any line $a'$ joining a point of $j$ on $a$ to a point of $j$ not on $a$ evidently has the required points $J_1J_2$.

Let $t$ be a triangle about $J_1$ such that one of its sides meets the linear segment $J_1J_2$ in a point $O$. Let $Q'$ and $Q''$ be two points of this side separated by $O$ and such that the linear interval $Q'Q''$ contains no point of $j$. The existence of these points depends on the theorem that $j$ is a geometrically perfect set.

$J_1$ and $J_2$ decompose $j$ into two segments which with the linear interval $J_1J_2$ constitute two closed curves $j'$ and $j''$. Assign the notation so that the first point, $J_1'$, after $Q''$ in the sense $Q'OQ''$ in which the boundary of $t$ meets $j$ shall be a point of $j'$. It follows that the first point $J_1''$ after $Q'$ in the sense $Q''OQ'$ in which the boundary of $t$ meets $j$ is a point of $j''$. For if it were a point of $j'$, the closed curve composed of the boundary of $t$ from $J_1''$ to $J_1'$ in the sense $Q'OQ''$ and the arc common to $j$ and $j'$ between $J_1'$ and $J_1''$ would cross the linear segment $J_1J_2$ of $j''$ simply in $O$ and would meet $j''$ in no other point. This would contradict corollary 1, theorem 9.

Thus $J_1''$ is a point of $j''$. Let $J_0''$ be the first point after $J_1'$ in the sense $Q'OQ''$ in which the boundary of $t$ meets $j''$. By the continuity of $j$, there

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exists a segment of the boundary of $t$ just preceding $J_0''$ in the sense $Q'OQ''$ and containing no point of $j'$ or $j''$. Let $X$ be any point of this segment. The broken line $b''$ composed of the boundary of $t$ in the sense $Q'OQ''$ from $Q''$ to $X$ does not meet $j''$. Likewise $X$ is joined to $Q'$ by a simple curve $c'$ composed of the linear segment $XJ_0''$, the common part of $j''$ and $j$ from $J_0''$ to $J_1''$* and the part of the boundary of $t$ from $J_1''$ to $Q'$ in the sense $Q'OQ''$. Thus $c'$ cannot meet $j'$ and, applying theorem 3, $c'$ can be replaced by a broken line $b'$ joining $X$ to $Q'$ without meeting $j'$. We are now ready to complete the proof of our lemma by showing that $X$ cannot be joined to $O$ by a broken line not meeting $j$.

In the sense from $X$ to $O$ any such broken line would meet the linear segments $J_1J_2$ and $Q'Q''$ in some first point $O_1$. If $O_1$ were on $J_1J_2$, some point $S$ preceding $O_1$ in the sense from $X$ to $O$ could be joined to a point $B$ of $Q'Q''$ by a segment not meeting $j'$ or $j''$. Call $b$ the resulting broken line from $X$ to $B$. In case $O_1$ were not on $J_1J_2$ it would be on $Q'Q''$ and different from $O$, and $b$ would be the broken line from $X$ to $O_1 = B$.

If $B$ were on the same side of the line $J_1J_2$ with $Q''$ then the polygon composed of $b$ and $b'$ and $BQ'$ would be crossed by $j'$ in $O$ and would meet $j'$ in no other point, contradicting theorem 9. If $B$ were on the opposite side of the line $J_1J_2$ from $Q''$ the polygon composed of $b$ and $b''$ and $BQ''$ would be crossed by $j''$ in $O$ and would meet $j''$ in no other point. $X$ and $O$ are therefore two points that cannot be joined by a broken line not meeting $j$.

## § 7. *Finite accessibility.*

DEFINITION 11. A point $C$ of a curve $c$ is *finitely accessible* from a point $P$ not on $c$ if there is a broken line from $C$ to $P$ not meeting $c$ except in $C$.

LEMMA B. *If $P$ is a point not on a simple closed curve $j$, and $J_1$ and $J_3$ are any two points of $j$ finitely accessible from $P$ or limit points† of the points finitely accessible from $P$, then there exists a pair of points $J_2$ and $J_4$ finitely accessible from $P$ that separate $J_1$ and $J_3$.*

*Proof.* Let $t_1$ be a triangle about $J_1$ not including $J_3$, and $t_3$ a triangle about $J_3$ not including any point of $t_1$. By condition $C$ of the definition of $j$, there is a segment of $j$ including $J_1$ and lying wholly within $t_1$; by theorem 4 there is a triangle $t_1'$ about $J_1$ within $t_1$ and including no point of $j$ not on this segment. Thus every segment of $j$ with end points on $t_1$ which meet $t_1'$ must include $J_1$. Similarly there is within $t_3$ a triangle $t_3'$ such that every segment of $j$ with end points on $t_3$ which meets $t_3'$ must include $J_3$.

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* Of course it may happen that $J_0'' = J_1''$. In this case $c'$ is a broken line.

† On a simple closed curve the notions of ordinal and geometrical limit points are interchangable: therefore we drop the distinction.

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Let $J_1$ be a point finitely accessible from P within $t$ and $J_2$ a point finitely accessible from P within $t_2$. The points $J_1$ and $J_2$ are thus joined by a broken line $b_1$ meeting $j$ only in $J_1$ and $J_2$, which without loss of generality may be supposed simple. On this broken line let P be the first point in the sense from $J_1$ to $J_2$ in which it meets the boundary of $t$. P lies on an interval $i$ of the boundary of $t$ containing no points of $j$ but such that its end points are points of $j$. Let $P_2$ be the last point in the sense from $J_1$ to $J_2$ in which $b_1$ meets the interval $i$. In case $P_2$ is distinct from P replace the portion of the broken line from P to $P_2$ by the portion of $i$ from P to $P_2$, calling the new broken line $b_2$. If $b_2$ crosses the interval $i$, $J_3$ and $J_4$ are the end points of the interval. If $b_2$ does not cross $i$ there must be some point $P_3$ beyond $P_2$ in the sense from $J_1$ to $J_2$ in which $b_2$ meets the boundary of $t$. The point $P_3$ must lie on an interval $i_2$ of the boundary of $t$ analogous to $i$. Proceed with $i_2$ as with $i$. Since $J_1$ is inside $t$ and $J_2$ outside $t$ and since $b$ has but a finite number of sides, we must by repeating the process above come to a first interval $i_k$ in which the boundary of $t$ is crossed by a reduced broken line $b_k$ from $J_1$ to $J_2$ in a point $P_k$ or a point pair $P_k P_{k-1}$. The end points of the point free interval $i_k$ of the boundary of $t$ are now to be shown to be the required $J_3$ and $J_4$.

We prove first that $J_3$ and $J_4$ separate $J_1'$ and $J_2'$. If this were not so, let the simple closed curve formed by $b_k$ and the arc $J_3 J_4'$ of $j$ not including $J_3$ and $J_4$ be denoted by $j_k$. Also let $j_1$ denote the simple closed curve formed by $i_k$ and the arc $J_3 J_4$ of $j$ not including $J_3'$ and $J_4'$. The simple closed curve $j_1$ would cross the arc $i_k$ of $j_1$ in the point $P_k$ or point pair $P_k P_{k-1}$ and would meet $j_1$ in no other point, contrary to corollary 1, theorem 9.

Hence $J_1'$ and $J_2'$ are on different arcs of $j$ with end points $J_3$ and $J_4$. But by the construction of the triangle $t_1'$, $J_1$ must be on the same arc with $J_1'$ and by the construction of $t_2'$, $J_2$ must be on the same arc with $J_2'$. Hence $J_3 J_4$ separate $J_1 J_2$.

THEOREM 10. *The set of points of a simple curve $j$ finitely accessible from a point* P *not on $j$ is everywhere dense on $j$.*

*Proof.* Denote by $[J]$ the set of points of $j$ which are either finitely accessible from P or are limit points of the set of finitely accessible points. The theorem amounts to showing that $[J']$ is identical with $j$. But if any point $J_0$ of $j$ should not belong to $[J']$ it would lie on an arc of $j$ free of points $J'$ and having two points of $[J']$ as end points. This would contradict lemma $B$.

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§ 8. *Decomposition of a plane by a simple closed curve.*

LEMMA *C*. Any simple closed curve of which one arc is a linear interval decomposes its plane into two open regions.

*Proof.* It has been shown (corollary 2, theorem 9) that $j$ decomposes the plane into at least two regions. The regions are open because a supposed frontier point of the set of points [P] that can be joined to a point $P_0$ not on $j$ could, if not itself a point of $j$, be surrounded by a triangle not meeting $j$ and containing points of [P]; it would therefore be an interior point of [P], contrary to hypothesis.

By theorem 10 every point $O$ of the straight arc of $j$ is finitely accessible from any point of the plane. Thus if there were three distinct regions there would be three segments meeting in $O$ and one lying in each of the three regions. But as two of these must lie on the same side of the straight segment of $j$ they could be joined by a straight segment not meeting $j$, contrary to the hypothesis that the three regions are separated from one another by $j$. Hence $j$ decomposes the plane into two and only two open regions.

THEOREM 11. *Every simple closed curve, $j$, decomposes its plane into two open regions.*

*Proof.* By lemma *A* the curve decomposes the plane into at least two regions which by the reasoning of the first paragraph of the proof of lemma *C* are open regions. Let P be any point not on $j$ and let $PJ_1$ and $PJ_2$ be two linear intervals meeting $j$ only in $J_1$ and $J_2$. $J_1$ and $J_2$ exist because $j$ is a perfect set of points. Let $Q$ be any point not on $j$ and not in the same region with P and let $J_3$ be a point on $j$ such that the linear segment $QJ_3$ does not meet $j$ and such that $J_3$ is distinct from $J_1$ and $J_2$. Then $QJ_3$ does not meet $PJ_1$ or $PJ_2$ and $Q$ can by theorem 10 be joined by a broken line not meeting $PJ_1$, $PJ_2$, $QJ_3$, or $j$ except in $J_4$ to a point $J_4$ of $j$ in the order $J_1J_2J_3J_4$. The broken line $J_1PJ_2$, the points between $J_2$ and $J_3$ in the sense $J_1J_2J_3$, the broken line $J_3QJ_4$ and the points between $J_4$ and $J_1$ in the sense $J_1J_2J_3$ constitute a simple closed curve $j'$ of the type which we have proved to decompose the plane into two and only two regions. The points of the segments $J_1J_2$ and $J_3J_4$ in the sense $J_1J_2J_3$, are not points of $j'$ and must lie both in the same region or in opposite regions with respect to $j'$. If they were in the same region a point in the region not containing the segments $J_1J_2$ and $J_3J_4$ could by theorem 10 be joined by broken lines not meeting $j$ to P and $Q$, thus contradicting the hypothesis that P and $Q$ are in different regions.

Having shown that the arcs $J_1J_2$ and $J_3J_4$ (in the fixed sense $J_1J_2J_3$) are in opposite regions with respect to $j'$ we are ready to complete the proof that $j$ does not decompose the plane into more than two regions. A point $R$ in a sup-



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posed third region could be joined because of theorem 10 by a broken line not meeting $j$ except in its end-point to a point $J_5$ of $J_1J_2$ and by a similar broken line to a point $J_6$ of $J_3J_4$. Since $R$ would not be in the same region with P or $Q$ these broken lines would not meet the broken line part of $j'$. Thus we should have two points $J_5$ and $J_6$ in opposite regions with respect to $j'$ joined by a broken line not meeting $j'$ contrary to lemma $C$. Hence $j$ decomposes the plane into not more than two, and therefore into exactly two, open regions.

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# GENERAL PROJECTIVE THEORY OF SPACE CURVES*

BY

E. J. WILCZYNSKI†

The theory of the invariants of a linear differential equation is not new. LAGUERRE, BRIOSCHI, HALPHEN, FORSYTH, BOUTON, FANO and others have written important papers on the subject. But in almost all of these papers the point of view is essentially analytical. It is only in the brilliant contributions of HALPHEN that one finds the idea of founding a geometry of curves upon this theory. But even HALPHEN's papers do not give the ground work for a thorough comprehension of the subject. This is to be found in the geometric interpretation of the semi-covariants which we shall discuss in this paper. With this as a basis, the whole theory becomes clear and transparent, and HALPHEN's results can easily be connected with it.

I have myself shown in recent years, how the general projective theory of ruled surfaces depends upon the theory of the invariants and covariants of a system of two linear differential equations of the second order. The present paper is governed by similar ideas, and to some extent depends upon this other theory. It is the main purpose of this paper to bring up the general projective theory of curves, based upon the theory of invariants, to the same level of perfection as the corresponding theory of ruled surfaces. Some of the theorems which we shall find, have of course been known for a long time. But even for most of these our proofs will be new.

## § 1. *The invariants and covariants.*

We shall confine our attention to the linear differential equation of the fourth order

$$y^{(4)} + 4p_1 y^{(3)} + 6p_2 y'' + 4p_3 y' + p_4 y = 0, \tag{1}$$

where

$$y' = \frac{dy}{dx}, \qquad y'' = \frac{d^2y}{dx^2}, \quad \text{etc.}$$

If we make the transformations

$$y = \lambda(x)\bar{y}, \qquad \xi = \xi(x),$$

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† Of the Carnegie Institution of Washington.



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where $\lambda$ and $\mu$ are arbitrary functions, we obtain another equation of the same form as (1). The functions of the coefficients of (1), which remain invariant under these transformations, are the *invariants*. If they involve also the functions $y, y', y'', y^{(3)}$, they are called *covariants*.

To determine these invariant functions, it is best according to LIE's theory to make use of infinitesimal transformations. But this has been done in a number of papers by other authors and need not therefore be repeated. We shall, however, need the finite transformations for the later parts of this paper, and shall, therefore, write them down.

Consider first the simpler transformation

$$y = \lambda(x)\bar{y},$$

of the dependent variable alone. We shall speak of the corresponding invariant functions as *seminvariants* and *semi-covariants*. Let

$$\bar{y}^{(4)} + 4\pi_1\bar{y}^{(3)} + 6\pi_2\bar{y}'' + 4\pi_3\bar{y}' + \pi_4\bar{y} = 0$$

be the transformed equation. Then

$$(2)\qquad \begin{aligned} \pi_1 &= \frac{\lambda' + p_1\lambda}{\lambda}, \\ \pi_2 &= \frac{\lambda'' + 2p_1\lambda' + p_2\lambda}{\lambda}, \\ \pi_3 &= \frac{\lambda^{(3)} + 3p_1\lambda'' + 3p_2\lambda' + p_3\lambda}{\lambda}, \\ \pi_4 &= \frac{\lambda^{(4)} + 4p_1\lambda^{(3)} + 6p_2\lambda'' + 4p_3\lambda' + p_4\lambda}{\lambda}, \end{aligned}$$

whence one may deduce the absolute seminvariants

$$(3)\qquad \begin{aligned} P_2 &= p_2 - p_1' - p_1^2, \\ P_3 &= p_3 - p_1'' - 3p_1p_2 + 2p_1^3, \\ P_4 &= p_4 - 4p_1p_3 - 3p_2^2 + 12p_1^2p_2 - 6p_1^4 - p_1^{(3)}, \end{aligned}$$

and the relative semi-covariants, besides $y$ which is obviously itself a semi-covariant,

$$(4)\qquad \begin{aligned} z &= y' + p_1y, \\ \rho &= y'' + 2p_1y' + p_2y, \\ \sigma &= y^{(3)} + 3p_1y'' + 3p_2y' + p_3y. \end{aligned}$$

The absolute semi-covariants are $z/y$, $\rho/y$, $\sigma/y$. All other semi-covariants and seminvariants are functions of these and of the derivatives of $P_2$, $P_3$, $P_4$.

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From (4) we deduce the following equations, which we shall use later:

$$
(5)\qquad
\begin{aligned}
y' &= -p_1 y + z,\\
z' &= -P_2 y - p_1 z + \rho,\\
\rho' &= -(P_3 - P_2')y - 2P_2 z - p_1\rho + \sigma,\\
\sigma' &= -(P_4 - P_3')y - 3(P_3 - P_2')z - 3P_2\rho - p_1\sigma,
\end{aligned}
$$

and also

$$
(6)\qquad
\begin{aligned}
y'' &= (2p_1^2 - p_2)y - 2p_1 z + \rho,\\
y^{(3)} &= (-p_3 + 6p_1p_2 - 6p_1^3)y + (-3p_2 + 6p_1^2)z - 3p_1\rho + \sigma,\\
y^{(4)} &= (-p_4 + 8p_1p_3 - 36p_1^2p_2 + 6p_2^2 + 24p_1^4)y\\
&\qquad + (-4p_3 + 24p_1p_2 - 24p_1^3)z + (-6p_2 + 12p_1^2)\rho - 4p_1\sigma.
\end{aligned}
$$

We now proceed to make a transformation of the independent variable $\xi = \xi(x)$. We find, denoting the coefficients of the transformed equation by $\bar{p}_k$,

$$
(7)\qquad
\begin{aligned}
\bar{p}_1 &= \frac{1}{\xi'}(p_1 + \tfrac{3}{2}\eta),\\
\bar{p}_2 &= \frac{1}{(\xi')^2}[p_2 + 2\eta p_1 + \tfrac{1}{6}(4\mu + 9\eta^2)],\\
\bar{p}_3 &= \frac{1}{(\xi')^3}[p_3 + \tfrac{3}{2}\eta p_2 + (\mu + \tfrac{3}{2}\eta^2)p_1 + \tfrac{1}{4}(\mu' + 4\eta\mu + 3\eta^3)],\\
\bar{p}_4 &= \frac{1}{(\xi')^4}p_4,
\end{aligned}
$$

where we have put

$$
(8)\qquad \eta = \frac{\xi''}{\xi'}, \qquad \mu = \eta' - \tfrac{1}{2}\eta^2.
$$

We find further

$$
(9)\qquad
\begin{aligned}
\bar{z} &= \frac{1}{\xi'}(z + \tfrac{3}{2}\eta y),\\
\bar{\rho} &= \frac{1}{(\xi')^2}[\rho + 2\eta z + \tfrac{1}{6}(4\mu + 9\eta^2)y],\\
\bar{\sigma} &= \frac{1}{(\xi')^3}[\sigma + \tfrac{3}{2}\eta\rho + (\mu + \tfrac{3}{2}\eta^2)z + \tfrac{1}{4}(\mu' + 4\eta\mu + 3\eta^3)y],
\end{aligned}
$$

cogredient with (7).

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Making use of these equations, we find

$$
\begin{aligned}
\bar{P}_2 &= \frac{1}{(\xi')^2}[P_2 - \tfrac{5}{6}\mu], \\
(10)\quad \bar{P}_3 &= \frac{1}{(\xi')^3}[P_3 - 3\eta P_2 - \tfrac{5}{4}\mu' + \tfrac{10}{4}\eta\mu], \\
\bar{P}_4 &= \frac{1}{(\xi')^4}[P_4 - 6\eta P_3 - 4\eta' P_2 + 11\eta^2 P_2 + \tfrac{15}{2}\eta\mu' - \tfrac{15}{2}\eta^2\mu \\
&\qquad - \tfrac{3}{2}\mu'' + \tfrac{10}{6}\mu^2],
\end{aligned}
$$

whence

$$
\begin{aligned}
\bar{P}'_2 &= \frac{1}{(\xi')^3}[P'_2 - 2\eta P_2 - \tfrac{5}{6}\mu' + \tfrac{5}{3}\eta\mu], \\
(11)\quad \bar{P}''_2 &= \frac{1}{(\xi')^4}[P''_2 - 5\eta P'_2 - 2\mu P_2 + 5\eta^2 P_2 - \tfrac{5}{6}\mu'' + \tfrac{5}{3}\mu^2 - \tfrac{25}{6}\mu\eta^2 + \tfrac{25}{6}\mu'\eta], \\
\bar{P}'_3 &= \frac{1}{(\xi')^4}[P'_3 - 3\eta(P'_2 + P_3) - 3\mu P_2 + \tfrac{15}{2}\eta^2 P_2 - \tfrac{5}{4}\mu'' \\
&\qquad + \tfrac{25}{4}\eta\mu' - \tfrac{25}{4}\mu\eta^2 + \tfrac{10}{4}\mu^2].
\end{aligned}
$$

We find therefore the following invariants and covariants:

$$
\begin{aligned}
&\theta_3 = P_3 - \tfrac{3}{2}P'_2, \qquad \theta_4 = P_4 - 2P'_3 + \tfrac{6}{5}P''_2 - \tfrac{6}{25}P_2^2, \\
&\theta_{3\cdot 1} = 6\theta_3\theta''_3 - 7\theta'^2_3 - \tfrac{108}{5}P_2\theta_3^2, \\
(12)\qquad &C_2 = 10z^2 - 15y\rho - 12P_2y^2, \\
&C_3 = 10z^3 - 3C_2z - 9(5\sigma + 6P_2z + P_3y)y^2, \\
&C_4 = 2\theta_3 z + \theta'_3 y,
\end{aligned}
$$

where the index indicates the weight. In denoting one invariant of weight 8 by $\theta_{3\cdot 1}$ we follow a notation due to FORSYTH. An invariant may be regarded as a covariant of degree zero. With this understanding, it suffices to say that the effect of the complete transformation

$$\bar{y} = \lambda(x)y, \qquad \xi = \xi(x),$$

upon a covariant of degree $d$ and of weight $w$, is to transform it into $\bar{C}$, where

$$\bar{C} = \frac{\lambda^d}{(\xi')^w}C.$$

LIE's theory shows that all other invariants and covariants may be deduced from these by algebraic and differentiation processes.

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§ 2. *Canonical forms.*

Equations (2) show that if we make the transformation

$$y = e^{-\int p_1 dx}\bar{y},$$

the coefficients of the resulting equation for $\bar{y}$ will be

$$\pi_1 = 0, \qquad \pi_2 = P_2, \qquad \pi_3 = P_3, \qquad \pi_4 = P_4 + 3P_2^2.$$

We shall say that the equation has been put into the *semi-canonical* form.

From (10) we see that if $\xi(x)$ be chosen so that

$$\eta' - \tfrac{1}{2}\eta^2 = \mu = \tfrac{6}{5}P_2, \tag{13}$$

in the resulting equation $\bar{P}_2$ will be zero. Since $P_2$ is a seminvariant, any transformation of the form $\bar{y} = \lambda y$ will not disturb the equation $P_2 = 0$, and we may again choose $\lambda$ so as to make the coefficient of $d^3\bar{y}/d\xi^3$ vanish. It is therefore always possible to reduce the equation to the form

$$\frac{d^4\bar{y}}{d\xi^4} + 4\pi_3\frac{d\bar{y}}{d\xi} + \pi_4\bar{y} = 0,$$

which we shall call the LAGUERRE-FORSYTH *canonical form.* This is equivalent to assuming $p_1 = p_2 = 0$ in the original equation.

If $\theta_3 \neq 0$, we may transform the independent variable so as to make $\bar{\theta}_3 = 1$. In fact we have for an arbitrary transformation

$$\bar{\theta}_3 = \frac{1}{(\bar{\xi}')^3}\theta_3.$$

If, therefore, we put

$$\xi = \int \sqrt[3]{\theta_3}\, dx, \tag{14}$$

$\bar{\theta}_3$ will be equal to unity. We may again by a transformation of the form $\bar{y} = \lambda y$ make $p_1$ vanish. The canonical form which is characterized by the conditions

$$p_1 = 0, \qquad \theta_3 = 1,$$

we may properly denote as the HALPHEN *canonical form.*

In our geometrical discussions only the quantity

$$\eta = \frac{\xi''}{\xi'},$$

not $\xi$ itself will be of any importance. $\lambda$ also is an unimportant factor which has no geometrical significance. Equation (13) shows, therefore, that the reduc- to the LAGUERRE-FORSYTH form can always be accomplished in $\infty^1$ essentially different ways. It is important to remark that (13) is an equation of the RICCATI form, so that the cross-ratio of any four solutions is constant.

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The HALPHEN form on the other hand can be obtained in just one way, if it exists at all, i. e., if $\theta_3 \neq 0$. If $\theta_3$ vanishes, $\theta_4$ may be reduced to unity unless it also is equal to zero. The case when both $\theta_3$ and $\theta_4$ vanish, is especially simple. The LAGUERRE-FORSYTH form reduces to

$$\frac{d^4\bar{y}}{d\xi^4} = 0.$$

If two equations of the form (1) can be transformed into each other by a transformation of the kind here considered, we shall call them equivalent. Clearly, for equivalent equations, the corresponding absolute invariants are equal.

If equation (1) is given, the invariants $\theta_3$, $\theta_4$, $\theta_{3.1}$, etc., are known functions of $x$. Conversely, equations (12) show that if $\theta_3$, $\theta_4$, $\theta_{3.1}$ are given as arbitrary functions of $x$, provided that $\theta_3 \neq 0$, $P_2$, $P_3$ and $P_4$ are determined uniquely. If $\theta_3 = 0$, then $\theta_{3.1} = 0$ also, and we must assign a further condition. The function

$$\theta_{4.1} = 8\theta_4\theta_4'' - 9(\theta_4')^2 - \frac{80}{3}P_2\theta_4^2 \tag{15}$$

is also an invariant. If $\theta_3 = 0$, and $\theta_4$, $\theta_{4.1}$ are given, $P_2$, $P_3$, $P_4$ are determined uniquely. If both $\theta_3$ and $\theta_4$ vanish, all invariants are zero, and the equation may be reduced to the form

$$\frac{d^4\eta}{d\xi^4} = 0.$$

As we may always assume that $p_1 = 0$, we see that *the differential equation* (1) *is essentially determined when its invariants are given as functions of* $x$.

The LAGRANGE adjoint of (1) is

$$\begin{aligned} u^{(4)} - 4p_1u^{(3)} + 6(p_2 - 2p_1')u'' - 4(p_3 - 3p_2' + 3p_1'')u' \\ + (p_4 - 4p_3' + 6p_2'' - 4p_1^{(3)})u = 0. \end{aligned} \tag{16}$$

If $y_1, \cdots, y_4$ constitute a fundamental system of (1), the minors of $x_1, \cdots, x_4$ in the determinant

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \end{vmatrix}$$

multiplied by a common factor, which does not interest us, form a fundamental system of (16).

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If we denote the seminvariants of (16) by $\Pi_2$, $\Pi_3$, $\Pi_4$, we have

$$(17) \qquad \Pi_2 = P_2, \qquad \Pi_3 = -P_3 + 3P_2', \qquad \Pi_4 = P_4 - 4P_3' + 6P_2'',$$

whence follows reciprocally

$$P_2 = \Pi_2, \qquad P_3 = -\Pi_3 + 3\Pi_2', \qquad P_4 = \Pi_4 - 4\Pi_3' + 6\Pi_2''.$$

The invariants of (16) differ from the invariants of (1) only in this that the sign of $\theta_3$ is changed.

## § 3. *Geometrical interpretation.*

If the functions $y_1, \cdots, y_4$ constitute a fundamental system of (1) we may interpret them as the homogeneous coördinates of a point $P_y$ of a curve $C_y$ in ordinary space. The coefficients $p_1, p_2, p_3, p_4$ of (1) are invariants of the general projective group. The transformation $y = \lambda\bar{y}$ does not change the ratios $y_1 : y_2 : y_3 : y_4$, and therefore leaves the curve $C_y$ invariant. The transformation $\xi = \xi(x)$ merely changes the parameter in terms of which the coördinates are expressed. It is clear therefore that any system of equations invariant under these transformations expresses a projective property of the curve $C_y$ in the vicinity of the point $P$.

The LAGRANGE adjoint of (1) may be taken to represent the same curve in tangential coördinates, or else a reciprocal curve in point coördinates.

We may therefore state the results of § 2 as follows: *If the invariants of a curve are given as functions of $x$, the curve is determined except for projective transformations. If the invariants of two curves, except those of weight three, are respectively equal to each other, while the invariants of weight three differ only in sign, the two curves are dualistic to each other. Those curves are self-dual for which $\theta_3 = 0$.*

Moreover these latter curves are the only curves which are self-dual in the restricted sense that a dualistic transformation exists which converts every point of the curve into the tangent plane of that point, and vice versâ, while every tangent is converted into itself.

If we put $y = y_k (k = 1, 2, 3, 4)$ into the expressions for $z$, $\rho$, $\sigma$ we obtain three other points $P_z$, $P_\rho$, $P_\sigma$, which describe curves $C_z$, $C_\rho$, $C_\sigma$ as $x$ varies, curves which are closely connected with $C_y$. $P_z$ is clearly a point on the tangent of $C_y$ constructed at $P_y$; $P_\rho$ is in the plane osculating $C_y$ at $P_y$, while $P_\sigma$ is outside of this plane. These four points are never coplanar except at those exceptional points of $C_y$ whose osculating planes are stationary, i. e., have more than three consecutive points in common with the curve.

In order to study the curve $C_y$ in the vicinity of $P_y$, it will therefore be convenient to introduce the tetrahedron $P_y P_z P_\rho P_\sigma$ as tetrahedron of reference, with the further convention that if any expressions of the form

$$u_k = \alpha_1 y_k + \alpha_2 z_k + \alpha_3 \rho_k + \alpha_4 \sigma_k \qquad (k = 1, 2, 3, 4)$$

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offer themselves the coördinates of the corresponding point $P_u$ shall be

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4).$$

In writing $u_k$ the index $k$ may be suppressed, so that a single expression

$$\alpha_1 y + \alpha_2 z + \alpha_3 \rho + \alpha_4 \sigma$$

represents the point $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ in this special system of coördinates.

As the independent variable $x$ is changed, the tetrahedron of reference is changed in accordance with equations (9). $P_y$ of course remains the same; $P_z$ is changed into $P_{\bar{z}}$, which may obviously be any point on the tangent: etc. Thus while an arbitrary transformation of the parameter $x$ does not affect the curve $C_y$ itself, it does very materially affect the semi-covariant curves $C_z$, $C_\rho$ and $C_\sigma$. It is clear however that two transformations $\xi = \xi(x)$, for which $\eta = \xi''/\xi'$ has the same value, are geometrically equivalent. We may also, without affecting the position of the points $P_z$, $P_\rho$, $P_\sigma$, assume that (1) is written in the semi-canonical form, so that $p_1 = 0$. For, in order to put (1) into the semi-canonical form, we need only multiply $y$ by a certain factors $\lambda$, which will then also appear multiplied into the semi-covariants $z$, $\rho$ and $\sigma$.

Let us then assume $p_1 = 0$. We shall have $z = y'$. If we differentiate (1) and eliminate $y$ between the resulting equation and (1), we shall find

$$\begin{aligned} &(P_4 + 3P_2^2)z^{(4)} - (P_4' + 6P_2P_2')z^{(3)} + 6P_2(P_4 + 3P_2^2)z'' \\ (18)\qquad &+ [(6P_2' + 4P_3)(P_4 + 3P_2^2) - 6P_2(P_4' + 6P_2P_2')]z' \\ &+ [(4P_3' + P_4 + 3P_2^2)(P_4 + 3P_2^2) - 4P_3(P_4' + 6P_2P_2')]z = 0, \end{aligned}$$

if $P_4 + 3P_2^2 \neq 0$. If $P_4 + 3P_2^2 = 0$ we find

$$(19)\qquad z^{(3)} + 6P_2 z' + 4P_3 z = 0.$$

Equation (18) determines the curve $C_z$ in the same way as (1) determines $C_y$. But if $P_4 + 3P_2^2 = 0$, $z$ satisfies (19) showing that the curve $C_z$ is in this case a plane curve. Therefore, if the variable $\xi$ be so chosen as to make $\bar{P}_4 + 3\bar{P}_2^2 = 0$, the corresponding curve $C_{\bar{z}}$ is a plane section of the developable surface whose cuspidal edge is $C_y$. In harmony with this, equations (10) show that the most general value of $\eta$, which satisfies the condition $\bar{P}_4 + 3\bar{P}_2^2 = 0$, contains three arbitrary constants, as it should since there are $\infty^3$ planes in space.

We shall need to consider the ruled surfaces generated by those edges of our tetrahedron which meet in $P_y$. Of these we know one immediately, namely the developable which has $C_y$ as its edge of regression, and of which $P_yP_z$ is a generator. The ruled surface generated by $P_yP_\rho$ clearly has $C_y$ as an asymptotic curve; for, the plane $P_yP_\rho P_z$ is both osculating plane of $C_y$ at $P_y$, and

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tangent plane of the surface at $P_y$. If we assume $p_1 = 0$, this ruled surface may be studied by means of the equations

$$(20)\qquad \begin{aligned} &y'' + P_2 y - \rho = 0, \\ &\rho'' + (4P_3 - 2P_2')y' + (P_4 - P_2'' - 2P_2^2)y + 5P_2\rho = 0, \end{aligned}$$

in accordance with the general theory of ruled surfaces as developed in former papers of the author. To prove (20) we need only differentiate twice the expression for $\rho$, express $y^{(3)}$ and $y^{(4)}$ in terms of $y$, $z$, $\rho$, $\sigma$, and eliminate $z$ and $\sigma$.

The ruled surface generated by $P_y P_\sigma$ is especially important. We have,

$$(21)\qquad \sigma = y^{(3)} + 3p_1 y^{(2)} + 3p_2 y' + p_3 y,$$

whence

$$(22)\qquad \begin{aligned} \sigma' &= y^{(4)} + 3p_1 y^{(3)} + (3p_1' + 3p_2)y'' + (3p_2' + p_3)y' + p_3' y, \\ \sigma'' &= y^{(5)} + 3p_1 y^{(4)} + 3(2p_1' + p_2)y^{(3)} \\ &\qquad + (3p_1'' + 6p_2' + p_3)y'' + (3p_2'' + 2p_3')y' + p_3'' y. \end{aligned}$$

From (21) we find

$$(23)\qquad \begin{aligned} y^{(3)} &= \sigma - 3p_1 y'' - 3p_2 y' - p_3 y, \\ y^{(4)} &= \sigma' - 3p_1\sigma - 3(p_1' + p_2 - 3p_1^2)y'' \\ &\qquad - (3p_2' + p_3 - 9p_1p_2)y' - (p_3' - 3p_1p_3)y. \end{aligned}$$

If we substitute these values in (1) we obtain the equation

$$(24)\qquad \begin{aligned} &3(p_2 - p_1' - p_1^2)y'' + 3(p_3 - p_2' - p_1p_2)y' + \sigma' \\ &\qquad + (p_4 - p_1p_3 - p_3')y + p_1\sigma = 0, \end{aligned}$$

where the coefficient of $y''$ is $3P_2$.

Let us differentiate both members of this equation, and eliminate $y''$ and $y^{(3)}$ by means of (23) and (24). We shall find

$$(25)\qquad \begin{aligned} 3P_2\sigma'' &= (q_3 r_3 + 3P_2 q_4)y' + (3P_2 q_1 - q_3)\sigma' \\ &\qquad + (r_4 q_3 + 3P_2 q_5)y + (3P_2 q_2 - p_1 q_3)\sigma, \end{aligned}$$

where

$$(26)\qquad \begin{aligned} &q_1 = -p_1, \qquad q_2 = 2p_1' - 3p_2 + 3p_1^2, \qquad q_3 = -3P_3 + 3p_1 P_2, \\ &q_4 = -2p_3' + 3p_2'' + 3p_1p_2' - 6p_1'p_2 - p_4 + p_1p_3 + 9p_2^2 - 9p_1^2p_2, \\ &q_5 = -(p_4' - p_3'' - p_1p_3' - p_1'p_3) + 3p_3(p_2 - p_1' - p_1^2), \\ &r_3 = -3(p_3 - p_2' - p_1p_2), \\ &r_4 = -(p_4 - p_3' - p_1p_3). \end{aligned}$$

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Equations (24) and (25) define the ruled surface generated by $P_y P_\sigma$. If we assume $p_1 = 0$, we find

$$(27)\qquad \begin{aligned} y'' + p_{11} y' + p_{12}\sigma' + q_{11} y + q_{12}\sigma = 0, \\ \sigma'' + p_{21} y' + p_{22}\sigma' + q_{21} y + q_{22}\sigma = 0, \end{aligned}$$

where

$$p_{11} = \frac{P_3 - P_2'}{P_2}, \qquad p_{12} = \frac{1}{3P_2}, \qquad q_{11} = \frac{P_4 + 3P_2^2 - P_3'}{3P_2}, \qquad q_{12} = 0,$$

$$(28)\qquad \begin{aligned} p_{21} &= \frac{1}{P_2}[-3P_3^2 + 3P_3P_2' + 2P_2P_3' - 3P_2P_2'' + P_2P_4 - 6P_2^3], \\ p_{22} &= -\frac{P_3}{P_2}, \\ q_{21} &= \frac{1}{P_2}[-P_3P_4 - 6P_2^2P_3 + P_3P_3' + P_2P_4' + 6P_2^2P_2' - P_2P_3''], \\ q_{22} &= 3P_2. \end{aligned}$$

If (1) is written in the LAGUERRE-FORSYTH form, $P_2 = 0$. In that case the two equations (27) reduce to the single equation

$$(29)\qquad P_3 y' + \tfrac{1}{3}\sigma' + \tfrac{1}{3}(P_4 - P_3')y = 0,$$

which proves that *in this case the surface generated by $P_y P_\sigma$ is developable.* For in this case the tangents constructed respectively to $C_y$ at $P_y$ and to $C_\sigma$ at $P_\sigma$ are coplanar. Moreover, only if $P_2 = 0$ will the surface generated by $P_y P_\sigma$ be a developable.

Let

$$\tau = \lambda y + \mu\sigma$$

represent the point at which $P_y P_\sigma$ intersects the edge of regression of the developable. Then, since $P_y P_\sigma$ must be tangent to the edge of regression, we shall have $\tau' = \alpha y + \beta\sigma$, or

$$(\lambda' - \alpha)y + (\mu' - \beta)\sigma + \lambda y' + \mu\sigma' = 0.$$

But according to (29)

$$\sigma' = -(P_4 - P_3')y - 3P_3 y',$$

so that

$$(\lambda' - \alpha)y + (\mu' - \beta)\sigma + \lambda y' - \mu[(P_4 - P_3')y + 3P_3 y'] = 0,$$

where for $y'$ we could also write $z$. Such a relation between $P_y, P_z, P_\sigma$ would, however, make these three points collinear, and, therefore, $P_y, P_z, P_\rho, P_\sigma$ coplanar, unless all of the coefficients are zero. We have seen, however, that this

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can happen only for such points $P_y$ at which the osculating plane is stationary. We must have, therefore,

$$\lambda' - \alpha - \mu(P_4 - P_3') = 0, \qquad \mu' - \beta = 0, \qquad \lambda - 3\mu P_3 = 0,$$

whence

$$\lambda = 3\mu P_3, \qquad \beta = \mu', \qquad \alpha = \lambda' - \mu(P_4 - P_3').$$

*We see, therefore, that*

$$\tau = 3P_3 y + \sigma \tag{30}$$

*represents the edge of regression of the developable to which the ruled surface generated by $P_y P_\sigma$ reduces when $P_2 = 0$.*

If $p_1 = 0$ and $P_2 = 0$, equations (2) and (10) show that the most general transformations of the variables, which do not disturb these conditions, satisfy the equations

$$\frac{\lambda'}{\lambda} + \tfrac{3}{2}\frac{\xi''}{\xi'} = 0, \qquad \mu = \eta' - \tfrac{1}{2}\eta^2 = 0,$$

which give on integration

$$\lambda = \frac{C}{(\xi')^{\frac{3}{2}}}, \qquad \eta = \frac{-2c}{1 + cx}.$$

If we transform $\tau$ under this assumption, we find that it is converted into

$$\bar{\tau} = \frac{1}{(\xi')^3 \lambda}\left[\sigma + \tfrac{3}{2}\eta\rho + \tfrac{3}{2}\eta^2 z + (\tfrac{3}{4}\eta^3 + 3P_3)y\right], \tag{30a}$$

where $\eta$ may have any numerical value.

Let us recapitulate. *The ruled surfaces generated by $P_y P_{\bar{\sigma}}$ are infinite in number. Their general expression involves an arbitrary function $\eta$. Among these surfaces there exists a single infinity of developables. If $P_2 = 0$, the surface generated by $P_y P_\sigma$ is one of these, and the locus of $P_\tau$ is its edge of regression, where*

$$\tau = 3P_3 y + \sigma, \tag{30}$$

*$P_\tau$ being the point where $P_y P_\sigma$ intersects the edge of regression. If we construct all the $\infty^1$ lines $P_y P_{\bar{\sigma}}$ through $P_y$, which are generators of the above mentioned family of developables, and mark upon each of them the point $P_{\bar{\tau}}$ where it intersects the cuspidal edge of the developable to which it belongs, the locus of these points is a twisted cubic curve. The equations of this curve referred to a parameter $\eta$ and to the fundamental tetrahedron $P_y P_z P_\rho P_\sigma$ are*

$$x_1 = 3P_3 + \tfrac{3}{4}\eta^3, \qquad x_2 = \tfrac{3}{2}\eta^2, \qquad x_3 = \tfrac{3}{2}\eta, \qquad x_4 = 1. \tag{31}$$

We shall see later that this cubic has five consecutive points in common with the curve $C_y$ at $P_y$, i. e., that it has at this point with $C_y$ a contact of the fourth

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order. We shall speak of it as the *torsal cubic of* $P_y$, on account of its connection with the developables which we have just been considering.

Equations (31) give the parametric equations of the torsal cubic referred to a special tetrahedron of reference for which $P_2 = 0$. We shall need its equations in a more general form. These may be easily obtained. Consider the expression

$$(32) \qquad \lambda = (3\theta_3 + \tfrac{3}{10}P_2' + \tfrac{6}{5}P_2 t + \tfrac{3}{4}t^3)y + (\tfrac{6}{5}P_2 + \tfrac{3}{2}t^2)z + \tfrac{3}{2}t\rho + \sigma$$

in which $t$ may for the moment be regarded as a parameter independent of $x$. Denote by $\bar{\lambda}$ the corresponding expression formed from the quantities $\bar{P}_2$, $\bar{P}_3$, etc., $\bar{y}$, $\bar{z}$, $\bar{\rho}$, $\bar{\sigma}$ after the general transformation $\xi = \xi(x)$. We shall find that $(\xi')^3\bar{\lambda}$ is equal to $\lambda$ after $t$ has been replaced by $t\xi' + \eta = t_1$. But of course this transformation may be chosen so as to make $\bar{P}_2 = 0$, which would make $\bar{\lambda}$ identical with $\bar{\tau}$ except for the notation.

*We see, therefore, that the expression* $\lambda$, *or the equations*

$$(33) \qquad \begin{aligned} x_1 &= 3\theta_3 + \tfrac{3}{10}P_2' + \tfrac{6}{5}P_2\eta + \tfrac{3}{4}\eta^3, \\ x_2 &= \tfrac{6}{5}P_2 + \tfrac{3}{2}\eta^2, \qquad x_3 = \tfrac{3}{2}\eta, \qquad x_4 = 1, \end{aligned}$$

*represent the torsal cubic referred to the fundamental tetrahedron* $P_y P_z P_\rho P_\sigma$ *when this is chosen in as general a way as is compatible with its definition.*

If in (32) $t$ is chosen as a function of $x$, as $x$ varies we obtain a curve on the surface formed by the totality of torsal cubics. If in particular $t$ satisfies as function of $x$ the differential equation

$$t' - \tfrac{1}{2}t^2 = \tfrac{6}{5}P_2,$$

we obtain the cuspidal edge of one of the developables.

## § 4. *The osculating cubic, conic and linear complex.*

A space cubic is determined by six of its points provided that no four of these points are coplanar. If, therefore, we take upon $C_y$, besides $P_y$, five other points, we shall in general obtain a perfectly definite space cubic determined by these six points. As these points approach coincidence with $P_y$, this cubic will in general approach a limit, which shall be called the *osculating cubic.* We proceed to find its equations.

Let $P_y$ correspond to the value of $x = a$, which we shall suppose is an ordinary point for our differential equation. Then $y$ may be developed by Taylor's theorem into a series proceeding according to powers of $x - a$. By putting $x - a = x'$ the development will be in powers of $x'$. We may therefore assume in the first place that $a = 0$. Let us assume, further, that $p_1 = 0$ and $P_2 = 0$. Then we shall have from (5) and (6),

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$$(34)\quad \begin{aligned} &y' = z, \qquad y'' = \rho, \qquad y^{(3)} = -P_3 y + \sigma, \qquad y^{(4)} = -P_4 y - 4P_3 z, \\ &y^{(5)} = -P_4' y - (P_4 + 4P_3')z - 4P_3\rho. \end{aligned}$$

In accordance with the definition of our coördinates, we denote the coefficients of $y, z, \rho, \sigma$ in this expansion carried as far as $x^5$, by $y_1, y_2, y_3, y_4$. We may of course multiply these quantities by a common factor since the coördinates are homogeneous. We shall multiply by 120 so as to clear of fractions. This gives

$$(35)\quad \begin{aligned} y_1 &= 120 - 20P_3x^3 - 5P_4x^4 - P_4'x^5 + \cdots, \\ y_2 &= 120x - 20P_3x^4 - (4P_3' + P_4)x^5 + \cdots, \\ y_3 &= 60x^2 - 4P_3x^5 + \cdots, \\ y_4 &= 20x^3 + \cdots. \end{aligned}$$

We see at once that the following equations are exact up to terms no higher than the fifth order:

$$(36)\qquad 3y_2y_4 - 2y_3^2 = 0, \qquad 5(2y_1y_3 - y_2^2) - 6P_3y_3y_4 = 0.$$

These equations must be satisfied by the coördinates of any point of the osculating cubic, since this must have contact of the fifth order with $C_y$ at $P_y$. *They are therefore its equations, referred to this special tetrahedron of reference. In terms of a parameter $t$ we may write*

$$(37)\quad x_1 = 15 + 12P_3t^3, \qquad x_2 = 30t, \qquad x_3 = 30t^2, \qquad x_4 = 20t^3.$$

The equation

$$3x_2x_4 - 2x_3^2 = 0$$

is that of a cone whose vertex is $P_y$ and which contains the osculating cubic. It may also be obtained by determining that cone of the second order with its vertex at $P_y$ which has the closest possible contact with $C_y$, viz. contact of the fifth order. We shall speak of it as the *osculating cone.* We notice at once that the torsal cubic also lies upon the osculating cone. This is shown by equations (31) which are referred to the same system of coördinates as that employed here. If we put $\eta = 1/t$ in (31) and if we multiply by $20t^3$, (31) becomes

$$(31a)\quad x_1 = 15 + 60P_3t^3, \qquad x_2 = 30t, \qquad x_3 = 30t^2, \qquad x_4 = 20t^3,$$

which differs from (37) only in having $5P_3$ in place of $P_3$.

By a method of reasoning precisely similar to that of the last paragraph, we find that *the expression*

$$(38)\quad (12P_3 - 12P_2' + 24P_2\tau + 15\tau^3)y + 20(\tfrac{6}{5}P_2 + \tfrac{3}{2}\tau^2)z + 30\tau\rho + 20\sigma$$

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*represents an arbitrary point of the osculating cubic when the tetrahedron of reference is not restricted to the condition* $P_2 = 0$. For this expression remains invariant under the general transformation $\xi = \xi(x)$, and reduces to (37) for $P_2 = 0$, if $\tau = 1/t$.

The equation of the plane, which osculates the osculating cubic at the point whose parameter is $\tau$, turns out to be

$$u_1 x_1 + u_2 x_2 + u_3 x_3 + u_4 x_4 = 0,$$

where

$$(39) \qquad \begin{aligned} &u_1 = -20, \qquad u_2 = 30\tau, \qquad u_3 = 16P_2 - 30\tau^2, \\ &u_4 = 12P_3 - 12P_2' - 36P_2\tau + 15\tau^3. \end{aligned}$$

For every value of $\tau$ this intersects the osculating plane, $x_4 = 0$, in a straight line

$$-20x_1 + 30\tau x_2 + (16P_2 - 30\tau^2)x_3 = 0.$$

The envelope of these lines will be obtained by eliminating $\tau$ between this equation and that obtained from it by partial differentiation with respect to $\tau$; the latter equation is

$$30x_2 - 60\tau x_3 = 0.$$

*We thus find*

$$(40) \qquad -40x_1x_3 + 15x_2^2 + 32P_2x_3^2 = 0,$$

*the equation of the osculating conic, which may be defined as a part of the intersection of the developable of the osculating cubic with the osculating plane.* The other part of this intersection is the tangent, which must be counted twice.

It is not without interest to verify that (37) represents the osculating cubic by another method. We have from (35) as non-homogeneous coördinates of the points of $C_y$ in the vicinity of $P_y$

$$(41) \qquad \begin{aligned} &\frac{y_2}{y_1} = x - \tfrac{1}{120}(P_3' - 4P_4)x^5 + \cdots, \qquad \frac{y_3}{y_1} = \tfrac{1}{2}x^2 + \tfrac{1}{20}P_3x^5 + \cdots, \\ &\frac{y_4}{y_1} = \tfrac{1}{6}x^3 + \cdots. \end{aligned}$$

From (37) we find for the points of the osculating cubic

$$\frac{x_2}{x_1} = 2t - \tfrac{8}{5}P_3t^4 + \cdots, \qquad \frac{x_3}{x_1} = 2t^2 - \tfrac{8}{5}P_3t^5 + \cdots, \qquad \frac{x_4}{x_1} = \tfrac{4}{3}t^3 + \cdots.$$

If we put

$$t = \tfrac{1}{2}x[1 + \tfrac{1}{10}P_3x^3 - \tfrac{1}{30}(P_3' + P_4)x^4 + \cdots],$$

these two expansions coincide up to terms of the fifth order. For the torsal cubic we have according to (31$a$),

$$\frac{x_2}{x_1} = 2t - 8P_3t^4 \cdots, \qquad \frac{x_3}{x_1} = 2t^2 - 8P_3t^5 + \cdots, \qquad \frac{x_4}{x_1} = \tfrac{4}{3}t^3 + \cdots.$$

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If we put into these equations

$$t = \tfrac{1}{2}x(1 + ax + bx^2 + cx^3 + dx^4 + \cdots),$$

we find that these expansions will agree with (41) up to terms of the fourth order for

$$a = 0, \qquad b = 0, \qquad c = \tfrac{1}{2}P_3.$$

but that it is impossible to make them agree with (41) any farther unless $P_3 = 0$. *In general, therefore, the torsal cubic has with $C_y$ a contact of the fourth order. Only if $\theta_3 = 0$ may the order of contact be higher. In that case the torsal and osculating cubics coincide.*

We proceed to deduce the equation of the *osculating linear complex*, i. e., of that linear complex determined by five consecutive tangents of the curve. We assume again $p_1 = 0$ and $P_2 = 0$. Denote by $Y$ and $Z$ the expansions of $y$ and $z$ in the vicinity of $P_y$. Then we have up to terms of the fourth order

$$Y = y(1 - \tfrac{1}{6}P_3x^3 - \tfrac{1}{24}P_4x^4) + z(x - \tfrac{1}{6}P_3x^4) + \tfrac{1}{2}\rho x^2 + \tfrac{1}{6}\rho x^3,$$

$$Z = y(-\tfrac{1}{2}P_3x^3 - \tfrac{1}{6}P_4x^3 - \tfrac{1}{24}P'_4x^4) + z[1 - \tfrac{2}{3}P_3x^3 - \tfrac{1}{24}(4P'_3 + P_4)x^4]$$
$$+ \rho(x - \tfrac{1}{6}P_3x^4) + \sigma\tfrac{1}{2}x^2.$$

If we denote the coefficients of $y, z, \rho, \sigma$ in these two expressions by $y_1, \cdots, y_4$ and $z_1, \cdots, z_4$, respectively, the Plückerian line-coördinates of the tangent will be

$$\omega_{ik} = y_i z_k - y_k z_i,$$

whence

$$\omega_{12} = 1 - \tfrac{1}{6}P_3x^3 + \tfrac{1}{8}P_4x^4 + \cdots, \quad \omega_{13} = x - \tfrac{1}{12}P_3x^4 + \cdots, \quad \omega_{14} = \tfrac{1}{2}x^2 + \cdots,$$

$$\omega_{23} = \tfrac{1}{2}x^2 + \cdots, \qquad \omega_{24} = \tfrac{1}{3}x^3 + \cdots, \qquad \omega_{34} = \tfrac{1}{12}x^4 + \cdots.$$

Therefore, *the equation of the osculating linear complex, referred to the special tetrahedron of reference, is*

$$\omega_{14} - \omega_{23} = 0. \tag{42}$$

We might have obtained this complex in another way. For, it is clear that the null-system of the osculating cubic will be the same as that determined by the osculating linear complex. We shall, instead, set up the null-system of the torsal cubic in its general form. We shall see that *the linear complex determined by the torsal cubic coincides with the osculating linear complex.*

We have the equations of the torsal cubic

$$\begin{aligned} x_1 &= 60\theta_3 + 6P'_2 + 24P_2\eta + 15\eta^3, \\ x_2 &= 24P_2 + 30\eta^2, \qquad x_3 = 30\eta, \qquad x_4 = 20. \end{aligned} \tag{43}$$

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The coördinates of the plane, which osculates the torsal cubic at the point whose parameter is $\eta$, are

$$(44)\qquad \begin{aligned} &u_1 = -180, \qquad u_2 = 270\eta, \qquad u_3 = 144P_2 - 270\eta^2, \\ &u_4 = 540\theta_3 + 54P_2' - 324P_2\eta + 135\eta^3. \end{aligned}$$

If we put in (43) $\eta = \eta_k (k = 1, 2, 3)$ we obtain three points on the cubic. The coördinates of their plane must be proportional to

$$(45)\qquad \begin{aligned} &v_1 = -180, \qquad v_2 = 90(\eta_1 + \eta_2 + \eta_3), \\ &v_3 = 144P_2 - 90(\eta_2\eta_3 + \eta_3\eta_1 + \eta_1\eta_2), \\ &v_4 = 540\theta_3 + 54P_2' - 108P_2(\eta_1 + \eta_2 + \eta_3) + 135\eta_1\eta_2\eta_3, \end{aligned}$$

for each of these expressions must be a symmetric function of $\eta_1, \eta_2, \eta_3$ of order not higher than the third, and for $\eta_1 = \eta_2 = \eta_3 = \eta$ we must have $v_k$ proportional to $u_k$. Similarly, the point in which the three osculating planes at $\eta_1, \eta_2, \eta_3$ intersect, must have its coördinates proportional to

$$(46)\qquad \begin{aligned} &\omega x_1 = 60\theta_3 + 6P_2' + 8P_2(\eta_1 + \eta_2 + \eta_3) + 15\eta_1\eta_2\eta_3, \\ &\omega x_2 = 24P_2 + 10(\eta_2\eta_3 + \eta_3\eta_1 + \eta_1\eta_2), \\ &\omega x_3 = 10(\eta_1 + \eta_2 + \eta_3), \qquad \omega x_4 = 20. \end{aligned}$$

If we eliminate $\eta_1, \eta_2, \eta_3$ between (45) and (46) and change slightly the factor of proportionality, we find

$$(47a)\qquad \begin{aligned} &\omega v_1 = -x_4, \qquad \omega v_3 = -x_2 + 2P_2x_4, \\ &\omega v_2 = +x_3, \qquad \omega v_4 = +x_1 - 2P_2x_3, \end{aligned}$$

or

$$(47b)\qquad \begin{aligned} &\omega' x_1 = +2P_2v_2 + v_4, \qquad \omega' x_3 = +v_2, \\ &\omega' x_2 = -2P_2v_1 - v_3, \qquad \omega' x_4 = -v_1, \end{aligned}$$

as the equations of the null-system defined by the torsal cubic.

A point $y_1, y_2, y_3, y_4$ lies in the plane corresponding to $x_1, x_2, x_3, x_4$ if

$$\sum_{k=1}^{4} v_k y_k = 0.$$

Therefore, the lines which pass through the point $x_1, x_2, x_3, x_4$ and lie in the plane corresponding to it in the null-system, satisfy the equation

$$(48)\qquad \omega_{14} - 2P_2\omega_{34} - \omega_{23} = 0.$$

If the tetrahedron of reference be so chosen as to make $P_2 = 0$, this equation is identical with (42). *Therefore* (48) *represents the osculating linear complex*

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*when the tetrahedron of reference is general. The osculating and torsal cubics are curves of this complex.*

If $P_2$ is finite the complex (48) is not special. We see therefore that only those values of $x$ for which $P_2 = \infty$ can give points of the curve at which five consecutive tangents have a straight line intersector.

Let us proceed to deduce the equation of the osculating linear complex belonging to a point of $C_y$ infinitesimally close to $P_y$. If we change $x$ by an infinitesimal amount $\delta x$, we find for the coördinates of the vertices of the new tetrahedron of reference

$$\begin{aligned}
\bar{y} &= y + y'\delta x = (1 - p_1\delta x)y + z\delta x,\\
\bar{z} &= z + z'\delta x = -P_2\delta x\cdot y + (1 - p_1\delta x)z + \rho\delta x,\\
\bar{\rho} &= \rho + \rho'\delta x = -(P_3 - P_2')\delta x\cdot y - 2P_2\delta x\cdot z + (1 - p_1\delta x)\rho + \sigma\delta x,\\
\bar{\sigma} &= \sigma + \sigma'\delta x = -(P_4 - P_3')\delta x\cdot y - 3(P_3 - P_2')\delta x\cdot z\\
&\qquad - 3P_2\delta x\cdot\rho + (1 - p_1\delta x)\sigma.
\end{aligned}$$

Therefore, if a point has the coördinates $\bar{x}_1, \cdots, \bar{x}_4$ in the new system of coördinates, and $x_1, \cdots, x_4$ in the old, we shall have

$$x_1 = (1 - p_1\delta x)\bar{x}_1 - P_2\delta x\cdot\bar{x}_2 - (P_3 - P_2')\delta x\cdot\bar{x}_3 - (P_4 - P_3')\delta x\cdot\bar{x}_4,$$

. . . . . .

Therefore the infinitesimal changes in the coördinates in the sense new minus old, will be

$$(49)\qquad \begin{aligned}
\delta x_1 &= [p_1x_1 + P_2x_2 + (P_3 - P_2')x_3 + (P_4 - P_3')x_4]\delta x,\\
\delta x_2 &= [-x_1 + p_1x_2 + 2P_2x_3 + 3(P_3 - P_2')x_4]\delta x,\\
\delta x_3 &= (-x_2 + p_1x_3 + 3P_2x_4)\delta x,\\
\delta x_4 &= (-x_3 + p_1x_4)\delta x.
\end{aligned}$$

Referred to the new tetrahedron of reference, the equation of the complex osculating $C_y$ at the point corresponding to $x + \delta x$ will be

$$(50)\qquad \bar{\omega}_{14} - 2\bar{P}_2\bar{\omega}_{34} - \bar{\omega}_{23} = 0,$$

where

$$\bar{P}_2 = P_2 + P_2'\delta x, \qquad \bar{\omega}_{ik} = \bar{x}_i\bar{y}_k - \bar{x}_k\bar{y}_i,$$
$$\bar{x}_i = x_i + \delta x_i, \qquad \bar{y}_i = y_i + \delta y_i,$$

if $\bar{x}_i$ and $\bar{y}_i$ denote the coördinates of two points on a line of the complex referred to the new tetrahedron of reference. Making the calculations we find

$$\begin{aligned}
\bar{\omega}_{14} &= \omega_{14} + [-\omega_{13} + 2p_1\omega_{14} - P_2\omega_{42} + (P_3 - P_2')\omega_{34}]\delta x,\\
\bar{\omega}_{34} &= \omega_{34} + (2p_1\omega_{34} + \omega_{42})\delta x,\\
\bar{\omega}_{23} &= \omega_{23} + [2p_1\omega_{23} - 3P_2\omega_{42} - \omega_{13} - 3(P_3 - P_2')\omega_{34}]\delta x.
\end{aligned}$$



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If we substitute in (50), we find as the *equation of the linear complex osculating $C_y$ at a point infinitesimally close to $P_y$*,

$$(51) \qquad (\omega_{14} - 2P_2\omega_{34} - \omega_{23})(1 + 2p_1\delta x) + 4\theta_3\omega_{34}\delta x = 0.$$

This coincides with the linear complex osculating $C_y$ at $P_y$, if, and only if, $\theta_3 = 0$.

Therefore, *if the invariant $\theta_3$ vanishes identically, the tangents of the curve $C_y$ belong to a linear complex.* If it does not vanish identically, those values of $x$ for which it does vanish correspond to points of the curve at which the osculating linear complex hyperosculates the curve.

This result, which has been known a long time, is here derived in a novel manner, which has above all the advantages of clearness. It may also be obtained by setting up the linear differential equation of the sixth order satisfied by the six line coördinates

$$\omega_{ik} = y_i y_k' - y_k y_i',$$

of the tangent, and noting that this reduces to the fifth order if, and only if, $\theta_3 = 0$. This is the method of HALPHEN.

A former result may now be stated as follows: *The osculating and torsal cubics of all points of a curve coincide, if, and only if, the curve belongs to a linear complex.*

§ 5. *Geometrical definition of the fundamental tetrahedron of reference.*

We have seen that there exists for every point of the curve $C_y$ a tetrahedron whose vertices $P_y$, $P_z$, $P_\rho$, $P_\sigma$ are determined by the choice of the independent variables $x$. In order that we may be able to obtain a clear insight into the geometry of the curve, it is necessary that we may be able to define this tetrahedron by purely geometrical considerations. As a consequence of our preceding results we are now able to do this.

We have already noticed that $P_z$ is a point on the tangent, and by a properly chosen transformation $\xi = \xi(x)$ it may be transformed into any other point of the tangent. When the independent variable has been definitely chosen to be $x$, we obtain, therefore, a point $P_z$ on the tangent which is not, in general, distinguished by any geometrical property from any other point of the tangent. Its position may serve as a geometric image of the independent variable.

Consider the osculating conic

$$x_4 = 0, \qquad 40x_1x_3 - 32P_2x_3^2 - 15x_2^2 = 0.$$

The polar of any point $(x_1', x_2', x_3', 0)$ of the osculating plane with respect to it, is the straight line

$$x_4 = 0, \qquad 20x_3'x_1 - 15x_2'x_2 + (20x_1' - 32P_2x_3')x_3 = 0.$$

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Therefore, the polar of $P_z$, whose coördinates are $(0, 1, 0, 0)$, is the line $x_2 = 0$, $x_4 = 0$. In other words:

*The line $P_y P_\rho$ is the polar of $P_z$ with respect to the osculating conic.*

We shall speak of the curves $C_z$, $C_\rho$, $C_\sigma$ as the derivative curves of $C_y$ with respect to $x$, of the first, second and third kind respectively. The ruled surfaces which are obtained by joining the points of $C_y$ to the corresponding points of $C_z$, $C_\rho$, $C_\sigma$ shall be called derivative ruled surfaces of the first, second and third kind respectively. Then, the derivative ruled surface of the first kind is unique. It is simply the developable whose cuspidal edge is $C_y$. Let us consider the derivative ruled surface $S$ of the second kind generated by $P_y P_\rho$. The curve $C_y$ is, of course, an asymptotic curve upon it. This surface is characterized by the equations (20), where $p_1$ has been assumed equal to zero.

According to the general theory of ruled surfaces,* the asymptotic tangents to $S$ at the points $P_y$ and $P_\rho$ are obtained by joining these points to $2z$ and

$$2\sigma - 4P_2 z + 2P_3 y$$

respectively. Therefore the asymptotic tangent to $S$ at any point $(\alpha_1, 0, \alpha_3, 0)$ of $P_y P_\rho$ joins this point to

$$2P_3\alpha_3 y + (2\alpha_1 - 4P_2\alpha_3)z + 2\alpha_3\sigma.$$

Hence, the equation of the plane tangent to $S$ at $(\alpha_1, 0, \alpha_3, 0)$ is

$$-\alpha_3 x_2 + (\alpha_1 - 2P_2\alpha_3)x_4 = 0.$$

To the same point of $P_y P_\rho$ there corresponds a plane in the osculating linear complex. According to (47$a$) this is the plane

$$\alpha_3 x_2 + (\alpha_1 - 2P_2\alpha_3)x_4 = 0.$$

Therefore, if at any point of the generator of the derived ruled surface of the second kind we construct the tangent plane as well as the plane which corresponds to it in the osculating linear complex, these planes form an involution. The double planes of this involution are the osculating plane $(x_4 = 0)$, and a plane $(x_2 = 0)$ which contains $P_\sigma$, the point of the derivative curve of the third kind which corresponds to $P_y$.

The point which corresponds to this latter plane, is

$$\beta = 2P_2 y + \rho. \tag{52}$$

According to (47$a$) we have further, corresponding to the point $P_z$ or (0, 1, 0, 0), the plane $x_3 = 0$, which also contains $P_\sigma$. The line $P_y P_\sigma$ is now completely determined, as follows:

* WILCZYNSKI, *Covariants*, etc. Transactions of the American Mathematical Society, vol. 3 (1902), p. 434.

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*The generator of the derived ruled surface of the third kind is the intersection of the following two planes: 1st, the plane corresponding to $P_\alpha$ in the osculating linear complex; 2d, that plane which is tangent to the derived ruled surface of the second kind at the same point which corresponds to it in the osculating linear complex.*

It still remains to determine the position of $P_\rho$ and $P_\sigma$ on the lines $P_y P_\rho$ and $P_y P_\sigma$.

The osculating conic intersects $P_y P_\rho$ in $P_y$ and in $P_\alpha$ where

$$\alpha = 4P_2 y + 5\rho. \tag{53}$$

The cross-ratio of the four points $P_y$, $P_\rho$, $P_\alpha$, $P_\beta$, is

$$(\alpha, y, \beta, \rho) = \tfrac{5}{2}.$$

*If, upon the generator of the derived ruled surface of the second kind, there be marked its intersections with the osculating conic, and the point $P_\beta$ whose tangent plane coincides with the plane corresponding to it in the osculating linear complex, the point $P_\rho$ is determined by the condition that the cross-ratio of these four points shall be equal to $\frac{5}{2}$.*

If $P_2 = 0$ this definition of $P_\rho$ breaks down. In that case, however, $P_\alpha$ and $P_\beta$ coincide with $P_\rho$. Therefore, *if the derived ruled surface of the third kind is a developable, $P_\rho$ is that point on the generator of the derived surface of the second kind where this generator intersects the osculating conic the second time. At this point the plane, tangent to the ruled surface, and the plane, corresponding to it in the osculating linear complex, coincide.*

If we use the notations of the theory of ruled surfaces,* we find from (20),

$$u_{12} = 4, \qquad u_{21} = 8P_3' - 4P_4 + 8P_2^2, \qquad u_{11} - u_{22} = 16P_2,$$
$$(u_{11} - u_{22})^2 + 4u_{12}u_{21} = -64(P_4 - 2P_3' - 6P_2^2).$$

But $u_{11} - u_{22} = 0$ is the condition that $C_y$ and $C_\rho$ shall be harmonically divided by the branches of the flecnode curves of the ruled surface, while $(u_{11} - u_{22})^2 + 4u_{12}u_{21} = 0$ is the condition under which the two branches of the flecnode curve coincide.† Therefore, we obtain the following theorem.

*If the derived ruled surface of the third kind is a developable, the intersections of the generator of the derived ruled surface of the second kind with the osculating conic give rise to two curves upon this surface harmonically conjugate with respect to the two branches of its flecnode curve.*

*If $\theta_4 = 0$, the second intersection of the generator of this surface with the osculating conic is a point of its flecnode curve. Moreover, the two branches of the flecnode curve must then coincide.*

* WILCZYNSKI, *Covariants*, etc., loc. cit., p. 445. *Invariants*, etc., Transactions of the American Mathematical Society, vol. 2 (1901), p. 6.

† Loc. cit., p. 444.

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It is to be noted that we have here a *geometrical interpretation for the invariant equation* $\theta_4 = 0$. We shall find two other, quite different interpretations for this condition later on. By the method of HALPHEN still another meaning can be attached to this equation, not however so purely geometrical.

We may, if we wish, make use of the torsal cubic in our further constructions. For, it is now defined entirely by geometrical considerations. If, in fact, we trace upon the developable, whose edge of regression is $C_y$ an arbitrary curve $C_{\bar{z}}$, we now know how the corresponding ruled surfaces of the second and third kind may be constructed. They depend upon an arbitrary function of $x$, as does the curve $C_{\bar{z}}$. Among the surfaces of the third kind there exists a single one-parameter family of developables. Upon that generator of each of these developables which passes through $P_y$ we mark the point where it intersects the cuspidal edge of the developable to which it belongs. The locus of these points is the torsal cubic.

We notice incidentally that *the reduction of equation* (1) *to the Laguerre-Forsyth canonical form is equivalent to the determination of one of the developables of the third kind.* Since this reduction is made by solving an equation of the RICCATI form we notice further the following theorem. *The four curves on the developable of* $C_y$ *which correspond to any four of the developables of the third kind, intersect all of the tangents of* $C_y$ *in point-rows of the same cross-ratio.*

Let us consider the developable surface of the torsal cubic, which is given by equations (44). We are going to find its intersection with the plane $P_y P_z P_\sigma$, or $x_3 = 0$. The intersection of the plane $u_1, \cdots, u_4$, which osculates the cubic at the point whose parameter is $\eta$, with the plane $x_3 = 0$ is the line

$$-180x_1 + 270\eta x_2 + (540\theta_3 + 54P_2' - 324P_2\eta + 135\eta^3)x_4 = 0$$

of this plane. As $\eta$ changes this line envelops a curve, the required intersection. Its equation will be found by eliminating $\eta$ between the above equation and this other one

$$270x_2 + (-324P_2 + 405\eta^2)x_4 = 0$$

obtained from it by differentiation with respect to $\eta$. This elimination may be easily performed. The result is

$$(54)\quad F = 8(5x_2 - 6x_4)(5x_1 - 6x_4)^2 + 15x_4\{10x_1 - (30\theta_3 + 3P_2')x_4\}^2 = 0.$$

This plane cubic together with the tangent $P_y P_z$ gives the complete intersection of the plane $P_y P_z P_\sigma$ with the developable of the torsal cubic. It has a cusp at $P_z$, and the equation of its cusp tangent is

$$5x_1 - 6x_4 = 0,$$

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as one may find by the general theory of plane curves. It intersects $P_y P_\sigma$ in the point

$$\kappa = 24y + 20\sigma. \tag{55}$$

The tangent to the plane cubic at $P_y$ is

$$10x_2 + 3x_4 = 0.$$

It intersects the cubic again in the point

$$(12 + 30\theta_3 + 3P_2')y - 6z + 20\sigma.$$

If this point be joined to $P_z$ by a straight line, the latter will intersect $P_y P_\sigma$ in the point

$$\lambda = (12 + 30\theta_3 + 3P_2')y + 20\sigma. \tag{56}$$

The plane $P_y P_z P_\sigma$ is tangent to the torsal cubic. It intersects it once more in the point corresponding to $\eta = 0$, viz., $(60\theta_3 + 6P_2')y + 24P_2 z + 20\sigma$. A line joining this point to $P_z$ intersects $P_y P_\sigma$ in

$$\mu = (60\theta_3 + 6P_2')y + 20\sigma. \tag{57}$$

Consider the four points $P_\kappa$, $P_\lambda$, $P_\mu$ and $P_y$. We have

$$\lambda = \frac{\mu + \kappa}{2}, \qquad (30\theta_3 + 3P_2' - 12)y = \frac{\mu - \kappa}{2};$$

so that $P_\lambda$ is the harmonic conjugate of $P_y$ with respect to $P_\kappa$ and $P_\mu$.

The osculating cubic differs from the torsal cubic only in having $2(P_3 - P_2')$ in place of $10\theta_3 + P_2'$. Consequently the plane cubic in which its developable intersects the plane $P_y P_z P_\sigma$ is

$$\bar{F} = 8(5x_2 - 6x_4)(5x_1 - 6x_4)^2 + 15x_4\{10x_1 - 6(P_3 - P_2')x_4\}^2 = 0. \tag{58}$$

If we denote by $P_{\bar\kappa}$, $P_{\bar\lambda}$, $P_{\bar\mu}$ the points constructed with respect to this curve in the same way as $P_\kappa$, and $P_\lambda$ and $P_\mu$ were with respect to $F = 0$, we find

$$\bar\kappa = \kappa,\ \bar\lambda = (12 + 6P_3 - 6P_2')y + 20\sigma,\ \bar\mu = (12P_3 - 12P_2')y + 20\sigma, \tag{59}$$

the cusp and its tangent being common to the two curves, as well as the tangent at $P_y$. I refrain from formulating explicitly the various theorems which may be obtained from these equations.

In order to obtain a simple construction for $P_\sigma$, we shall consider finally the developable generated by the motion of the plane $P_y P_\rho P_\sigma$. The equation of this plane is $x_2 = 0$. As $x$ changes into $x + \delta x$, $y$, $z$, $\rho$ change into $y + y'\delta x$, $\rho + \rho'\delta x$, $\sigma + \sigma'\delta x$ respectively, where $y'$, $\rho'$, $\sigma'$ are given by equations (5). The equation of the plane of these points, referred to the tetrahedron $P_y$, $P_z$, $P_\rho$, $P_\sigma$, is

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$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ 1-p_1\delta x & \delta x & 0 & 0 \\ -(P_3-P_2')\delta x & -2P_2\delta x & 1-p_1\delta x & \delta x \\ -(P_4-P_3')\delta x & -3(P_3-P_2')\delta x & -3P_2\delta x & 1-p_1\delta x \end{vmatrix} = 0,$$

which becomes, when developed,

$$x_1\delta x - x_2(1-3p_1\delta x) - 2x_3P_2\delta x - 3x_4(P_3-P_2')\delta x = 0.$$

Therefore the equations of the generator of the developable, generated by the motion of the plane $P_y P_\rho P_\sigma$, are

$$(60) \qquad x_2 = 0, \qquad x_1 - 2P_2x_3 - 3(P_3-P_2')x_4 = 0.$$

It intersects the generator $P_y P_\rho (x_2 = x_4 = 0)$ of the derived ruled surface of the second kind in that point

$$(52) \qquad \beta = 2P_2 y + \rho$$

whose tangent plane coincides with the plane corresponding to it in the osculating linear complex. Its intersection with $P_y P_\sigma$, the generator of the derived ruled surface of the third kind, is

$$(61) \qquad \gamma = 3(P_3-P_2')y + \sigma.$$

The generator of the developable joins $P_\beta$ to $P_\gamma$. We wish to determine its edge of regression. If

$$\delta = l\beta + m\gamma$$

is the point where $P_\beta P_\gamma$ meets the edge of regression, we must have

$$\delta' = r\beta + s\gamma, \qquad \text{or} \qquad l\beta' + m\gamma' = \bar{r}\beta + \bar{s}\gamma.$$

We proceed to determine the ratio of $l$ to $m$. We find

$$\beta' = (3P_2' - 2p_1P_2 - P_3)y - p_1\rho + \sigma,$$
$$\gamma' = -[P_4 - 4P_3' + 3P_2'' + 3p_1(P_3-P_2')]y - 3P_2\rho - p_1\sigma.$$

We may eliminate $\rho$ and $\sigma$ by (52) and (61). This gives

$$\beta' = -4\theta_3 y - p_1\beta + \gamma,$$
$$\gamma' = -(P_4 - 4P_3' + 3P_2'' - 6P_2^2)y - 3P_2\beta - p_1\gamma.$$

We may therefore put

$$l = P_4 - 4P_3' + 3P_2'' - 6P_2^2, \qquad m = -4\theta_3;$$

so that

$$(62) \qquad \delta = (P_4 - 4P_3' + 3P_2'' - 6P_2^2)\beta - 4\theta_3\gamma$$

gives the edge of regression. This gives the following theorem:

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*The developable, generated by the plane of the generators of the derived ruled surfaces of the second and third kind, has its edge of regression upon the derived ruled surface of the second kind, if, and only if, the curve $C_y$ belongs to a linear complex.*

We may write in place of (51)

$$\gamma = 60(P_3 - P_2')y + 20\sigma.$$

We have from (59)

$$\bar{\mu} = 12(P_3 - P_2')y + 20\sigma.$$

Therefore, the cross-ratio of the four points $P_\gamma$, $P_y$, $P_{\bar{\mu}}$ and $P_\sigma$ is

$$(\gamma, y, \bar{\mu}, \sigma) = 5.$$

We have found finally a geometrical definition for $P_\sigma$, which we may recapitulate as follows. *The plane of the tangent and the generator of the third derived ruled surface intersects the osculating cubic in $P_y$ counted twice and one other point. If the latter point be joined to $P_z$ by a straight line we obtain a certain point $P_{\bar{\mu}}$ as the intersection of this line with $P_yP_\sigma$. The generator of the developable, generated by the plane of the generators of the derived ruled surfaces of the second and third kind, intersects $P_yP_\sigma$ in another point $P_\gamma$. $P_\sigma$ may now be found as that point of $P_yP_\sigma$ which makes the cross-ratio*

$$(P_\gamma, P_y, P_{\bar{\mu}}, P_\sigma) = 5.$$

We have shown how to construct the fundamental tetrahedron when $P_z$ is given. If $P_\rho$ is given, $P_z$ can be found at once as the pole of $P_yP_\rho$ with respect to the osculating conic. If $P_\sigma$ is given we may find first its polar plane with respect to the osculating linear complex which is

$$-x_1 + 2P_2x_3 = 0,$$

and therefore passes through $P_z$, but not through $P_y$. $P_z$ can therefore be found at once at the intersection of this plane with the tangent to $C_y$ at $P_y$.

We see therefore, that *any one of the three points $P_z$, $P_\rho$, $P_\sigma$ determines uniquely the others.*

## § 6. *Some further properties of the derived ruled surfaces of the second and third kind.*

Let us suppose $p_1 = P_2 = 0$, so that the derived ruled surface of the third kind is a developable, and let us consider the derived ruled surface of the second kind which corresponds to it. We proceed to deduce the equation of its osculating linear complex.

Let $Y$ and $R$ denote the developments of $y$ and $\rho$ in the vicinity of the ordi-

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nary point $x = a$, and replace again $x - a$ by $x$ in the developments. Then we shall find

$$Y = y_1 y + y_2 z + y_3 \rho + y_4 \sigma, \qquad R = \rho_1 y + \rho_2 z + \rho_3 \rho + \rho_4 \sigma,$$

where

$$(63)\quad \begin{aligned} \rho_1 &= -P_3 x - \tfrac{1}{2} P_4 x^2 - \tfrac{1}{6} P_4' x^3 + \tfrac{1}{24}(4P_3^2 - P_4'') x^4 + \cdots, \\ \rho_2 &= -2P_3 x^2 - \tfrac{1}{6}(4P_3' + P_4) x^3 - \tfrac{1}{12}(2P_3'' + P_4') x^4 + \cdots, \\ \rho_3 &= 1 - \tfrac{2}{3} P_3 x^3 - \tfrac{1}{24}(8P_3' + P_4) x^4 + \cdots, \\ \rho_4 &= x - \tfrac{1}{6} P_3 x^4 + \cdots, \end{aligned}$$

while $y_1, \cdots, y_4$ have been computed before. Denote by

$$\omega_{ik} = y_i \rho_k - y_k \rho_i$$

the Plückerian coördinates of the line joining the two points. We find

$$\begin{aligned} \omega_{12} &= -P_3 x^2 - \tfrac{1}{3}(2P_3' - P_4) x^3 - \tfrac{1}{12}(2P_3'' - P_4') x^4 + \cdots, \\ \omega_{13} &= 1 - \tfrac{1}{3} P_3 x^3 - \tfrac{1}{6}(2P_3' - P_4) x^4 + \cdots, \\ \omega_{14} &= x - \tfrac{1}{6} P_3 x^4 + \cdots, \\ \omega_{23} &= x + \tfrac{1}{6} P_3 x^4 + \cdots, \\ \omega_{24} &= x^2 + \cdots, \qquad \omega_{34} = \tfrac{1}{3} x^3 + \cdots. \end{aligned}$$

Let

$$a\omega_{12} + b\omega_{12} + c\omega_{14} + d\omega_{23} + e\omega_{24} + f\omega_{34} = 0$$

be the equation of the osculating linear complex of the surface in question. Then, the coefficients of all powers of $x$ up to and including $x^4$ must be zero, if we substitute the above developments of $\omega_{ik}$ into the left member. This gives us the following equations:

$$\begin{aligned} b = 0, \qquad c + d = 0, \qquad -P_3 a + e = 0, \\ -\tfrac{1}{3}(2P_3' - P_4) a - \tfrac{1}{3} P_3 b + \tfrac{1}{3} f = 0, \\ -\tfrac{1}{12}(2P_3'' - P_4') a - \tfrac{1}{6}(2P_3' - P_4) b - \tfrac{1}{6} P_3 c + \tfrac{1}{6} P_3 d = 0, \end{aligned}$$

whence the ratios of the coefficients may be easily deduced.

*We find thus the equation of the linear complex osculating the derived ruled surface of the second kind which corresponds to a developable of the third kind; it is*

$$(64)\quad -4P_3\omega_{12} - (P_4' - 2P_3'')(\omega_{14} - \omega_{23}) - 4P_3^2\omega_{24} + 4(P_4 - 2P_3')P_3\omega_{34} = 0.$$

It coincides with the osculating linear complex of $C_y$ if and only if $P_3 = 0$, i. e., if $C_y$ belongs to a linear complex. This result is also obvious for geometrical reasons.

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The coördinates $v_k$ of the plane, which corresponds to a point $x_1, x_2, x_3, x_4$ in the linear complex (64), are given by

$$
(65)\qquad
\begin{aligned}
\omega v_1 &= 4P_3x_2 + (P_4' - 2P_3'')x_4,\\
\omega v_2 &= -4P_3x_1 - (P_4' - 2P_3'')x_3 + 4P_3^2x_4,\\
\omega v_3 &= (P_4' - 2P_3'')x_2 - 4P_3(P_4 - 2P_3')x_4,\\
\omega v_4 &= -(P_4' - 2P_3'')x_1 - 4P_3^2x_2 + 4P_3(P_4 - 2P_3')x_3,
\end{aligned}
$$

where $\omega$ is a proportionality factor.

Let us consider at the same time the osculating linear complex of $C_y$. The lines common to the two complexes form a congruence whose directrices we propose to find. This we can do quite easily by writing down the equations which express that, for a point on one of the directrices, the two planes corresponding to it in the two complexes must coincide. The right members of (65) must, for such a point be equal to

$$-\omega x_4, \qquad +\omega x_3, \qquad -\omega x_2, \qquad +\omega x_1$$

respectively, where $\omega$ is a proportionality factor.

The four equations obtained in this way can be satisfied only if their determinant vanishes, which gives

$$(66)\qquad 2P_3'' - P_4' - \omega = \pm 4P_3\sqrt{2P_3' - P_4},$$

whence the following equations for the two directrices

$$
(67)\qquad
\begin{aligned}
\pm\sqrt{2P_3' - P_4}\,x_1 - P_3x_2 - (2P_3' - P_4)x_3 &= 0,\\
-x_2 \pm \sqrt{2P_3' - P_4}\,x_4 &= 0,\\
-x_1 \pm \sqrt{2P_3' - P_4}\,x_3 + P_3x_4 &= 0,
\end{aligned}
$$

of which three equations only two are independent, and where we have assumed $P_3 \neq 0$. In fact, if $P_3$ were zero the two complexes would coincide and the congruence would be indeterminate.

Since we have assumed $P_2 = 0$, the quantity under the square root is $-\theta_4$. We find a second interpretation for the condition $\theta_4 = 0$. *If $\theta_4 = 0$, the congruence has coincident directrices.* We may combine this with our former result to the following theorem.

*Choose as derived ruled surface of the third kind one of the developables of the single family which exists. Consider the osculating linear complex of the corresponding ruled surface $S$ of the second kind. Let the directrices of the congruence, which this complex has in common with the osculating linear complex of the fundamental curve $C_y$, coincide. Then the two branches of the*

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*flecnode curve of $S$ coincide and the generator of $S$ which passes through $P_y$ will intersect the osculating conic of $C_y$ in $P_y$, and a second point whose locus is the flecnode curve of the surface $S$.*

I refrain from formulating the converse. The above conditions are fulfilled if and only if $\theta_4 = 0$.

Let us consider one of the directrices (67), for example that one which corresponds to the plus sign of the square root. Then we see that

$$(68)\qquad \begin{aligned} \alpha &= P_3\sqrt{2P_3' - P_4}\,y + P_3\rho, \\ \beta &= \sqrt{2P_3' - P_4}\,(P_3 y + \sqrt{2P_3' - P_4}\,z + \sigma) \end{aligned}$$

are two points on the directrix. We have multiplied each expression by a factor so as to have $\alpha$ and $\beta$ of the same weight. If now we change the independent variable, but in such a way as not to disturb the condition $P_2 = 0$, we shall get in (64) a single infinity of complexes, and in (67) two families of lines, the directrices of the single infinity of congruences which thus result. We are going to study to some extent the two ruled surfaces thus generated.

Put

$$(69)\qquad k = \sqrt{2P_3'' - P_4}.$$

Making the transformations which preserve $P_2 = 0$, for which we must have

$$\mu = \eta' - \tfrac{1}{2}\eta^2 = 0,$$

we find that $\alpha$ and $\beta$ are transformed into $\bar\alpha$ and $\bar\beta$, where

$$\begin{aligned} (\xi')^5\bar\alpha &= \alpha + 2\eta P_3 z + \tfrac{3}{2}\eta^2 P_3 y, \\ (\xi')^5\bar\beta &= \beta + \tfrac{3}{2}\frac{k}{P_3}\eta\alpha + k(\tfrac{3}{2}\eta^2 z + \tfrac{3}{4}\eta^3 y). \end{aligned}$$

The point $m\bar\alpha + n\bar\beta$ will be an arbitrary point on the line joining $P_{\bar\alpha}P_{\bar\beta}$. We find therefore, the equations of our surface referred to two parameters $\eta$ and $m/n$,

$$(70)\qquad \begin{aligned} x_1 &= \tfrac{3}{2}mP_3\eta^2 + (m+n)P_3 k + n(\tfrac{3}{2}\eta k^2 + \tfrac{3}{4}\eta^3 k), \\ x_2 &= 2m\eta P_3 + n(k^2 + \tfrac{3}{2}\eta^2 k), \\ x_3 &= mP_3 + \tfrac{3}{2}kn\eta, \\ x_4 &= nk. \end{aligned}$$

The nature of this surface may be easily determined. We have, returning to the two curves $C_\alpha$ and $C_\beta$ upon it, for $C_\alpha$

$$\alpha_1 = k + \tfrac{3}{2}\eta^2, \qquad \alpha_2 = 2\eta, \qquad \alpha_3 = 1, \qquad \alpha_4 = 0,$$

if we put $n = 0$, $m = 1/P_3$ in (70). If we put $m = 0$, $n = 1/k$ we find for $C_\beta$

$$\beta_1 = P_3 + \tfrac{3}{2}k\eta + \tfrac{3}{4}\eta^3, \qquad \beta_2 = k + \tfrac{3}{2}\eta^2, \qquad \beta_3 = \tfrac{3}{2}\eta, \qquad \beta_4 = 1.$$

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But $(\alpha_k, \beta_k)$ are simultaneous solutions of the equation

$$\frac{d\beta}{d\eta} = \tfrac{3}{2}\alpha,$$

which proves that *the ruled surface which we are considering is a developable whose edge of regression is the twisted cubic* $C_\beta$.

The curve $C_\alpha$ is a conic, the intersection of the developable of the cubic with the osculating plane of $C_y$. Its equations are

$$x_4 = 0, \qquad -3x_2^2 + 8x_1x_3 - 8kx_3^2 = 0.$$

We notice that for $\theta_4 = 0$ *it coincides with the osculating conic*, a further interpretation of this condition. *In general, the two conics have a contact of third order at* $P_y$.

If, in these equations, we change $k$ into $-k$ we obtain the developable, cubic and conic associated with the second directrix of our congruence. A considerable number of other configurations are suggested by the combinations of these various curves and surfaces. I will refrain, however, from any further study in this direction.

The curve $C_y$ is an asymptotic curve upon every derived ruled surface of the second kind. Moreover, the most general derived ruled surface of the second kind depends upon one arbitrary function, as does also the most general ruled surface containing $C_y$ as an asymptotic curve. It is easy to see that the derived ruled surface of the second kind may be made to coincide with any ruled surface upon which $C_y$ is an asymptotic curve if the independent variable be properly chosen.

Upon the derived ruled surface of the third kind, $C_y$ can never be an asymptotic curve. It may, however, be one branch of the flecnode curve. In fact, if we form the quantities $u_{ik}$ of the theory of ruled surfaces for system (27), we find

$$u_{12} = -\frac{P_2'}{P_2^2}.$$

But $u_{12} = 0$ is the condition that $C_y$ may be a branch of the flecnode curve on the surface generated by $P_yP_\sigma$. Suppose that the variable has been so chosen as to make $P_2' = 0$. The most general transformation which is possible, leaving this relation invariant, must, according to (11) satisfy the condition

$$-2\eta P_2 - \tfrac{5}{6}\mu' + \tfrac{5}{3}\eta\mu = 0,$$

or

$$-5\eta'' - 12\eta P_2 + 15\eta\eta' - 5\eta^3 = 0, \tag{71}$$

a differential equation of the second order for $\eta$. Moreover two different solutions of this equation always give rise to two distinct ruled surfaces. For, let

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$\eta_1$ and $\eta_2$ be two such solutions, and let $\sigma_1$, $\sigma_2$ be the corresponding values of $\sigma$. Then, according to (9),

$$\sigma_\kappa = \frac{1}{(\xi_\kappa')^3}\left[\sigma + \tfrac{3}{2}\eta_\kappa\rho + (\mu_\kappa + \tfrac{3}{2}\eta_\kappa^2)z + \tfrac{1}{4}(\mu_\kappa' + 4\eta_\kappa\mu_\kappa + 3\eta_\kappa^3)y\right] \quad (\kappa = 1, 2),$$

But if the same ruled surface corresponds to $\eta_1$ and $\eta_2$, the three points $y$, $\sigma_1$ and $\sigma_2$ must be collinear. We must therefore be able to reduce

$$(\xi_1')^3\sigma_1 - (\xi_2')^3\sigma_2 = \tfrac{3}{2}(\eta_1 - \eta_2)\rho + \cdots$$

to a multiple of $y$. But this is clearly only possible if $\eta_2 = \eta_1$.

We see, therefore, that *there are $\infty^2$ derived ruled surfaces of the third kind upon which $C_y$ is one branch of the flecnode curve.*

If $P_2'$ is not zero, our problem leads to the differential equation

$$\frac{d}{dx}(\eta' - \tfrac{1}{2}\eta^2) = \tfrac{6}{5}P_2' - \tfrac{12}{5}P_2\eta + 2\eta(\eta' - \tfrac{1}{2}\eta^2), \tag{72}$$

which is of the second order and third degree.

I have shown in a former paper that the most general ruled surface which has $C_y$ as one branch of its flecnode curve contains an arbitrary function in its general expression.* I have also shown that together with any such surface, its flecnode surface and each member of a single infinity of surfaces determined by these two, also contains $C_y$ as one branch of its flecnode curve. One might imagine that there could be based upon these theorems a transformation theory of equation (72). This is not the case however. For, if one of the surfaces containing $C_y$ as a branch of its flecnode curve is a derived ruled surface of the third kind, its flecnode surface is not, nor is any member of the family of ruled surfaces just mentioned.

Corresponding to the $\infty^2$ solutions of (72) or of

$$\mu' = \tfrac{6}{5}P_2' - \tfrac{12}{5}P_2\eta + 2\eta\mu,$$

we find $\infty^2$ positions for $P_{\bar{\sigma}}$, viz.:

$$4(\xi')^3\bar{\sigma} = (\tfrac{6}{5}P_2' - \tfrac{12}{5}P_2\eta + 6\eta\mu + 3\eta^3)y + (4\mu + 6\eta^2)z + 6\eta\rho + 4\sigma.$$

The locus of these points is a cubic surface

$$27P_2'x_4^3 - 36P_2x_3x_4^2 + 90x_2x_3x_4 - 90x_1x_4^2 - 40x_3^3 = 0, \tag{73}$$

which contains $P_yP_s$, the tangent of $C_y$ as a double line. It is therefore a ruled surface. It is in fact a CAYLEY's cubic scroll.

*If one derived surface of the third kind is known upon which $C_y$ is a branch*

* *General theory of curves on ruled surfaces*, Transactions of the American Mathematical Society, vol. 6 (1905), p. 78.

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*of the flecnode curve, two others may be found by merely solving a quadratic equation.*

In fact, suppose that a solution $\eta$ of (72) be known. We may make a transformation of the independent variables, $\xi = \xi(x)$ such that

$$\frac{\xi''}{\xi'} = \eta.$$

In the resulting equation $P_2' = 0$. If we again denote the independent variable by $x$, (72) becomes

$$\eta'' - \eta\eta' = -\tfrac{12}{5}P_2\eta + 2\eta(\eta' - \tfrac{1}{2}\eta^2),$$

where $P_2$ is a constant, since $P_2' = 0$. But we may satisfy this equation by putting $\eta =$ const., which gives the equation

$$\eta^3 + \tfrac{12}{5}P_2\eta = 0,$$

whence

$$\eta = 0, \qquad \pm\sqrt{-\tfrac{12}{5}P_2}.$$

The root $\eta = 0$ gives the original solution. The other two are new.

## §7. *The principal tangent plane of two space curves. The covariants. Transition to Halphen's investigations.*

HALPHEN has introduced a very important notion, which we shall now proceed to explain.

Let there be given two space curves having at a point $P$ a contact of the $n$th order. If these curves be projected from any center $Q$ upon a plane, the projections will also have, in general, a contact of the $n$th order at the point corresponding to $P$. HALPHEN shows that there exists a plane passing through the common tangent of the two curves such that if the center of projection be taken anywhere within it, the contact of the projections will be of an order higher than $n$. This plane he calls the *principal tangent plane* of the two curves.*

We shall follow HALPHEN in determining the principal tangent plane at $P_y$ of the curve $C_y$ and its osculating cubic. This will lead us to an especially simple form for the development of the equations of the curve, which is also due to HALPHEN and on the basis of which he draws his further conclusions. It will also enable us to substitute for our system of covariants $C_2$, $C_3$, $C_4$ another system whose geometrical significance will be apparent, and in terms of which $C_2$, $C_3$, $C_4$ may be expressed.

* HALPHEN, *Sur les invariants différentiels des courbes gauches*, Journal de l'Ecole Polytechnique, vol. 47 (1880), p. 25.

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Assuming $P_2 = 0$, the equations of the osculating cubic, referred to the tetrahedron $P_y P_s P_\rho P_\sigma$, are

$$3x_2x_4 - 2x_3^2 = 0, \qquad 2x_3(5x_1 - 3P_3x_4) - 5x_2^2 = 0.$$

Let us put

$$\bar{x}_1 = x_2, \qquad \bar{x}_2 = \tfrac{2}{5}x_3, \qquad \bar{x}_3 = \tfrac{6}{25}x_4, \qquad \bar{x}_4 = 5x_1 - 3P_3x_4, \tag{74}$$

and

$$x = \frac{\bar{x}_1}{\bar{x}_4}, \qquad y = \frac{\bar{x}_2}{\bar{x}_4}, \qquad z = \frac{\bar{x}_3}{\bar{x}_4}, \tag{75}$$

Then the equations of the cubic reduce to

$$y = x^2, \qquad z = x^3. \tag{76}$$

The relation of the new tetrahedron of reference to the cubic is quite simple. The plane $\bar{x}_3 = 0$ is the osculating plane at $P$; $\bar{x}_2 = 0$ is some other plane through the tangent; this plane intersects the cubic in another point $Q$; the plane tangent to the cubic at $Q$ and passing through $P$ is $\bar{x}_1 = 0$; the osculating plane at $Q$ is $\bar{x}_4 = 0$.

Since $\bar{x}_2 = 0$ may be chosen in an infinity of ways we see that the reduction of the equations of a space cubic to the form (76) may be accomplished in an infinity of ways.

For the curve $C_y$ we have

$$\begin{aligned}
y_1 &= 1 - \frac{P_3}{3!}x^3 - \frac{P_4}{4!}x^4 - \frac{P_4'}{5!}x^5 - \frac{P_4'' - 4P_3^2}{6!}x^6 \\
&\qquad - \frac{P_4^{(3)} - 5P_3P_4 - 12P_3P_3'}{7!}x^7 + \cdots, \\
y_2 &= x - \frac{P_3}{3!}x^4 - \frac{P_4 + 4P_3'}{5!}x^5 - \frac{2P_4' + 4P_3''}{6!}x^6 \\
&\qquad + \frac{16P_3^2 - 3P_4'' - 4P_3^{(3)}}{7!}x^7 + \cdots, \\
y_3 &= \tfrac{1}{2}x^2 - \frac{4P_3}{5!}x^5 - \frac{1}{6!}(P_4 + 8P_3')x^6 - \frac{1}{7!}(3P_4' + 12P_3'')x^7 + \cdots, \\
y_4 &= \tfrac{1}{6}x^3 - \frac{4}{6!}P_3x^6 - \frac{1}{7!}(P_4 + 12P_3')x^7 \cdots.
\end{aligned} \tag{77}$$

If we refer it to the same tetrahedron to which we have just referred the osculating cubic, and if we put

$$\xi = \frac{\bar{y}_1}{\bar{y}_4}, \qquad \eta = \frac{\bar{y}_2}{\bar{y}_4}, \qquad \zeta = \frac{\bar{y}_3}{\bar{y}_4},$$

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we shall find

$$(78)\qquad \begin{aligned} \eta &= \xi^2 + \frac{125}{36}(8P_3' - 5P_4)\xi^6 + \frac{625}{7.36}(8P_3'' - 5P_4')\xi^7 + \cdots, \\ \zeta &= \xi^3 - \frac{25}{3}P_3\xi^6 + \frac{125}{7.12}(36P_3' - 25P_4)\xi^7 + \cdots. \end{aligned}$$

If we put

$$(79)\qquad \xi = \epsilon X, \qquad \eta = \epsilon^2 Y, \qquad \zeta = \epsilon^3 Z, \qquad \epsilon^3 = -\frac{3}{25P_3},$$

assuming, therefore, that $P_3$ is not zero, this becomes

$$(80)\qquad \begin{aligned} Y &= X^2 + \lambda_6 X^6 + \lambda_7 X^7 + \cdots, \\ Z &= X^3 + \mu_6 X^6 + \mu_7 X^7 + \cdots, \end{aligned}$$

where

$$(81)\qquad \begin{aligned} \lambda_6 &= \tfrac{125}{36}(8P_3' - 5P_4)\epsilon^4, & \lambda_7 &= \frac{625}{7.36}(8P_3'' - 5P_4')\epsilon^5, \\ \mu_6 &= 1, & \mu_7 &= \frac{125}{7.12}(36P_3' - 25P_4)\epsilon^4. \end{aligned}$$

But we can make a further change of coördinates without disturbing the form (80) of the development. For the plane $\bar{x}_2 = 0$ may be taken to be any plane through the tangent. It may be taken in such a way as to make $\lambda_6$ vanish, as we shall presently see. The plane thus obtained will obviously be the principal tangent plane of the curve $C_y$ and its osculating cubic.

Instead of working out the transformation geometrically we shall put HALPHEN

$$(82a)\qquad \bar{X} = \frac{\xi_1}{\omega_1}, \qquad \bar{Y} = \frac{\eta_1}{\omega_1}, \qquad \bar{Z} = \frac{Z_1}{\omega_1},$$

where

$$(82b)\qquad \begin{aligned} \omega_1 &= 1 + 3pX + 3p^2Y + p^3Z, \qquad \eta_1 = Y + pZ, \\ \xi_1 &= X + 2pY + p^2Z, \end{aligned}$$

the quantity $p$ being arbitrary. We shall find with him

$$(83)\qquad \begin{aligned} \bar{Y} &= \bar{X}^2 + \Lambda_6\bar{X}^6 + \Lambda_7\bar{X}^7 + \cdots, \\ \bar{Z} &= \bar{X}^3 + M_6\bar{X}^6 + M_7\bar{X}^7 + \cdots, \end{aligned}$$

as the result of applying this transformation to any system of equations of the form (80). The coefficients of (83) are

$$(84)\qquad \begin{aligned} M_6 &= \mu_6, & M_7 &= \mu_7 + 3p\mu_6, \\ \Lambda_6 &= \lambda_6 + p\mu_6, & \Lambda_7 &= \lambda_7 + p\mu_7 + p(p\mu_7 - \lambda_6). \end{aligned}$$

For

$$p = -\frac{\lambda_6}{\mu_6},$$

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this reduces to the canonical form required, in which we may also assume $\mu_6 = 1$ if the curve does not belong to a linear complex.

If we combine the various transformations which we have made successively we get the following result: *If we introduce non-homogeneous coördinates by putting*

$$X = \frac{\bar{y}_1}{\bar{y}_4}, \qquad Y = \frac{\bar{y}_2}{\bar{y}_4}, \qquad Z = \frac{\bar{y}_3}{\bar{y}_4}, \tag{85a}$$

*where*

$$\begin{aligned} \bar{y}_1 &= 25\epsilon^2 y_2 + 20p\epsilon y_3 + 6p^2 y_4, \\ \bar{y}_2 &= 10\epsilon y_3 + 6p y_4, \\ \bar{y}_3 &= 6y_4, \\ \bar{y}_4 &= 125\epsilon^3 y_1 + 75p\epsilon^2 y_2 + 30p^2\epsilon y_3 + (6p^3 - 75P_3\epsilon^3) y_4, \end{aligned} \tag{85b}$$

*and where*

$$\epsilon^3 = -\frac{3}{25P_3}, \qquad p = -\tfrac{125}{36}(8P_3' - 5P_4)\epsilon^4, \tag{85c}$$

*the development of the equations of the curve $C_y$ may be written in the canonical form*

$$\begin{aligned} Y &= X^2 + \Lambda_7 X^7 + \cdots, \\ Z &= X^3 + X^6 + M_7 X^7 + \cdots, \end{aligned} \tag{86}$$

*where $\Lambda_7, M_7, \cdots$ are given by* (84) *together with* (81). *This transformation is valid if $C_y$ does not belong to a linear complex.*

The coefficients of (86) are absolute invariants. We find, in fact,

$$\Lambda_7 = \frac{625\epsilon^5}{7.12\cdot 36\theta_3}(25\theta_4^2 + 20\theta_8 - 4\theta_{3.1}), \qquad M_7 = \frac{625\epsilon^4}{42}\theta_4, \tag{87}$$

where

$$\theta_8 = 4\theta_4\theta_3' - 3\theta_3\theta_4'. \tag{88}$$

From (86) and (76) it is clear that the plane $Y = 0$ or $\bar{y}_2 = 0$ is the principal plane of the curve $C_y$ and its osculating cubic. In the original system of coördinates its equation will, therefore, be

$$4P_3x_3 + (8P_3' - 5P_4)x_4 = 0. \tag{89}$$

*If $C_y$ belongs to a linear complex, it coincides with the osculating plane.*

The point which corresponds to (89) in the osculating linear complex is

$$(5P_4 - 8P_3')y + 4P_3 z, \tag{90}$$

or in invariant form

$$(\theta_3' + \tfrac{5}{2}\theta_4)y + 2\theta_3 z, \tag{91}$$



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an expression whose covariance may be verified directly. We shall call this point the *principal point of the tangent.* We thus obtain a curve on the developable of $C_y$ which may be called *its principal curve.*

If we make a transformation such that

$$3\eta = \theta_3' + \tfrac{5}{2}\theta_4, \tag{92}$$

the point $P_{\bar{z}}$ describes the principal curve on the developable of $C_y$. The points $P_{\bar{\rho}}$ and $P_{\bar{\sigma}}$ also describe perfectly definite curves, whose expressions may be obtained from (9) by substituting for $\eta$ the expression (92). *Any covariant may be expressed in terms of $y$ and the three which we have just determined, which may, therefore, serve to replace the covariants $C_2$, $C_3$ and $C_4$.*

If $C_y$ belongs to a linear complex these four curves all coincide, so that a different set of fundamental covariants must then be selected.

In this exceptional case our fundamental tetrahedron $P_y P_z P_\rho P_\sigma$ gives rise to a most remarkable configuration. If we put again $P_2 = 0$, we have in this case also $P_3 = 0$. $P_y P_\sigma$ generates one of the developables of the third kind of which $C_\sigma$ is the cuspidal edge, while $P_y P_z$ of course generates the developable of which $C_y$ is the cuspidal edge. The surface generated by $P_y P_\rho$ is not developable. Its equations become

$$\rho'' + P_4 y = 0, \qquad y'' - \rho = 0.$$

It belongs to the same linear complex as $C_y$, and $C_y$ and $C_\rho$ are the two branches of its complex curve which is at the same time an asymptotic curve. $P_z P_\rho$ generates a developable, since $z' = \rho$, of which $C_z$ is the cuspidal edge. $P_\rho P_\sigma$ generates a developable, since $\rho' = \sigma$, of which $C_\rho$ is the cuspidal edge. Finally $P_z P_\sigma$ generates a ruled surface whose equations are

$$\sigma'' - \frac{P_4'}{P_4}\sigma' + P_4 z = 0, \qquad z'' - \sigma = 0,$$

upon which $C_z$ and $C_\sigma$ are asymptotic lines.

We find, therefore, the following theorem.

*If the curve belongs to a linear complex, we may, in an infinity of ways, choose the fundamental tetrahedron so that four of its edges give rise to developables whose cuspidal edges are described by the four vertices. The other two edges of the tetrahedron will then give rise to ruled surfaces upon each of which the vertices of the tetrahedron trace a pair of asymptotic curves. The latter coincide with the two branches of the complex curve for the derived surface of the second kind.*

It does not seem convenient to develop the geometrical theory of the covariants for this case at the present moment, as it will appear naturally in connec-

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tion with some further investigations which I have not yet completed, and which I hope to present on some future occasion.

Those results which HALPHEN has deduced for the general case, and which have not been mentioned in this paper might now be directly transcribed from his memoir. For we have connected our own theory with his by obtaining the canonical form (86) for the development of the equations of the curve. In the case $\theta_3 = 0$, a different canonical form must be used, which has also been indicated by HALPHEN, but not extensively employed. The discussion of this we shall leave for a later occasion together with the investigations mentioned above.

CAMBRIDGE, ENGLAND,
*June* 20, 1904.

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# SUR LES OPÉRATIONS LINÉAIRES*

## (DEUXIÈME NOTE†)

PAR

MAURICE FRÉCHET

*Nouvelles représentations d'une opération linéaire.*

*Remarque sur le théorème de M. Hadamard.* D'après ce théorème, toute opération linéaire $U_f$ portant sur une fonction $f(x)$ continue entre $a$ et $b$ peut se mettre sous la forme

$$U_f = \lim_{n=\infty} \int_a^b K_n(y) f(y)\, dy,$$

où les $K_n(y)$ sont des fonctions continues entre $a$ et $b$. Il nous sera utile d'observer qu'on peut même supposer que les fonctions $K_n$ soient nulles en $a$ et $b$. Il suffit de remplacer $K_n(y)$ par $K_n(y) + T_n(y)$ en appelant $T_n(y)$ une fonction continue nulle de $a + \epsilon_n$ à $b - \epsilon_n$ et égale à

$$\frac{y - a - \epsilon_n}{\epsilon_n} K_n(a) \qquad \text{et} \qquad \frac{b - \epsilon_n - y}{\epsilon_n} K_n(b)$$

dans les intervalles respectifs $(a,\ a + \epsilon_n)$, $(b - \epsilon_n,\ b)$. En prenant, par exemple

$$\epsilon_n = \frac{1}{n(|K_n(a)| + 1)(|K_n(b)| + 1)},$$

on voit que l'on a

$$\left| \int_{b-\epsilon_n}^b \frac{b\epsilon_n - y}{\epsilon_n} K_n(b)\, dy \right| = \frac{|K_n(b)|}{2n(|K_n(a)| + 1)(|K_n(b)| + 1)} < \frac{1}{2n}.$$

Par suite, l'intégrale

$$\int_{b-\epsilon_n}^b T_n(y)\, dy$$

tend vers 0 avec $1/n$ et il en est de même de

$$\int_a^b T_n(y)\, dy.$$

* Presented to the Society December 29, 1904. Received for publication December 2, 1904.

† Voir Transactions of the American Mathematical Society, vol. 5 (1905), pp. 493–499. Nous conservons ici les mêmes notations.



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*On peut aussi supposer que les $K_n(y)$ soient des polynomes.* En effet puisque ce sont des fonctions continues, on peut trouver pour chaque valeur de $n$ un polynome $P_n(y)$ tel que l'on ait, entre $a$ et $b$,

$$|P_n(y) - K_n(y)| < \frac{1}{n}.$$

Et l'on aura

$$\lim_{n=\infty} \int_a^b P_n(y) f(y)\, dy = \lim_{n=\infty} \int_a^b K_n(y) f(y)\, dy = U_f.$$

Mais on ne pourra pas toujours supposer en même temps que $P_n(y)$ soit nul en $a$ et $b$.

La remarque précédente nous fournit un nouveau développement de l'opération linéaire la plus générale portant sur une fonction continue. En effet, posons

$$P_n(y) = a_0^n + a_1^n y + \cdots + a_{p_n}^n y^{p_n}.$$

On aura

$$U_f = \lim_{n=\infty} \left[ a_0^n U_f^{(0)} + a_1^n U_f^{(1)} + \cdots + a_{p_n}^n U_f^{(p_n)} \right], \tag{21}$$

avec

$$U_f^{(r)} = \int_a^b y^r f(y)\, dy. \tag{22}$$

On peut obtenir un autre développement en s'appuyant sur ce que toute fonction continue entre 0 et 1 peut être représentée par le développement uniformément convergent

$$f(y) = \lim_{n=\infty} \left[ R_{0,n}(y) f(0) + \cdots R_{r,n}(y) f\left(\frac{r}{n}\right) + \cdots + R_{n,n}(y) f\left(\frac{n}{n}\right) \right],$$

où les $R_{r,n}(y)$ sont certains polynomes déterminés indépendamment de la fonction $f$.* De sorte qu'en appliquant l'opération $U$ terme à terme dans le cas où $a = 0$, $b = 1$, l'on aura

$$U_f = \lim_{n=\infty} \left[ b_0^n f(0) + \cdots + b_r^n f\left(\frac{r}{n}\right) + \cdots + b_n^n f\left(\frac{n}{n}\right) \right], \tag{23}$$

les nombres

$$b_r^n = \int^b f(y) R_{r,n}(y)\, dy$$

restant indépendants de la fonction $f(y)$.

Cette représentation de l'opération $U_f$ est intéressante en ce qu'elle donne un procédé pour calculer effectivement $U_f$ connaissant seulement les valeurs de $f(y)$ en tous les points de l'ensemble $E$ des points d'abscisses commensurables.

* Voir E. Borel, *Leçons sur les fonctions de variables réelles*, p. 80, Gauthier-Villars, Paris, 1904.

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On pourrait d'ailleurs l'étendre au cas où $E$ est un ensemble dénombrable partout dense entre 0 et 1.

Mais les deux développements (21) et (23) ont l'inconvénient de ne pas être chacun seul de son espèce: autrement dit, les coefficients $a_r^n$, $b_r^n$ ne sont pas déterminés d'une façon unique par l'opération $U_f$. Ainsi, par exemple, la formule (23) subsisterait en remplaçant les $b_r^n$ par les quantités $b_r^n + d_r^n/n^2$ pourvu que les nombres arbitraires $d_r^n$ soient bornés dans leur ensemble.

Au contraire, *le développement* (4′)* *en série indéterminée n'est valable que pour une suite de valeurs* $u_0, u_1, \cdots$ *bien déterminées par l'opération* $U_f$, comme on s'en assure facilement.

*Décomposition d'une opération linéaire, correspondant à une décomposition de l'intervalle.*

Soit $c$ un nombre quelconque compris entre $a$ et $b$ et soient $\phi_1$ et $\phi_2$ deux fonctions continues définies de la façon suivante: $f(x)$ étant une fonction continue de $a$ à $b$ qui coïncide avec $f_1(x)$ de $a$ à $c$ et avec $f_2(x)$ de $c$ à $b$, nous prendrons

$$\phi_1(x) = f(x) - \tfrac{1}{2}f(c) \text{ de } a \text{ à } c, \qquad \phi_1(x) = \tfrac{1}{2}f(c) \text{ de } c \text{ à } b,$$

puis

$$\phi_2(x) = f(x) - \phi_1(x).$$

Alors, on aura

$$U_f = U_{\phi_1} + U_{\phi_2}.$$

Or la fonction continue $\phi_1$ est déterminée connaissant seulement la fonction $f_1(x)$ entre $a$ et $c$. Donc $U_{\phi_1}$ peut être considérée comme une opération fonctionnelle $V_{f_1}$ portant sur la fonction continue $f_1(x)$ définie entre $a$ et $c$. De plus, on voit facilement que cette opération $V_{f_1}$ est distributive et continue, c'est à dire linéaire. *On pourra ainsi écrire*

$$U_f \equiv V_{f_1} + W_{f_2}, \tag{24}$$

*$V$ et $W$ étant deux opérations linéaires définies respectivement dans les intervalles* $(a, c)$, $(c, b)$.

Étant donnée une telle décomposition, on peut en trouver immédiatement une infinité d'autres: il suffit de remplacer respectivement $V_{f_1}$ et $W_{f_2}$ par $V_{f_1} + Af_1(c)$ et $W_{f_2} - Af_2(c)$, $A$ étant une constante arbitraire. Il n'y a pas d'autres décompositions. Autrement dit, *si* $v_{f_1}$, $V_{f_1}$ *et* $w_{f_2}$, $W_{f_2}$ *sont des opérations linéaires définies respectivement dans les intervalles* $(a, c)$ *et* $(c, b)$ *et telles que l'on ait*

$$U_f \equiv v_{f_1} + w_{f_2} \equiv V_{f_1} + W_{f_2},$$

*on a nécessairement*

$$v_{f_1} \equiv V_{f_1} + Af(c), \qquad w_{f_2} \equiv W_{f_2} - Af(c),$$

*Transactions, loc. cit., p. 496.

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*A étant une constante arbitraire.* En effet, les deux membres de l'identité évidente

$$v_{f_1} - V_{f_1} = W_{f_2} - w_{f_2}$$

représentent une même opération qui ne varie pas quand on remplace $f_1$ et $f_2$ par deux fonctions continues quelconques ayant en $c$ la valeur $f(c)$. Par conséquent, cette opération est une fonction de $f(c)$, au sens ordinaire: soit, $G[f(c)]$. Et comme on doit avoir

$$G[f(c)] + G[F(c)] = G[f(c) + F(c)],$$

$G$ est nécessairement de la forme $Af(c)$.

Comme on a

$$U_f = \lim_{n=\infty} \int_a^b K_n(y) f(y)\, dy,$$

on serait tenté de dire que l'une des décompositions précédentes est fournie par les formules,

$$(25) \qquad v_{f_1} = \lim_{n=\infty} \int_a^c K_n f_1\, dy \qquad w_{f_2} = \lim_{n=\infty} \int_c^b K_n f_2\, dy.$$

Mais cette conclusion serait prématurée, car il n'est pas certain que les seconds membres de (25) soient convergents. Cependant, nous allons montrer qu'on peut toujours choisir les fonctions $K_n$ *continues de $a$ à $b$* de façon à ce qu'il en soit ainsi. En effet; soit

$$U_f = V_{f_1} + W_{f_2}$$

l'une des décompositions possibles de $U_f$. D'après notre première remarque, on peut trouver des fonctions $H_n(y)$, $L_n(y)$ respectivement continues dans $(a, c)$ et $(c, b)$ *mais nulles en $c$*, de façon que l'on ait

$$V_{f_1} = \lim_{n=\infty} \int_a^c H_n f_1\, dy, \qquad W_{f_2} = \lim_{n=\infty} \int_c^b L_n f_2\, dy.$$

Or la fonction $K_n(y)$ qui coïncide avec $H_n$ de $a$ à $c$ et avec $L_n$ de $c$ à $b$ *est continue de $a$ à $b$* et telle que l'on ait

$$U_f = \lim_{n=\infty} \int_a^b K_n f\, dy,$$

—ce qui démontre la proposition.

On généralise immédiatement ce qui précéde au cas où l'on diviserait l' intervalle $(a, b)$ en un nombre fini quelconque d'intervalles contigus.

Etant donnée une opération fonctionnelle *continue* $U_f$ quelconque portant sur la fonction $f(x)$ continue entre $a$ et $b$, on ne peut pas toujours la mettre sous la forme d'une somme de deux opérations *continues* $V_{f_1}$, $W_{f_2}$, déterminées quand on connait les valeurs $f_1(x)$ et $f_2(x)$ de $f(x)$ dans les intervalles $(a, c)$, $(c, b)$.

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Mais il ne faudrait pas croire cependant que les opérations *linéaires* sont les seules opérations continues qui jouissent de cette importante propriété. Soit par exemple, l'opération continue $U_f = [f(\alpha)]^2$ où $\alpha$ est un nombre fixe entre $a$ et $b$. Appelons $V_{f_1}$, l'opération continue égale à $f_1^2(\alpha)$ si $\alpha \leqq c$, nulle si $\alpha > c$ et $W_{f_2}$ l'opération continue égale à 0 dans le premier cas, à $f_2^2(\alpha)$ dans le second. On a $U_f = V_{f_1} + W_{f_2}$ et cependant l'opération $U_f$ n'est pas linéaire.

*Importance du champ fonctionnel dans lequel on définit une opération linéaire.*

Nous avons supposé jusqu' ici que les opérations linéaires sur lesquelles nous raisonnions faisaient correspondre un nombre $U_f$ à *toute* fonction $f(x)$ continue entre $a$ et $b$ et cela sans savoir si $U_f$ était définie pour d'autres fonctions. Cette hypothèse est essentielle, comme nous allons le voir. Tout d'abord, observons que l'on peut construire des opérations linéaires définies pour toute fonction $f(x)$ *uniforme* entre $a$ et $b$. Il suffit de prendre par exemple $U_f = f(a)$.

Pour donner un exemple plus général, considérons un ensemble dénombrable $E$ de points entre $a$ et $b$: $c_1, c_2, c_3, \cdots$ et soit

$$A_1 + A_2 + \cdots + A_n + \cdots$$

une série absolument convergente. L'opération

$$U_f = A_1 f(c_1) + \cdots + A_n f(c_n) + \cdots$$

sera définie et linéaire pour toute fonction $f(x)$ uniforme et bornée entre $a$ et $b$.

Or, *on ne peut étendre à de telles opérations le théorème de* M. HADAMARD. En effet son énoncé même suppose qu'on peut calculer l'intégrale

$$\int_a^b K_n f dy;$$

il faut pour cela que $f$ soit intégrable. Par conséquent, il ne peut avoir de sens que dans le champ des fonctions intégrables. Mais cette restriction n'est pas encore suffisante. *Étant donnée une opération linéaire $U_f$ définie pour toute fonction sommable* $f(x)$ entre $a$ et $b$, on ne peut pas toujours trouver une suite de fonctions mesurables et bornées $K_n(x)$ telles que l'on ait*

$$U_f = \lim_{n=\infty} \int_a^b K_n(y) f(y) dy. \tag{26}$$

(On voit que nous n'imposons même plus aux $K_n$ la condition d'être continues). Il suffit pour le voir de prendre par exemple $U_f = f(a)$. Si l'on applique la formule (26) à la fonction sommable $f(x)$ qui est égale à 1 pour $x = a$ et nulle partout ailleurs, on arrive à une impossibilité.

* Voir LEBESGUE, *Leçons sur l'intégration*, p. 115.

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D'ailleurs, il faut observer, inversement, que le théorème de M. HADAMARD n'est pas uniquement applicable dans le champ des fonctions continues; sa démonstration même est valable lorsque $f(x)$ n'a qu'un nombre fini de discontinuités de première espèce, et c'est dans ce cas général qu'il avait été énoncé par son auteur.

Un autre genre de difficultés se présente quand on considère des opérations linéaires qui ne sont pas définies dans tout le champ des fonctions continues, comme l'opération $U_f = f'_c$. On peut être ainsi amené à considérer ce que nous appellerons des *opérations linéaires d'ordre $n$*, c'est à dire des opérations distributives portant seulement sur les fonctions ayant leurs $n$ premières dérivées continues et telles que $U_f$ ait pour limite $U_\phi$ lorsque *$f$ et ses $n$ premières dérivées* tendent uniformément vers $\phi$ et ses $n$ premières dérivées.

Si $U_f$ est une telle opération, le théorème de M. Hadamard lui est applicable et on peut même mettre $U_f$ sous la forme suivante

$$U_f = A_0 f(a) + A_1 f'_a + \cdots + A_{n-1} f_a^{(n-1)} + \lim_{p=\infty} \int_a^b K_p f^{(n)} dy,$$

les $A_i$ désignant certaines constantes et les $K_p$ certaines fonctions continues de $a$ à $b$. Il suffit pour le voir, d'appliquer l'opération $U_f$ à chaque terme de l'identité

$$f(x) \equiv f(a) + (x-a) f'_a + \cdots + \frac{(x-a)^{n-1}}{(n-1)!} f_a^{(n-1)} + \int_a^x \cdots \int_a^x f^{(n)}(x)(dx)^n.$$

Au contraire, si l'on décompose $(a, b)$ en deux intervalles, le raisonnement qui nous a conduit à établir la formule (24) n'est plus valable, car les fonctions $\phi_1$ et $\phi_2$ ne sont pas dérivables jusqu'à l'ordre $n$. Mais le résultat est encore exact, si nous prenons de $a$ à $c$

$$\phi_1(x) = f(x) - \tfrac{1}{2}\left[ f(c) + (x-c) f'_c + \cdots + \frac{(x-c)^n}{n!} f_c^{(n)} \right]$$

et de $c$ à $b$

$$\phi_1(x) = \tfrac{1}{2}\left[ f(c) + (x-c) f'_c + \cdots + \frac{(x-c)^n}{n!} f_c^{(n)} \right];$$

puis $\phi_2 \equiv f - \phi_1$.

Seulement, si l'on a, entre des opérations linéaires d'ordre $n$, la relation

$$v_{f_1} + w_{f_2} \equiv V_{f_1} + W_{f_2},$$

on n'aura plus nécessairement

$$v_{f_1} - V_{f_1} = A f(c),$$

mais

$$v_{f_1} - V_{f_1} = A_0 f(c) + A_1 f'_c + \cdots + A_n f_c^{(n)},$$

$A_0, A_1, \cdots, A_n$ désignant certaines constantes.

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*Equations fonctionnelles.* Il peut arriver que l'on ait à chercher les fonctions $\theta(x)$ telles que $U_\theta$ prenne une valeur donnée A. La solution est bien simple. Si nous supposons en effet, l'opération $U$ non identiquement nulle dans le champ $H$ des fonctions auxquelles elle s'applique, il existe au moins une fonction $\psi(x)$ de ce champ, non identiquement nulle entre $a$ et $b$ et telle que $U_\psi$ soit différent de zéro. Il y a au moins une abscisse $c$ de l'intervalle $(a, b)$ qui n'annule pas $\psi(x)$. Ceci étant, il est extrêmement facile de voir qu'*on obtiendra toutes les racines $\theta(x)$ de l'équation $U_\theta = A$, chacune une fois et une seule, par la formule*

$$\theta(x) = f(x) + \frac{A - U_f}{U_\psi}\psi(x),$$

*où $f(x)$ est remplacée successivement par toutes les fonctions distinctes, nulles en $c$, du champ $H$.* Le raisonnement s'applique aussi bien quand le champ $H$ est celui des fonctions continues entre $a$ et $b$ qu'aux opérations linéaires d'ordre quelconque.

PARIS, *Novembre*, 1904.

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# SURFACES WHOSE GEODESICS MAY BE REPRESENTED IN THE PLANE BY PARABOLAS*

BY

EDWARD KASNER

The general problem of geodesic representation, whose preliminary discussion was given by the author in the Transactions for 1903,† is essentially the the same as what may be termed the *inverse problem of geodesics*, proposed by LIE ‡ as follows: Given the equation of the geodesics in the form $F(u, v, \lambda, \mu) = 0$, determine the corresponding surfaces.

For the existence of such a surface, it is necessary that the differential equation of given system should be of the form

$$y'' = Ay'^3 + By'^2 + Cy' + D, \tag{1}$$

where the coefficients are functions of $x$, $y$. We shall refer to any equation of this form as one of *cubic type*. Such equations are of interest independently of their connection with geodesics: the type is invariant with respect to arbitrary point transformation.

In order that (1) shall represent a system of possible geodesics, it is further necessary that the functions $A$, $B$, $C$, $D$ satisfy certain relations expressed by the consistency of a system of partial differential equations.§ For our purpose, it is not necessary to have these relations explicitly. We note merely that the equations of *geodesic type* constitute only a subclass of the general cubic class (1)—a subclass which is itself invariant under the group of all point transformations.

A more special subclass is formed by those equations of the second order which are equivalent, under point transformation, to $y'' = 0$. Such an equation may be characterized by the fact that its complete integral can be put into the form

$$f_1(x, y) + \text{const.}\, f_2(x, y) + \text{const.}\, f_3(x, y) = 0,$$

* Presented to the Society December 30, 1902. Received for publication February 25, 1905.

† *The generalized Beltrami problem concerning geodesic representation*, vol. 4, pp. 149–152.

‡ *Untersuchungen über geodetischen Curven*, Mathematische Annalen, vol. 20 (1882), in particular p. 445.

§ Cf. R. LIOUVILLE, *Sur la charactère auquel se reconnaît l'équation différentielle d'un système géodésique*, Comptes Rendus, vol. 108 (1889), p. 495.



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of the first degree in the two parameters. This subclass includes then all linear systems, in the general sense of the term. It is a part of the geodesic class, corresponding, by BELTRAMI's theorem, to the surfaces of constant curvature.* The necessary and sufficient conditions on the coefficients of (1), for such a system, are †

$$(BD - D_y)_y - (AD + \tfrac{1}{3}B_x - \tfrac{2}{3}C_y)_x + C(AD + \tfrac{1}{3}B_x - \tfrac{2}{3}C_y) - D(AC + A_x) = 0,$$

$$-(AC + A_x)_x + (AD + \tfrac{2}{3}B_x - \tfrac{1}{3}C_y)_y + B(AD + \tfrac{2}{3}B_x - \tfrac{1}{3}C_y) - A(BD - D_y) = 0.$$

In the present paper we consider problems of this kind: Given a system of plane curves involving three or more parameters, determine the two-parameter systems capable of representing the geodesics of a surface, and investigate the corresponding surfaces. A more general problem is the determination of all the integrals of form (1) satisfying a given differential equation of the third or higher order.‡ The only case that has received treatment is the case of circles; here the only systems of geodesic type are linear, so that the corresponding surfaces are merely those of constant curvature. §

The main part of the paper, §§ 1–7, is devoted to what seems to be the simplest case of our problem, namely, the three-parameter system of vertical parabolas

$$y = \lambda x^2 + \mu x + \nu, \tag{2}$$

or the corresponding differential equation

$$y''' = 0. \tag{3}$$

We shall find that this leads to interesting and important results; in particular, it is the first case which leads to surfaces other than those of constant curvature.

In § 1, we find all the doubly infinite systems of parabolas whose differential equation is of the cubic form (1). In addition to linear systems, certain quadratic systems satisfy the conditions of the problem (§ 2). The systems are further classified and characterized geometrically (§ 3, § 4). In § 5 the existence

* Cf. Transactions, vol. 4 (1903), p. 119.

† R. LIOUVILLE, Journal de l'Ecole Polytechnique, vol. 57 (1887), p. 219. For an application of these equations cf. E. KASNER, *A characteristic property of isothermal systems*, Mathematische Annalen, vol. 59 (1904), p. 352.

‡ In general no solution exists, since the functions $A$, $B$, $C$, $D$ must satisfy an infinite number of conditions. An example where the inconsistency of the conditions may be verified very simply is $y''' = y^6$. The same holds for equations of higher order.

§ Cf. BUSSE, Berliner Sitzungsberichte, 1896. It may be shown, by an investigation similar to that in § 1 of the present paper, that the only systems of circles of the cubic type (1) are the linear systems.

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of corresponding surfaces is verified, namely, those of constant curvature, and those which are geodesically representable on surfaces with the linear element

$$ds^2 = v(du^2 + dv^2).$$

In § 7 the group of the geodesics is determined in order to obtain the most general representation in which geodesics are pictured by parabolas. In § 6 the problem is specialized by adding the requirement that the representation shall be conformal; the surfaces are then either developables or applicable on a certain surface of revolution. Finally, in § 8, certain more general problems in conformal representation are treated; in this section the discussion is merely outlined since no new classes of surfaces are brought to light.

The surfaces of variable curvature which satisfy the conditions of our problem present themselves, in whole or in part, in many important investigations. We mention here the following, detailed references being given later: (1) Systems of geodesics admitting infinitesimal transformations (LIE, KOENIGS). (2) Systems of geodesic circles admitting contact transformations (LIE). (3) Surfaces for which the partial differential equation $\Delta\theta = 1$ admits integrals of the first and second degree in the first derivatives (DARBOUX). (4) Surfaces whose element is reducible to the LIOUVILLE form in an infinite number of ways (RAFFY, KOENIGS).

## § 1. *Differential equations of the problem.*

We first obtain the conditions that an equation of cubic type (1) shall represent a double infinity of parabolas (2). For this purpose we differentiate (1) and find that

$$y''' = \alpha_1 y'^5 + \alpha_2 y'^4 + \alpha_3 y'^3 + \alpha_4 y'^2 + \alpha_5 y' + \alpha_6, \tag{4}$$

where

$$\begin{aligned} &\alpha_1 = 3A^2, && \alpha_4 = 3AD + 3BC + B_x + C_y, \\ &\alpha_2 = 5AB + A_y, && \alpha_5 = 2BD + C^2 + C_x + D_y, \\ &\alpha_3 = 4AC + 2B^2 + A_x + B_y, && \alpha_6 = CD + D_x. \end{aligned} \tag{5}$$

Since for our parabolas equation (3) must be satisfied, we have then the conditions

$$\alpha_1 = 0,\ \alpha_2 = 0,\ \alpha_3 = 0,\ \alpha_4 = 0,\ \alpha_5 = 0,\ \alpha_6 = 0.$$

The first of these shows that

$$A = 0; \tag{6}$$

the second is then satisfied identically; and the others give the following four equations for the determination of $B$, $C$, $D$:

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$$
(7)\qquad
\begin{aligned}
B_y + 2B^2 &= 0,\\
B_x + C_y + 3BC &= 0,\\
C_x + D_y + C^2 + 2BD &= 0,\\
D_x + CD &= 0.
\end{aligned}
$$

It will be convenient to divide the discussion of this system with reference to the vanishing or non-vanishing of $B$.

*Discussion for $B = 0$.*

Putting $B = 0$ into $(7_2)$, we find that $C$ is a function of $x$ alone, say the function $X$. The remaining equations of the system then give

$$
(8)\qquad D_y + X' + X^2 = 0, \qquad D_x + XD = 0.
$$

Eliminating $D$, we obtain

$$
X'' + 3XX' + X^3 = 0,
$$

whose solution is

$$
C = X = \frac{2(ax + b)}{ax^2 + 2bx + c}.
$$

The value of $D$ is now found, from $(7_4)$, to be

$$
D = \frac{-2ay + 2d}{ax^2 + 2bx + c}.
$$

These values of $C$ and $D$, together with $A = 0$, $B = 0$, satisfy all our conditions. The corresponding equation (1) is

$$
(9)\qquad (ax^2 + 2bx + c)y'' = 2(ax + b)y' - 2(ay - d).
$$

*Discussion for $B \neq 0$.*

The integration of $(7_1)$ gives

$$
(10)\qquad B = (2y + X)^{-1},
$$

where $X$ is an arbitrary function of $x$. We now simplify the remaining equations of the set (7) by introducing new variables $u$, $v$ as follows:

$$
(11)\qquad u = x, \qquad v = (2y + X)^{-\frac{1}{2}}.
$$

The result of the transformation, noting that $B = v^2$, is found to be

$$
(12)\qquad
\begin{aligned}
vC_v - 3C + U'v^2 &= 0,\\
-v^3 D_v + 2v^2 D + C^2 + C_u - \tfrac{1}{2}U'v^3 C_v &= 0,\\
D_u - \tfrac{1}{2}U'v^3 D_v + CD &= 0,
\end{aligned}
$$

where $U$ denotes the result of substituting $u$ for $x$ in the function $X$.

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The solution of the first of these equations is

$$C + U'v^2 + U_1 v^3, \tag{13}$$

where $U_1$ is an arbitrary function of $u$. The other equations may now be written

$$\begin{aligned} D_u &= -U_1 v^3 D + \tfrac{1}{2} U' v^2 R, \\ D_v &= 2v^{-1} D + v^{-1} R, \end{aligned} \tag{14}$$

where $R$ is defined by

$$R = U'' + U_1' v + \tfrac{1}{2} U' U_1 v^3 + U_1^2 v^4. \tag{14'}$$

Equating the two values of $D_{uv}$ found from (14), we have

$$3 U_1 D + S = 0, \tag{15}$$

where

$$S = U_1 R - \tfrac{1}{2} U' R_v + v^{-3} R_u. \tag{15'}$$

We now proceed to show that if $U_1$ does not vanish, the equations (14) are inconsistent. For, under the assumption that $U_1 \neq 0$, (15) may be written

$$D = -\frac{S}{3U_1}.$$

Substituting this in $(14_2)$, we find

$$S_v - 2v^{-1} S + 3 U_1 v^{-1} R = 0.$$

This must be satisfied identically if equations (14) are to be consistent. Substituting the values of $R$ and $S$ given in (14′) and (15′), the coefficient of every power of $v$ must vanish. For our purpose it is sufficient to select the coefficient of $v^3$, which is found to be $5 U_1^3$. This, however, cannot vanish, in virtue of the assumption made at the outset.

We may therefore assume that

$$U_1 = 0, \tag{16}$$

Then equation (15) gives $S = 0$. From (14′), $R$ reduces to $U''$; and thereby, from (15′), $S$ reduces to $v^{-3} U'''$. Equating the two values of $S$ thus obtained, we have $U''' = 0$, or

$$U = au^2 + 2bu + c, \tag{17}$$

where $a$, $b$, $c$ are arbitrary constants.

Substituting the values of $U$ and $U_1$ in (14), we find

$$D_u = 2av^2(au + b), \qquad D_v = 2b^{-1}(D + a).$$

From the derivation these equations must be consistent; their common solution, containing a new arbitrary constant $d'$, is in fact

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(18) $$D = av^2(au^2 + 2bu + d') - a.$$

The value of $C$ given in (13) reduces, in virtue of (16) and (17), to

(19) $$C = 2v^2(au + b).$$

This completes the discussion of the set (7) for $B \neq 0$. The solution found, given in (10), (19), (18), may be reduced by means of the substitution formulas (11) and the value

$$X = ax^2 + 2bx + c,$$

to the form

(19) $$B = \frac{1}{2y + ax^2 + 2bx + c},$$

$$C = \frac{2(ax + b)}{2y + ax^2 + 2bx + c},$$

$$D = \frac{-2(ay - d)}{2y + ax^2 + 2bx + c},$$

where $d$ takes the place of $a(d' - c)$.

The differential equation (1), thus found, is

(20) $$(2y + ax^2 + 2bx + c)y'' = y'^2 + 2(ax + b)y' - 2(ay - d).$$

We may state the result obtained at this stage as follows:

*The only equations of cubic type* (1) *which represent a doubly infinite system of parabolas* (2) *are those of the forms* (9) *and* (20). *Both these forms are included in the complete solution*

(21) $$(2ty + ax^2 + 2bx + c)y'' = ty'^2 + 2(ax + b)y' - 2(ay - d)$$

*containing four constants* $a : b : c : d : t$.

## § 2. *General character of the systems of parabolas.*

There is no difficulty in integrating (21); for it is known, from the derivation, that the integral is of the form (2). We find that *our systems of parabolas are represented by*

$$y = \lambda x^2 + \mu x + \nu,$$

*where* $\lambda$, $\mu$, $\nu$ *are connected by a relation of the form*

(21′) $$t(\mu^2 - 4\lambda\nu) - 2c\lambda + 2b\mu - 2a\nu + 2d = 0.$$

The character of the system, we shall see, depends essentially upon the discriminant of this quadratic in $\lambda$, $\mu$, $\nu$, namely,

(22) $$\delta = 4t^2(b^2 - ac - 2td).$$

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We show first that, if $\delta = 0$, the equation (21) is equivalent, under the group of point transformations, to $y'' = 0$. This requires the discussion of the cases $t = 0$ and $b^2 - ac - 2td = 0$.

If $t = 0$, that is, if equation (21) is of the form (9), the corresponding relation (21′) reduces to the form

$$(9') \qquad -2c\lambda + 2b\mu - 2a\nu + 2d = 0.$$

This system of parabolas being linear, is evidently reducible to $y'' = 0$.

If $t$ is not 0, we may assume it to be unity, so that our equation is of form (20). The vanishing of $\delta$ requires $b^2 - ac - 2d = 0$. The transformation

$$(23) \qquad x_1 = x, \qquad 2y_1 = 2y + ax^2 + 2bx + c$$

then converts (20) into

$$(24) \qquad 2y_1 y_1'' = y_1'^2,$$

For this normal form the relation (21′) becomes

$$\mu^2 - 4\lambda\nu = 0.$$

We may therefore put

$$\lambda = \alpha^2, \qquad \mu = 2\alpha\beta, \qquad \nu = \beta^2,$$

so that the solution of (24) may be written

$$(24') \qquad y = (\alpha x + \beta)^2.$$

This represents the vertical parabolas touching the axis of $x$. In the theory of algebraic curves this would be termed a quadratic system since the parameters are involved to the second degree. But in the general (infinitesimal) theory * of systems of curves the system (24′) is linear since it may be written

$$\alpha x + \beta - \sqrt{y} = 0,$$

which involves the parameters linearly. Hence the equation (24) is equivalent to $y'' = 0$. This completes the prooof of the result stated above.

We pass now to the case $\delta \neq 0$. We take, as before, $t = 1$, and put

$$(25) \qquad \kappa^2 = ac - b^2 + d;$$

so that $\kappa$, by assumption, does not vanish. The transformation †

$$(26) \qquad x_1 = x, \qquad 2\kappa y_1 = 2y + ax^2 + 2bx + c$$

converts (20) into

$$(27) \qquad 2y_1 y_1'' = y_1'^2 + 1.$$

* See Bulletin of the American Mathematical Society, vol. 12 (1905), p. 307.

† If reality considerations are taken into account, it is necessary to separate the cases $\delta < 0$, and $\delta > 0$. In the latter case, $\kappa$ is imaginary; but another transformation gives the normal form $2yy'' = y'^2 - 1$. This represents the vertical parabolas whose foci are on the axis of $x$.



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The solution for this normal form is

$$4\beta(y-\beta)=(x-\alpha)^2,$$

which represents the parabolas having the axis of $x$ for directrix.

This is an essentially distinct case; the equation (27) is not equivalent to $y''=0$, since it does not satisfy the conditions of LIOUVILLE given in the introduction. We have then this result:

*With respect to the group of all point transformation, the systems of parabolas possessing a cubic differential equation* (1), *that is, the systems* (21), *divide into two distinct types. In the first type, the discriminant* $\delta$ *of the relation* (21′) *vanishes, and the system is linear, i. e., equivalent to* $y''=0$. *In the second type,* $\delta$ *does not vanish, the system is essentially quadratic, and equivalent to* (27).

## § 3. *Detailed classifications.*

The preceding paragraph contains the classification of our systems (21) with respect to the group of all point transformations. We can obtain a closer geometric survey of the systems by means of classifications based on certain finite (continuous) groups which suggest themselves in connection with our problem.

The totality of vertical parabolas (2) evidently admits the projective transformations of the form

$$G_5: \qquad x_1=e_1x+e_2, \qquad y_1=e_3y+e_4x+e_5,$$

which constitute a five-parameter group. It may be shown that it admits a more extensive group

$$G_6: \qquad x_1=e_1x+e_2, \qquad y_1=e_0x^2+e_3y+e_4x+e_5.$$

This group $G_6$ contains $G_5$ as a subgroup, the latter arising by putting $e_0=0$.

Since any point transformation converts an equation of cubic form (1) into an equation of the same form, and since the particular transformations $G_6$ also convert vertical parabolas into vertical parabolas, it follows that the transformations $G_6$ convert any solution of our problem, that is, any system (21), into a solution or system (21). The effect of a transformation $G_6$ upon the equation (21) containing the constants $t:a:b:c:d$, is to convert it into an equation of the same form with new constants $t_1:a_1:b_1:c_1:d_1$.

The induced transformation of the constants is found to be

$$G_6': \begin{aligned} t_1 &= e_3^2t,\\ a_1 &= 2e_3e_0t+e_3e_1a,\\ b_1 &= e_3e_4t+e_3e_1e_2a+e_3e_1b,\\ c_1 &= 2e_3e_5t+e_3e_2^2a+2e_3e_2b+e_3c,\\ d_1 &= \frac{e_4^2-4e_0e_5}{2}t+(e_1e_2e_4-e_0e_2^2-e_1^2e_5)a+(e_1e_4-2e_0e_2)b-e_0c+e_1^2d. \end{aligned}$$

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The induced group $G_5'$, corresponding to the projective transformations $G_5$, is obtained by simply putting $e_0 = 0$ in the above. The modulus of the linear transformation is $e_1^5 e_3^3$; this cannot vanish, since otherwise the original transformation $G_6$ would have a vanishing jacobian and thus degenerate. We may therefore assume that neither $e_1$ nor $e_3$ vanishes.

We are now in position to discuss the equivalence of our systems of parabolas, or the corresponding differential equations (21), with respect to the group $G_6$. From the importance of the quantity $\delta$ in the general discussion of § 2, one is led to expect its invariant character in the present connection. In fact, the factors of $\delta$ give rise to two (relative) invariants $t$ and $b^2 - ac - 2td$; for

$$t_1 = e_3^2 t,$$
$$b_1^2 - a_1 c_1 - 2t_1 d_1 = e_1^2 e_3^2 (b^2 - ac - 2td).$$

This shows that the systems for which $t = 0$ must be distinguished from those for which $t \neq 0$; and, similarly, the cases $b^2 - ac - 2td = 0$ and $b^2 - ac - 2td \neq 0$ are essentially distinct. It is also observed, from the form of $G_6'$ above, that when $t$ vanishes, $a_1 = e_3 e_1^2 a$, so that it is necessary then to distinguish further according to the vanishing or non-vanishing of $a$. *We are thus led to the following classification of the systems* (21) *with respect to* $G_6$:

$$\begin{array}{rl}
\text{I:} & t = 0,\ b^2 - ac - 2td = 0,\ a = 0; \\
\text{II:} & t = 0,\ b^2 - ac - 2td = 0,\ a \neq 0; \\
\text{III:} & t = 0,\ b^2 - ac - 2td \neq 0,\ a = 0; \\
\text{IV:} & t = 0,\ b^2 - ac - 2td \neq 0,\ a \neq 0; \\
\text{V:} & t \neq 0,\ b^2 - ac - 2td = 0; \\
\text{VI:} & t \neq 0,\ b^2 - ac - 2td \neq 0.
\end{array}$$

*No member of one of these classes can be transformed into a member of another class; while systems belonging to the same class are equivalent.*

The last statement may be proved by showing that for each class it is possible to set up a canonical form to which all the members of that class are equivalent. We omit detailed discussion and merely state the *canonical equations with their solutions*: *

$$\begin{array}{rcl}
\text{I:} & y'' = 2, & y = x^2 + \alpha x + \beta; \\
\text{II:} & x^2 y'' = 2xy' - 2y, & y = \alpha x^2 + \beta x; \\
\text{III:} & xy'' = y', & y = \alpha x^2 + \beta; \\
\text{IV:} & (x^2 - 1) y'' = 2xy' - 2y, & y = \alpha x^2 + \beta x + \alpha; \\
\text{V:} & 2yy'' = y'^2, & y = (\alpha x + \beta)^2; \\
\text{VI:} & 2yy'' = y'^2 + 1, & 4\beta y = (x - \alpha)^2 + 4\beta^2.
\end{array}$$

* The discussion for the cases where $t \neq 0$ is really contained in § 2. The transformations there employed, namely (23) and (26), belong to $G_6$. Hence (24) and (27) may be used as the canonical equations for V and VI respectively.

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By a transformation of the type $G_6$ any system (21) can be reduced to one of these forms.

In classes I–IV, the relation (21′) between $\lambda, \mu, \nu$ is of the first degree, while in classes V, VI the relation is of the second degree. Furthermore, classes I–V all belong to the first type described in the final theorem of § 2, while VI is identical with the second type.

We pass now to the projective group $G_5$. In the formulas for $G_6'$, we put then $e_0 = 0$, so that $a$ is a (relative) invariant as well as $t$ and $b^2 - ac - 2td$. The six classes considered above are of course distinct with respect to the smaller group now being considered; but V and VI each subdivide according to the vanishing or non-vanishing of $a$. A detailed discussion shows that the only other subdivision takes place in I; here $d = 0$ must be separated from $d \neq 0$.

*We then have the following complete classification of the equations* (21) *with reference to the projective group* $G_5$:

$\mathrm{I}_0$: this is I with $d = 0$, canonical form $y'' = 0$;

$\mathrm{I}_1$: this is I with $d \neq 0$;

II, III, IV: same as in the $G_6$ classification;

$\mathrm{V}_1$: this is V with $a = 0$;

$\mathrm{V}_2$: this is V with $a \neq 0$, canonical form $(2y + x^2)y'' = y'^2 + 2xy' - 2y$;

$\mathrm{VI}_1$: this is VI with $a = 0$;

$\mathrm{VI}_2$: this is VI with $a \neq 0$, canonical form $(2y + x^2)y'' = y'^2 + 2xy' - 2y + 1$.

As canonical forms for $\mathrm{I}_1$, II, III, IV, $\mathrm{V}_1$, $\mathrm{VI}_1$, we may take the forms for I, II, III, IV, V, VI given in the $G_6$ classification above. The case $\mathrm{I}_0$ might be omitted as trivial since the system does not consist of proper parabolas but of straight lines. These enter into the discussion since straight lines, as well as vertical parabolas, satisfy the equation $y''' = 0$.

In the first five of the nine classes the relation between $\lambda, \mu, \nu$ is linear. Hence: *Of the two parameter systems of curves* $y = \lambda x^2 + \mu x + \nu$ *defined by linear relations*

$$c\lambda - b\mu + a\nu - d = 0,$$

*there are five projectively distinct cases:*

$$\begin{aligned} \mathrm{I}_0: &\quad a = 0,\ b = 0,\ d = 0; \\ \mathrm{I}_1: &\quad a = 0,\ b = 0,\ d \neq 0; \\ \mathrm{II}: &\quad a \neq 0,\ b^2 - ac = 0; \\ \mathrm{III}: &\quad a = 0,\ b^2 - ac \neq 0; \\ \mathrm{IV}: &\quad a \neq 0,\ b^2 - ac \neq 0. \end{aligned}$$

*The first consists merely of straight lines, the others of proper parabolas. Systems belonging to the same class are equivalent under the projective transformations* $G_5$.

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§ 4. *Geometric determinations.*

In this section we characterize geometrically the systems (21) and the various classes discussed in the preceding section. The discussion is merely outlined, but the results are complete.

Consider first the systems where $t = 0$, i. e., the systems where the relation (21′) takes the linear form (9). Such a linear relation is obtained by taking a fixed parabola* $y = \lambda_0 x^2 + \mu_0 x + \nu_0$ and a variable parabola $y = \lambda x^2 + \mu x + \nu$; the straight line through their two points of intersection is

$$(\lambda - \lambda_0)y + (\lambda_0\mu - \mu_0\lambda)x + \lambda_0\nu - \nu_0\lambda = 0;$$

and the condition that this line passes through a fixed point $(x_0, y_0)$ is

$$(y_0 - \mu_0 x_0 - \nu_0)\lambda + \lambda_0 x_0 \mu + \lambda_0 \nu - \lambda_0 y_0 = 0;$$

which is of the required form. Hence

*Any system defined by a linear relation between $\lambda$, $\mu$, $\nu$ may be characterized as consisting of the $\infty^2$ parabolas intersecting a fixed parabola $\pi_0$ in pairs of points collinear with a fixed point $P_0$.*

The classes described at the close of § 3 (omitting $I_0$) are distinguished as follows: In cases II and IV the point $P_0$ is finite, while in I and III it is at infinity. In I and II the point $P_0$ is on the parabola $\pi_0$, while in III and IV it is not so situated.

We pass now to the systems for which $t \neq 0$, i. e., the systems for which the relation (21′) is really quadratic. To characterize these we make use of the canonical forms V and VI in § 3, and the transformations $G_5$ and $G_6$.

For V, the canonical equation represents the parabolas tangent to the axis of $x$. The transformation $G_5$ converts the axis into some straight line; and a quadratic transformation $G_6$ converts the axis into some parabola. Therefore

*Any system defined by a quadratic relation* (21′) *with vanishing discriminant consists of the $\infty^2$ parabolas tangent to either a fixed straight line* (*case* $V_1$) *or a fixed parabola* (*case* $V_2$).

For VI, the canonical equation represents the parabolas having the axis of $x$ for directrix. If a point is taken on this axis, the pairs of tangents drawn to the members of the system constitute an involution. Applying a transformation $G_5$, the axis of $x$ is converted into some line and the involution property still holds for any point of this line. The effect of a transformation $G_6$, on the other hand, is to replace the axis of $x$ by a parabola, and the pencil of lines through the point on the axis by a pencil of congruent parabolas through the corresponding point. The result is:

*Any system defined by a quadratic relation* (21′) *with non-vanishing discriminant may be obtained as follows: Take a pencil of straight lines* (*case* $VI_1$), *or a*

* The term parabola is used throughout in the sense of vertical parabola.

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*pencil of congruent parabolas (case* $VI_2$*), and establish an involution therein. The* $\infty^2$ *parabolas touching the pairs (of straight lines or parabolas) belonging to the involution constitute the required system.*

## § 5. *The corresponding surfaces.*

As was remarked in the introduction, the fact that an equation is of the cubic form (1) does not of itself show that it represents the geodesics of a surface. But for the systems of parabolas obtained we shall find that corresponding surfaces do exist.

At the close of § 2, it was seen that with respect to point transformation the systems are of two distinct types. In the first type, the equation is equivalent to $y'' = 0$; so that, by the familiar BELTRAMI theorem, the surfaces are merely those of constant curvature.

It remains then to discuss only the second type, whose canonical form is

$$2yy'' = y'^2 + 1. \tag{27}$$

To find a corresponding surface, it is sufficient to note that, when the element of length of a surface is written in the isothermal form

$$ds^2 = E(du^2 + dv^2), \tag{28}$$

the differential equation of the geodesics is

$$v'' = -L_u v'^3 + L_v v'^2 - L_u v' + L_v, \tag{29}$$

where

$$L = \tfrac{1}{2}\log E.$$

This becomes of the same form as (27) if $L$ is $\frac{1}{2}\log v$. The corresponding element is then

$$ds^2 = v(du^2 + dv^2). \tag{30}$$

Hence if the element of length of a surface can be reduced to the form (30), and the point $(u, v)$ of the surface be represented by the point $x = u$, $y = v$ in the plane, then the geodesics of the surface are pictured by the parabolas (27) having the axis of $x$ for directrix.

If two surfaces $S$ and $S_1$ can be represented upon a plane so that the geodesics are pictured by the same curves, it follows that it is possible to establish a point-to-point correspondence between $S$ and $S_1$ so that the geodesics correspond. When this is true we may say that $S$ and $S_1$ are geodesically equivalent. The problem of finding all pairs of geodesically equivalent surfaces was solved by DINI and LIE. The only cases where $S_1$ is not applicable on (isometric to) $S$, or a surface homothetic to $S$, arise for the LIOUVILLE surfaces, whose elements are reducible to the form

$$ds^2 = \{U(u) + V(v)\}(du^2 + dv^2).$$

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The element (30), obtained above, is of this type, hence there exist essentially distinct classes of surfaces with the same geodesic representation. It happens that LIE, in illustration of the general theory due to DINI and himself, has given the discussion of a particular case equivalent to (30). Through an oversight, however, LIE overlooked one form of solution.*

The resulting surfaces coincide, as KOENIGS has shown, with those surfaces whose geodesic equation, $\Delta\theta = 1$, admits precisely three homogeneous integrals of the second degree. Another equivalent problem is the determination of surfaces applicable on surfaces of revolution and having an element reducible to the LIOUVILLE form in an infinite number of ways. We may therefore make use of the forms obtained by RAFFY in his discussion of this question.†

If a surface can be represented geodesically upon a surface with the element (30), then its element is reducible to (30), or to one of the forms

$$ds^2 = \left(\frac{P}{v^2} + Q\right)(du^2 + dv^2), \tag{31}$$

$$ds^2 = (Pe^v + Qe^{2v})(du^2 + dv^2), \tag{32}$$

$$ds^2 = \frac{P + Q(e^v + e^{-v})}{(e^v - e^{-v})^2}(du^2 + dv^2); \tag{33}$$

so that (30), (31), (32), (33) together define a complete class of geodesically equivalent surfaces.

Applying this to our problem, we have the final result:

*The only surfaces which can be represented point by point upon a plane so that the geodesics are pictured by parabolas $y = \lambda x^2 + \mu x + \nu$ are:*

1°. *Surfaces of constant curvature. The system of parabolas is then linear (in the general sense), defined by a relation* (21′) *with vanishing discriminant* $\delta$.

2°. *Surfaces of variable curvature whose linear element is reducible to one of the forms* (30), (31), (32), (33). *These are all geodesically equivalent; and are applicable on surfaces of revolution. The corresponding system of parabolas is essentially quadratic, defined by a relation* (21′) *whose discriminant does not vanish.*

There is no difficulty in finding the explicit equations of the surfaces of revolution whose elements are of the above form. We give the result for the simplest case. The class represented by (30) is applicable on any one of the surfaces of revolution

$$x = h\sqrt{v}\cos\frac{u}{h}, \qquad y = h\sqrt{v}\sin\frac{u}{h}, \qquad z = \int\sqrt{\frac{4v^2 - h^2}{4v}}\,dv, \tag{34}$$

* KOENIGS, *Mémoire sur les lignes géodésiques*, Mémoires des Savants Etrangeres, vol. 31 (1894), p. 22.

† RAFFY, Comptes Rendus, vol. 108 (1889), p. 493; DARBOUX, *Théories des surfaces*, vol. 3 (1894), p. 39.

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where $h$ is an arbitrary constant. The equation of the meridian curve is

$$z = h^3 \int \sqrt{4x^4 - h^6}\, dx. \tag{34'}$$

Another geometric characterization of our surfaces may be obtained by making use of WEINGARTEN's theorem: any surface applicable on a surface of revolution (exclusive of the catenoid) is a nappe of the evolute surface of a $W$-surface. The following simple result is obtained without difficulty. Take a surface whose principal radii are connected by the relation

$$R_1 + 2R_2 = 0,$$

and construct the nappe of its evolute corresponding to the radius $R_1$.* This gives surface with the element (30).†

## § 6. *Conformal representation.*

Having investigated all cases where a surface is capable of representation upon a plane so that the geodesics correspond to parabolas, we now determine those cases where the representation is conformal.

For this purpose, we recall that the most general conformal representation of any surface on a plane is obtained by putting $ds^2$ into the isothermal form (28) and setting $x = u$, $y = v$. The plane curves corresponding to the geodesics are then defined by an equation of the form

$$y'' = -L_x y'^3 + L_y y'^2 - L_x y' + L_y. \tag{35}$$

Comparing this with (1), we may state the useful

LEMMA. *In order that a system of plane curves shall correspond by conformal representation to the geodesics of a surface, it is necessary and sufficient that the equation of the system be of the cubic form*

$$y'' = Ay'^3 + By'^2 + Cy' + D, \tag{1}$$

*and that the coefficients satisfy the relations*

$$A = C, \qquad B = D, \qquad A_y + B_x = 0. \tag{36}$$

The only equations (21) which satisfy these conditions are found to be

$$(2ty + c)y'' = t(y'^2 + 1). \tag{37}$$

If $t = 0$, this equation reduces to $y'' = 0$, so that the system consists of straight lines. The only surfaces whose geodesics can be represented conformally by straight lines are, it is known, the developable surfaces.

* The other nappe has an element of the form $ds^2 = v^4(du^2 + dv^2)$.

† A more general theorem is given by LIE, Mathematische Annalen, vol. 20 (1882), p. 389.

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If $t \neq 0$, the equation, by a translation of the $x$ axis, may be reduced to (27). The corresponding surfaces are then represented by (30). Our result is therefore:

*If a surface is capable of being represented conformally upon a plane in such a manner that its geodesics are pictured by parabolas $y = \lambda x^2 + \mu x + \nu$, then either (case 1°) the surface is developable, and the representing curves are straight lines; or (case 2°) the surface belongs to the class (30) of surfaces applicable on the surface of revolution (34), and the representing curves are parabolas with a common directrix.*

## § 7. *Groups and representations.*

We have obtained all the systems of parabolas and all the surfaces connected with our problem. It remains now to determine all the possible representations.

Consider first the case of surfaces of constant curvature. The geodesics of such a surface, being equivalent to straight lines $y'' = 0$, admit an eight-parameter group, isomorphic to the projective group of the plane. Hence, for a given surface and a given system of parabolas belonging to the first type described at the close of § 2, there are $\infty^8$ possible representations. The total number of systems of this type is, however, $\infty^3$. Therefore:

*A surface of constant curvature may be represented upon the plane so that its geodesics are pictured by parabolas (2) in $\infty^{11}$ ways. None of these representations is conformal except when the curvature is zero; then there are $\infty^4$ conformal representations.*

The corresponding discussion for the second type of surfaces obtained (those of variable curvature) is not so simple. We have seen that the system of parabolas may then be put into the canonical form (27). Our first step is to examine the group of point transformations which this equation admits.

If the equation (27) is invariant under an infinitesimal transformation with the symbol

$$\xi \frac{\partial t}{\partial x} + \eta \frac{\partial t}{\partial y},$$

the general theory of groups gives the condition *

$$\{\eta + y(\eta_y - 2\xi_x - 3\xi_y y')\}(1 + y'^2) - 2yy'\{\eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2\}$$
$$+ 2y^2\{-\xi_{yy}y'^3 + (\eta_{yy} - 2\xi_{xq})y'^2 + (2\eta_{xy} - \xi_{xx})y' + \eta_{xx}\} = 0.$$

Equating the coefficients of the various powers of $y'$ to zero, we find the following system of equations for the determination of the unknown functions $\xi$, $\eta$:

$$2y\xi_{yy} + \xi_y = 0, \qquad \eta - y\eta_y + 2y^2\eta_{yy} - 4y^2\xi_{xy} = 0,$$
$$3\xi_y + 2\eta_x + 2y(\xi_{xx} - 2\eta_{xy}) = 0, \qquad \eta + y(\eta_y - 2\xi_x) + 2y^2\eta_{xx} = 0.$$

* LIE-SCHEFFERS, *Differentialgleichungen*, p. 363.

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We shall omit the somewhat long but not particularly difficult discussion of these equations and simply state the solution

$$\xi = a_1 x^2 + a_2 x + a_3, \qquad \eta = 2a_1 xy + a_2 y. \tag{38}$$

Hence the equation (24) admits three independent infinitesimal transformations with the symbols

$$\frac{\partial f}{\partial x}, \qquad x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}, \qquad x^2\frac{\partial f}{\partial x} + 2xy\frac{\partial f}{\partial y}. \tag{39}$$

The finite equations of the continuous group generated by these infinitesimal transformations may be found by the integration of the simultaneous system

$$\frac{dx_1}{a_1 x_1^2 + a_2 x_1 + a_3} = \frac{dy_1}{2a_1 xy + a_2 y} = d\tau,$$

with the initial conditions $\tau = 0$, $x_1 = x$, $y_1 = y$.

When $a_1$ vanishes, it is found that

$$x_1 = a'x + b', \qquad y_1 = a'y. \tag{40}$$

When $a_1$ does not vanish, it may be taken equal to unity, and the integrals, after putting $\tau = \log t$, take the form

$$\frac{x_1 + r}{x_1 + s} = t\frac{x + r}{x + s}, \qquad \frac{(x_1 + r)(x_1 + s)}{y_1} = \frac{(x + r)(x + s)}{y} \tag{41}$$

or, explicitly,

$$x_1 = \frac{(st - r)x + rs(t - 1)}{(1 - t)x + s - rt}, \qquad y_1 = \frac{t(s - r)^2 y}{\{(1 - t)x + s - rt\}^2}. \tag{41'}$$

*Hence, equation* (27), *or the system of parabolas with a common directrix, admits the three-parameter continuous groups* $H_3$ *represented by* (40) *and* (41). *The transformations* (40) *constitute a two-parameter conformal subgroup.*

Consider now the possible representations of a surface $S$ defined by

$$ds^2 = v(du^2 + dv^2).$$

We have already seen that, by the representation

$$R: \qquad x = u, \qquad y = v,$$

the geodesics are pictured by the system (27),

$$\sigma: \qquad 2yy'' = y'^2 + 1.$$

Since $\sigma$ is transformed into itself by the group $H_3$, it follows that if $R$ is combined with one of these plane transformations, the resulting representation will still picture the geodesics of $S$ by the system $\sigma$. We thus obtain $\infty^3$ repre-

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sentations $RH_3$, corresponding to the one system $\sigma$. Of these, a double infinity, namely $RG_2$, are conformal.

But there are $\infty^4$ systems of the second type described in § 2, any one of which may be used instead of the system $\sigma$. All these are equivalent to $\sigma$ with respect to the group $G_6$ of § 3. To obtain all the systems from $\sigma$ it is in fact sufficient to employ the four-parameter subgroup

$$G_4: \qquad x_1 = x, \qquad y_1 = e_0 x^2 + e_3 y + e_4 x + e_5.$$

The most general representation of $S$ is therefore obtained by combining $G_4$ with the representations already found. The result may be indicated by $RH_3G_4$, and involves seven arbitrary constants. This yields no new conformal representations since $G_4$ contains no conformal transformations except identity.

*If a surface belongs to the class defined by* (30) *it may be represented upon the plane so that its geodesics are pictured by parabolas in* $\infty^7$ *ways, of which* $\infty^2$ *are conformal. The general representation is of the form* $RH_3G_4$, *while the conformal is* $RG_2$.

Let us pass now to the other surfaces belonging to the second type of § 5. Such a surface is geodesically equivalent to those just discussed. The geodesics will still admit a three-parameter group, so that there are $\infty^3$ representations for the one system $\sigma$. From § 6, none of these will be conformal. Our result is:

*If a surface belongs to the classes defined by* (31), (32), *or* (33), *it is possible to represent it in* $\infty^7$ *ways upon a plane so that its geodesics are pictured by parabolas. None of these representations is conformal.*

## § 8. *Additional results.*

We may extend the results obtained by making use of the following

*Lemma.* If an equation of cubic type (1) is an integral of an equation of the form

$$y^{(n)} + P_1 y^{(n-1)} + \cdots + P_{n-1} y' + P_n = 0, \tag{42}$$

where the coefficients are functions of $x$, $y$, then $A = 0$, that is, (1) reduces to a quadratic.

To prove this, it is sufficient to observe that, when the values of $y''$, $y'''$, $\cdots$, $y^{(n)}$ obtained from (1) in terms of $y'$ are substituted in (42), the coefficient of the highest power of $y'$ is a numerical multiple of $A^{n-1}$,

In particular, if the cubic equation is of the form (35), arising in connection with conformal representation, it follows that $L_x = 0$, so that $L$ is a function of $y$ alone. The corresponding linear element (28) then belongs to surfaces applicable on surfaces of revolution.

*If a surface can be represented conformally on a plane so that the curves depicting its geodesics satisfy an equation of the form* (42) (*of any order but linear in the derivatives*), *then its element is necessarily of the form*

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$$ds^2 = f(v)(du^2 + dv^2), \tag{43}$$

*so that the surface is applicable on a surface of revolution.*

The equation $y''' = 0$, which has been discussed, is of the form (42). We have seen (§ 6) that the corresponding surfaces are the developables and those with the element (30). We shall now show that no new surfaces arise in connection with any equation (42) of the third order.

The equation of the geodesics of (43) is

$$y'' = (1 + y'^2)Y, \tag{44}$$

where $Y$ is a function of $y$, namely,

$$Y = \frac{f'(y)}{2f(y)}. \tag{44'}$$

By successive differentiation, we find

$$\begin{aligned} y^{\mathrm{III}} &= y'(1 + y'^2)Y_1, && \text{where } Y_1 = 2Y^2 + Y'; \\ y^{\mathrm{IV}} &= (1 + y'^2)(Y_2 y'^2 + Y_3), && \text{where } Y_2 = 3YY_1 + Y_1', \quad Y_3 = YY_1; \\ y^{\mathrm{V}} &= y'(1 + y'^2)(Y_4 y'^2 + Y_5), && \text{where } Y_4 = 4YY_2 + Y_2', \\ & && \qquad Y_5 = 2Y(Y_2 + Y_3) + Y_3'; \text{ etc.} \end{aligned} \tag{45}$$

Taking the equation (42) for $n = 3$, and substituting the values of $y'$ and $y'''$, we have

$$y'(1 + y'^2)Y_1 + (1 + y'^2)YP_1 + y'P_2 + P_3 = 0.$$

Equating the coefficients of powers of $y'$, we obtain a simple set of conditions, with the solution

$$P_1 = 0, \qquad P_2 = 0, \qquad P_3 = 0, \qquad Y_1 = 0.$$

The first three values show that (42) reduces to $y''' = 0$: and the last shows that $Y$ is either a constant or of the form $(2y + c)^{-1}$. This leads, of course, to the surfaces considered in § 6.

*The only case in which an equation of the third order of the form* (42) *can be satisfied by a system of curves corresponding to the geodesics of a surface in conformal representation, is the case* $y''' = 0$; *the surfaces are then the developables and the class* (30) *described in* § 6.

The examination of the equations $y^{\mathrm{IV}} = 0$, $y^{\mathrm{V}} = 0$, $y^{\mathrm{VI}} = 0$ leads to the same conclusion: no new surfaces are obtained. The same is *probably* true of all equations of the form $y^{(n)} = 0$. There are, however, equations of the fourth order of the form (42) which lead to new solutions. We shall discuss these on another occasion.

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# THE DOUBLY PERIODIC SOLUTIONS OF POISSON'S EQUATION IN TWO INDEPENDENT VARIABLES*

BY

MAX MASON

The only doubly periodic solution † of LAPLACE's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is $u = c$, where $c$ is a constant. For if $u$ be such a solution, and $v$ the conjugate potential to $u$, then $u + iv$ would be a complex analytic function which has a value under a fixed finite limit for all values of $x$, $y$. But, as is well known, such a function is necessarily a constant.

It is the object of this paper to investigate, by the use of methods analogous to those of the potential theory, the doubly periodic solutions of POISSON's equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y), \tag{1}$$

where $f(x, y)$ is continuous and periodic in $x$ and in $y$ with the periods $a$ and $b$ respectively.‡ A "doubly periodic GREEN's function," $G$, will be formed from known functions, and the desired solution of (1) found by quadrature from $G$ and $f$.

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† A function $u$ will be called a solution of the differential equation within a region $\Omega$, provided that $u$ satisfies the differential equation at every point within $\Omega$. This definition requires the existence of the second derivatives of $u$ at every point in $\Omega$, and therefore the continuity of the first derivatives. By a doubly periodic solution we shall mean a doubly periodic function which is a solution of the equation in the period rectangle, and therefore in the entire plane. Such a function, in particular, has a value less than a fixed finite number for all values of $x$, $y$.

‡ In a recent article (Journal de Mathématiques, ser. 5, vol. 10 (1904), p. 445) I have considered by a different method the existence of periodic solutions of the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda A(x, y) u = f(x, y),$$

where $\lambda$ is a parameter.



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§ 1. *A doubly periodic Green's function and its law of reciprocity.*

Let $\Re$ denote the real part of the term before which it is written, and consider the function

$$\Re \log \frac{\sigma(z-\zeta)}{\sigma(z-\gamma)}\,;\; z = x + iy,\; \zeta = \xi + i\eta,\; \gamma = \alpha + i\beta,$$

where $\sigma(z)$ is the Sigma function of WEIERSTRASS, formed with the periods $a, ib$; and $\xi, \gamma$ are two points in the interior of the period rectangle $\Omega$ bounded by the lines $x = 0, y = a, x = 0, y = b$. This function is a solution of Laplace's equation within $\Omega$, except at the points $(\xi, \eta)$ and $(\alpha, \beta)$, and has the form

$$\log \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} - \log \frac{1}{\sqrt{(x-\alpha)^2 + (y-\beta)^2}} + g,$$

where $g$ is a solution of LAPLACE'S equation throughout $\Omega$.

Since for any integers $m, n$, the function $\sigma$ obeys the law

$$\sigma(z + ma + inb) = (-1)^{mn+m+n} e^{(m\eta_1 + n\eta_3)(2z + ma + inb)} \sigma(z),$$

where $\eta_1, \eta_3$ are certain complex constants,* we have

$$(2) \qquad \Re \log \frac{\sigma(z+ma+inb-\zeta)}{\sigma(z+ma+inb-\gamma)} = \Re \log \frac{\sigma(z-\zeta)}{\sigma(z-\gamma)} = m\Re\, 2\eta_1(\zeta-\gamma) - n\Re\, 2\eta_3(\zeta-\gamma).$$

Define a real function $V$ by the equation

$$V(x, y, \xi, \eta, \alpha, \beta) = \Re \log \frac{\sigma(z-\zeta)}{\sigma(z-\gamma)} + \frac{x}{a}\Re\, 2\eta_1(\zeta-\gamma) + \frac{y}{b}\Re\, 2\eta_3(\zeta-\gamma).$$

Since the last two terms are linear in $x$ and $y$, $V$ has the form

$$V(x, y, \xi, \eta, \alpha, \beta) = \log \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} - \log \frac{1}{\sqrt{(x-\alpha)^2 + (y-\beta)^2}} + S(x, y, \xi, \eta, \alpha, \beta),$$

where $S$ is a known solution of

$$\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} = 0$$

within $\Omega$. Furthermore $V$ is doubly periodic in $x, y$ with the periods $a, b$, since from (2) the equation results:

*See *e. g.*, BURKHARDT, *Elliptische Functionen*, p. 53.

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$$V(x+ma, y+nb, \xi, \eta, \alpha, \beta) = \Re \log \frac{\sigma(z+ma+inb-\zeta)}{\sigma(z+ma+inb-\gamma)}$$

$$+ \frac{x+ma}{a} \Re\, 2\eta_1(\zeta-\gamma) + \frac{y+nb}{b} \Re\, 2\eta_3(\zeta-\gamma)$$

$$= \Re \log \frac{\sigma(z-\zeta)}{\sigma(z-\gamma)} + \frac{x}{a} \Re\, 2\eta_1(\zeta-\gamma) + \frac{y}{b} \Re 2\eta_3(\zeta-\gamma)$$

$$= V(x, y, \xi, \eta, \alpha, \beta)^*.$$

*We shall call the function*

$$G(x, y, \xi, \eta, \alpha, \beta) = V(x, y, \xi, \eta, \alpha, \beta) - S(\alpha, \beta, \xi, \eta, \alpha, \beta)$$

*the doubly periodic Green's function for the periods $a$, $b$. This function has the following characteristics:*

1°. Except at $(\xi, \eta)$ and $(\alpha, \beta)$, $G$ is, within $\Omega$, a solution of the equation

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial^2 y} = 0.$$

2°. $G$ has the form

$$G = \log \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} - \log \frac{1}{\sqrt{(x-\alpha)^2 + (y-\beta)^2}} + R(x, y, \xi, \eta, \alpha, \beta),$$

where $R$ is a known function, which, with respect to the variables $x$, $y$, is a solution of LAPLACE's equation within $\Omega$, and which satisfies for all values of $\xi$, $\eta$ in $\Omega$, the equation

$$R(\alpha, \beta, \xi, \eta, \alpha, \beta) = 0.$$

3°. $G$ is doubly periodic in $x$, $y$ with the periods $a$, $b$.

*The functions $G$ and $R$ obey the following laws of reciprocity:*

$$G(x, y, \xi, \eta, \alpha, \beta) + \log \frac{1}{\sqrt{(x-\alpha)^2 + (y-\beta)^2}} = G(\xi, \eta, x, y, \alpha, \beta)$$

$$+ \log \frac{1}{\sqrt{(\xi-\alpha)^2 + (\eta-\beta)^2}},$$

$$R(x, y, \xi, \eta, \alpha, \beta) = R(\xi, \eta, x, y, \alpha, \beta).$$

* In the same manner may be formed a doubly periodic function

$$V(x, y, \xi_1, \eta_1, \cdots, \xi_n, \eta_n) = \sum_{i=1}^{n} c_i \log \sqrt{(x-\xi_i)^2 + (y-\eta_i)^2} + S(x, y, \xi_1, \eta_1, \cdots, \xi_n, \eta_n)$$

with any number of logarithmic singularities, where $S$ is a solution of LAPLACE's equation in $\Omega$ with respect to the variables $x$, $y$, provided that $c_1 + c_2 + \cdots + c_n = 0$.

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To prove these laws apply GREEN's theorem,

$$\int\int (v\Delta u - u\Delta v)dxdy = \int \left( v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right) ds,$$

where $n$ is the outward drawn normal, to the region $\Omega'$ formed by excluding from $\Omega$ the circles $c(\xi, \eta)$, $c(\xi', \eta')$, $c(\alpha, \beta)$ of radius $r$ about the points $(\xi, \eta)$, $(\xi', \eta')$, $(\alpha, \beta)$ respectively, and choose

$$u = G(x, y, \xi, \eta, \alpha, \beta), \qquad v = G(x, y, \xi', \eta', \alpha, \beta).$$

Since $u$ and $v$ are, within $\Omega'$, solutions of LAPLACE's equation the double integral over $\Omega'$ is zero. Furthermore, since $u$ and $v$ are doubly periodic in $x$, $y$, each assumes equal values at opposite points of the bounding lines of the rectangle $\Omega$, while at these points the normal derivative of each assumes values numerically equal but opposite in sign. Therefore the line integral over the sides of the rectangle $\Omega$ is zero, and we have, replacing $ds$ by $rd\theta$,

$$\int_{c(\xi,\eta)} \frac{1}{r} G(x, y, \xi', \eta', \alpha, \beta) rd\theta - \int_{c(\xi',\eta')} \frac{1}{r} G(x, y, \xi, \eta, \alpha, \beta) rd\theta$$

$$+ \int_{c(\alpha,\beta)} \frac{1}{r} \{ G(x, y, \xi, \eta, \alpha, \beta) - G(x, y, \xi', \eta', \alpha, \beta \} rd\theta + h = 0,$$

where

$$\lim_{r=0} h = 0.$$

But

$$G(x, y, \xi, \eta, \alpha, \beta) - G(x, y, \xi', \eta', \alpha, \beta) =$$

$$\log \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} - \log \frac{1}{\sqrt{(x-\xi')^2 + (y-\eta')^2}}$$

$$+ R(x, y, \xi, \eta, \alpha, \beta) - R(x, y, \xi', \eta', \alpha, \beta),$$

and

$$R(\alpha, \beta, \xi, \eta, \alpha, \beta) = 0, \qquad R(\alpha, \beta, \xi', \eta', \alpha, \beta) = 0.$$

We obtain therefore in the limit $r = 0$, by well known methods,

$$G(\xi, \eta, \xi', \eta', \alpha, \beta) - G(\xi', \eta', \xi, \eta, \alpha, \beta)$$

$$+ \log \frac{1}{\sqrt{(\alpha-\xi)^2 + (\beta-\eta)^2}} - \log \frac{1}{\sqrt{(\alpha-\xi')^2 + (\beta-\eta')^2}} = 0,$$

or writing $x$, $y$ for $\xi'$, $\eta'$,

$$G(x, y, \xi, \eta, \alpha, \beta) + \log \frac{1}{\sqrt{(x-\alpha)^2 + (y-\beta)^2}}$$

$$= G(\xi, \eta, x, y, \alpha, \beta) + \log \frac{1}{\sqrt{(\xi-\alpha)^2 + (\eta-\beta)^2}}.$$

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which is the law of reciprocity for $G$. If we replace $G$ in this equation by its expression from 2° we have immediately,

$$R(x, y, \xi, \eta, \alpha, \beta) = R(\xi, \eta, x, y, \alpha, \beta).$$

Thus $R$ is symmetrical with respect to $x$, $y$ and $\xi$, $\eta$ and is therefore a solution of

$$\frac{\partial^2 R}{\partial \xi^2} + \frac{\partial^2 R}{\partial \eta^2} = 0$$

within $\Omega$.

§ 2. *The doubly periodic solutions of*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y). \tag{1}$$

Suppose a doubly periodic solution $u$ of equation (1) exists, for the periods $a$, $b$, where $f$ is continuous and doubly periodic with the same periods. If a second solution of the same nature existed, the difference of the two would be a doubly periodic solution of LAPLACE's equation, and therefore a constant. It follows that a doubly periodic solution of (1) for the periods $a$, $b$ is uniquely determined if its value at a fixed point is given.

Apply GREEN's theorem to the period rectangle $\Omega$, choosing for $u$ a doubly periodic solution of (1) and taking $v = 1$. Since the integral over the boundary vanishes, the following equation results:

$$\int_0^b \int_0^a \Delta u \, dx \, dy = \int_0^b \int_0^a f(x, y) \, dx \, dy = 0.$$

This equation is a necessary condition for the existence of a doubly periodic solution of (1). We shall now show that it is also sufficient. Consider the function

$$\begin{aligned} u(\xi, \eta) &= -\frac{1}{2\pi} \int_0^b \int_0^a G(x, y, \xi, \eta, \alpha, \beta) f(x, y) \, dx \, dy \\ &= -\frac{1}{2\pi} \int_0^b \int_0^a \log \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} f(x, y) \, dx \, dy \\ &\quad + \frac{1}{2\pi} \int_0^b \int_0^a \log \frac{1}{\sqrt{(x-\alpha)^2 + (y-\beta)^2}} f(x, y) \, dx \, dy \\ &\quad - \frac{1}{2\pi} \int_0^b \int_0^a R(x, y, \xi, \eta, \alpha, \beta) \, dx \, dy. \end{aligned}$$

Since

$$\frac{\partial^2 R}{\partial \xi^2} + \frac{\partial^2 R}{\partial \eta^2} = 0,$$

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we see at once, from the potential theory, that $u(\xi, \eta)$ is a solution of

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = f(\xi, \eta).$$

Furthermore, putting $\xi = \alpha$, $\eta = \beta$ we have

$$u(\alpha, \beta) = 0,$$

since

$$R(x, y, \alpha, \beta, \alpha, \beta) = R(\alpha, \beta, x, y, \alpha, \beta) = 0.$$

From the law of reciprocity for $G$ we have

$$G(x, y, \xi + ma, \eta + nb) - G(x, y, \xi, \eta, \alpha, \beta)$$
$$= \log \frac{1}{\sqrt{(\xi + ma - \alpha)^2 + (\eta + nb - \beta)^2}} - \log \frac{1}{\sqrt{(\xi - \alpha)^2 + (\eta - \beta)^2}},$$

and therefore

$$u(\xi + ma, \eta + nb) - u(\xi, \eta)$$
$$= -\frac{1}{2\pi} \log \sqrt{\frac{(\xi - \alpha)^2 + (\eta - \beta)^2}{(\xi + ma - \alpha)^2 + (\eta + nb - \beta)^2}} \int_0^b \int_0^a f(x, y)\, dx dy.$$

Therefore, if

$$\int_0^b \int_0^a f(x, y)\, dx\, dy = 0,$$

the function $u$ possesses the periods $a$, $b$. We have therefore proved the theorems:

*The necessary and sufficient condition for the existence of a doubly periodic solution (periods $a$, $b$) of the equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y), \tag{1}$$

*where $f$ is a continuous doubly periodic function with the periods $a$, $b$, is that $f$ satisfy the equation*

$$\int_0^b \int_0^a f(x, y)\, dx\, dy = 0.$$

*If this condition is satisfied, then the doubly periodic solution of* (1), *with periods $a$, $b$, which assumes the value $C$ at $x = \alpha$, $y = \beta$ is uniquely determined, and is given by the formula:*

$$u(\xi, \eta) = -\frac{1}{2\pi} \int_0^b \int_0^a G(x, y, \xi, \eta, \alpha, \beta) f(x, y)\, dx\, dy + C,$$

*where $G$ is a known function, expressible in terms of Sigma functions.*

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*Multiplication of complexes.** When $E_1$ and $E_2$ are defined as above, the product $E_1 E_2$ is defined as the complex of all numbers linearly dependent on $X_p X'_q (p = 1, \cdots, m; q = 1, \cdots, m')$. It is easily shown from the definition of multiplication of complexes that the process is associative. Thus if we consider the system

$$\begin{array}{c|cc} & e_1 & e_2 \\ \hline e_1 & e_1 & e_2 \\ e_2 & e_2 & 0 \end{array}$$

and let $E_1 \equiv e_1$, $E_2 \equiv e_2$, then $E_1 E_2 = e_1 e_2 = e_2$. It can easily be seen that if the main system $E$ contains a modulus (say $e_1$) then $EE (= E^2) = E$; for, the modulus (unit) $e_1$ multiplied by $e_1 \cdots e_n$ on the right gives $e_1 \cdots e_n$ and therefore $E^2$ contains all the units of $E$. If $E$ does not contain a modulus it may happen that in $E^2$ some of the units $e_1, \cdots, e_n$ do not appear and therefore $E^2 \precsim E$. The condition that $E$ be a system, closed under multiplication, is $E^2 \precsim E$.

*Semireducibility.* The algebra $E$ is said to be semireducible† when by a proper choice of units the following conditions are fulfilled:

$$\left.\begin{array}{ll} (4) & X \cdot {}_2X = {}_2X' \\ (5) & {}_2X \cdot X = {}_2X'' \end{array}\right\} \text{for every } X, {}_2X,$$

the general number of $E$ being

$$X = {}_1X + {}_2X' = \sum_{j=1}^{m} x_j e_j + \sum_{k=m+1}^{n} x_k e_k.$$

It follows at once that the complex $E_2 = e_{m+1} \cdots e_n$ is closed under multiplication. The conditions (4) and (5) may be written

$$(4') \qquad E \cdot E_2 \precsim E_2,$$

$$(5') \qquad E_2 E \precsim E_2,$$

and, therefore, of course, $E_2 E_2 = E_2^2 \precsim E_2$. Under these conditions the subalgebra $E_2$ is said to be invariant in $E$.‡

THEOREM. *$E_2$ cannot contain the modulus of $E$ among its numbers.* For, from (4) and (5) it is impossible that any number ${}_2I$ of $E_2$ shall have the property that ${}_2I \cdot X = X \cdot {}_2I = X$ for every number $X$ of $E$.

*Accompanying system,§ complementary system.* If

$$E = E_1 + E_2 \equiv e_1 \cdots e_m e_{m+1} \cdots e_n,$$

*FROBENIUS, Berliner Sitzungsberichte, 1895, p. 164.

†Transactions, vol. 4 (1903), p. 440.

‡CARTAN, Annales de Toulouse, vol. 12 (1898).

§MOLIEN'S *Begleitendes System*, Mathematische Annalen, vol. 41 (1893), p. 93.

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we may write, in well known notation, $E = E_1 (\text{mod } E_2)$, $E = E_2 (\text{mod } E_1)$; similarly, if $X = {}_1X + {}_2X$, where $X$ belongs to $E$, ${}_1X$ to $E_1$, ${}_2X$ to $E_2$, we may write $X = {}_1X (\text{mod } E_2)$ or $X = {}_2X (\text{mod } E_1)$. Regarding as equal* any two numbers of $E$ which are equal, modulo $E_2$, we get, when $E_2$ is an invariant subsystem, a hypercomplex number system $K$ which is said to *accompany* $E$ and to be *complementary to the invariant subsystem* $E_2$ *with respect to* $E$. It is evidently sufficient to show that the system so defined is associative. If ${}_1X_\alpha$, ${}_1X_\beta$, ${}_1X_\gamma$ are three numbers of $E_1$ and if

$$ {}_1X_\alpha \cdot {}_1X_\beta = X_{\alpha\beta} = {}_1X_{\alpha\beta} + {}_2X_{\alpha\beta}, \qquad {}_1X_\beta \cdot {}_1X_\gamma = X_{\beta\gamma} = {}_1X_{\beta\gamma} + {}_2X_{\beta\gamma}, $$

then

$$ ({}_1X_\alpha \cdot {}_1X_\beta) \cdot {}_1X_\gamma = {}_1X_{\alpha\beta} \cdot {}_1X_\gamma + {}_2X_{\alpha\beta} \cdot {}_1X_\gamma = {}_1X_{(\alpha\beta)\gamma} + {}_2X' $$

if $E_2 E_1 \lessdot E_2$. Similarly

$$ {}_1X_\alpha \cdot ({}_1X_\beta \cdot {}_1X_\gamma) = {}_1X_{\alpha(\beta\gamma)} + {}_2X''. $$

Now by the associative law and the linear independence of the units we have

$$ {}_1X_{(\alpha\beta)\gamma} \equiv {}_1X_{\alpha(\beta\gamma)}, $$

and therefore

$$ ({}_1X_\alpha \cdot {}_1X_\beta) \cdot {}_1X_\gamma \equiv {}_1X_\alpha \cdot ({}_1X_\beta \cdot {}_1X_\gamma) (\text{mod } E_2). $$

The above may also be deduced from a consideration of the $\gamma_{i_1 i_2 i_3}$'s. Using (3) it is seen that

$$ (3_j) \qquad \sum_{i_3=1}^{n} (\gamma_{j_1 j_2 i_3} \gamma_{i_3 j_4 j_5} - \gamma_{j_1 i_3 j_5} \gamma_{j_2 j_4 i_3}) = 0 \qquad (j_1, j_2, j_4, j_5 = 1, \cdots, m). $$

Now if

$$ E_2 E_1 \lessdot E_2, \qquad E_1 E_2 \lessdot E_2, $$

or,

$$ {}_2X \cdot {}_1X = {}_2X', \qquad {}_1X \cdot {}_2X = {}_2X'', $$

that is to say, if every $\gamma_{kjj} = 0$ and every $\gamma_{jkj} = 0$, we may write $(3_j)$ in the form

$$ (3'_j) \qquad \sum_{j_3=1}^{m} (\gamma_{j_1 j_2 j_3} \gamma_{j_3 j_4 j_5} - \gamma_{j_1 j_3 j_5} \gamma_{j_2 j_4 j_3}) = 0.\dagger $$

## § 2.

By $E_1 \frown E_2$ is meant the common complex of $E_1$ and $E_2$. Thus if in (1), $E_1 \equiv e_1 \cdots e_m \cdots e_r$, $E_2 \equiv e_{m+1} \cdots e_r \cdots e_n$, then $E_1 \frown E_2 \equiv e_{m+1} \cdots e_r$. $E_1 \frown E_2 = 0$ states that $E_1$ and $E_2$ have no numbers in common.

* FROBENIUS, l. c., p. 634.

† According to MOLIEN, loc. cit., p. 92, it is necessary that $EE_2 \prec E_2$, $E_2 E \lessdot E_2$. Evidently, from the text, we can have an accompanying system with the milder conditions $E_2 E_1 \lessdot E_2$, $E_1 E_2 \lessdot E_2$.

It may be remarked at this point that the $\gamma_{ijj}$ are the structure constants of $K$.

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*Reducibility.* According to PEIRCE* an algebra $E$ is said to be mixed (or reducible) if new units can be introduced such that $E = E_1 + E_2$, where $E_1^2 \leqq E_1$, $E_2^2 \leqq E_2$, $E_1 E_2 \leq E_1 \frown E_2$, $E_2 E_1 \leqq E \frown E_2$.†

THEOREM I. *If $E_1$ is a maximal invariant subalgebra of $E$ and if there exists a second invariant subalgebra $E_1'$ of $E$, then either $E$ is reducible or else $E_1'$ is a subalgebra $E_1$.*

Since $E_1$ and $E_1'$ are invariant we have (*a*) $EE_1 \leqq E_1$, (*b*) $E_1 E_1' \leqq E_1$, (*c*) $EE_1' \leqq E_1'$, (*d*) $E_1' E \leqq E_1'$. We have now that $E_1 + E_1'$ is an invariant subalgebra of $E$ for $E(E_1 + E_1') \leqq E_1 + E_1'$, $(E_1 + E_1')E \leqq E_1 + E_1'$ and since $E_1$ is is maximal by hypothesis $E_1 + E_1' = E$ or else $E_1 + E_1' = E_1$. If $E_1 + E_1' = E$ the algebra is easily seen to be reducible. First, $E_1^2 \leqq E_1$ by (*a*), $E_1'^2 \leqq E_1'$ by (*c*), $E_1 E_1' \leqq E_1$ by (*b*) and $E_1 E_1' \leqq E_1'$ by (*c*) and therefore $E_1 E_1' \leqq E_1 \frown E_1'$. Similarly by (*b*) and (*d*), $E_1' E_1 = E_1 \frown E_1'$. In the second case, where $E_1 + E_1' = E_1$, it is evident that $E_1'$ is a subalgebra of $E_1$.

THEOREM II. *If $E_1$ and $E_1'$ are maximal invariant subalgebras of $E$, and if $E_1 \frown E_1' = F \neq 0$, then $F$ is a maximal invariant subalgebra of both $E_1$ and $E_1'$.*

By theorem I, $E = E_1 + E_1'$. Let

$$E = E_0 + E_1 \qquad (E_0 \frown E_1 = 0),$$
$$= E_0' + E_1' \qquad (E_0' \frown E_1' = 0).$$

Evidently then $E = E_0 + E_0' + F$. Because the products $E_1 F$, $FE_1$, $E_1' F$, $FE_1'$ must be contained in $E_1$ and in $E_1'$ it follows that

$$E_1 F \leqq F, \qquad FE_1 \leqq F, \qquad E_1' F \leqq F, \qquad FE_1' \leqq F,$$

and therefore that $F$ is invariant in both $E_1$ and $E_1'$.

Suppose now that $F$ is not maximal in $E_1$ and let $G + F$ (where $G \frown F = 0$) be a maximal invariant subsystem of $E_1$, i. e.,

$$E_1(G + F) \leqq G + F, \qquad (G + F)E_1 \leqq G + F.$$

Consider the complex $G + E_1'$. Clearly $G \frown E_1' = 0$ and

$$GE_1' \leqq F, \qquad E_1' G \leqq F,$$

since the products must both lie in $E_1$ and $E_1'$. Therefore, since

$$E_1 G \leqq G + F, \qquad E_1 E_1' \leqq F, \qquad E_1' G \leqq F, \qquad E_1'^2 \leqq E_1',$$

we have

$$E(G + E_1') = (E_1 + E_1')(G + E_1') \leqq G = E_1'.$$

It is shown in a similar manner that $(G + E_1')E \leqq G + E_1'$. Thus, if $F$ is not a maximal invariant subalgebra of $E_1$, $E_1'$ will not be a maximal invariant

* American Journal, vol. 4 (1881), p. 100.

† In SCHEFFERS's definition, Mathematische Annalen, vol. 39 (1891), p. 317, $E_1 \frown E_2 = 0$.

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subalgebra of $E$ (since $G + E_1'$ has been proved invariant and $G \frown E_1' = 0$), which is contrary to the hypothesis. It is shown in exactly the same way that $F$ is maximal in $E_1'$.

THEOREM III. *Let $E, E_1, E_2, \cdots$ be a normal series of subalgebras of $E$ (i. e., $E_r$ is a maximal invariant subalgebra of $E_{r-1}$, $E_0 = E$) and let $K_1, K_2, \cdots$ be a series of complementary algebras, such that $K_r$ accompanies $E_{r-1}$ and is complementary to $E_r$. Under these assumptions the series $K_1, K_2, \cdots$ is, apart from the order, independent of the choice of the series $E, E_1, E_2, \cdots$. In other words, if $E, E_1', E_2', \cdots$ is any other normal series, the complementary series of algebras $K_1', K_2', \cdots$ which it defines is the same as the series $K_1, K_2, \cdots$, apart from the sequence.*

According to the demonstration of theorem 1, we have (if $E_1 \neq E_1'$)

$$E = E_1 + E_1'. \tag{6}$$

Let $E_1 \frown E_1' = F_1$, then

$$E_1 = F_1 + D_1 \qquad (F_1 \frown D_1 = 0), \tag{7}$$

$$E_1' = F_1 + D_1' \qquad (F_1 \frown D_1' = 0),$$

and it is evident that

$$E = F_1 + D_1 + D_1'. \tag{8}$$

By theorem 2, $F_1$ is a maximal invariant subalgebra of $E_1$ and $E_1'$.

If $F_1, F_2, F_3$ is a normal series of $F_1$, then $E, E_1, F_1, F_2, \cdots$ and $E, E_1', F_1, F_2, \cdots$ are two new normal series of $E$. We will now have to prove that the proposition is true for these two series. For this purpose it is merely necessary to prove that the complementary algebras defined by $E, E_1, F_1$ and $E, E_1', F_1$ are the same. If

$$E \equiv e_1 \cdots e_{d_1} e_{d_1+1} \cdots e_t e_{t+1} \cdots e_n,$$

$$E_1 \equiv e_1 \cdots e_{d_1} e_{d_1+1} \cdots e_t,$$

$$E_1' \equiv e_{d_1+1} \cdots e_t e_{t+1} \cdots e_n,$$

then

$$E_1 \frown E_1' = F_1 \equiv e_{d_1+1} \cdots e_t,$$

$$D_1 \equiv e_1 \cdots e_{d_1}, \qquad D_1' = e_{t+1} \cdots e_n.$$

Now the complementary algebra of $E_1 = F_1 + D_1$ with respect to $E$ is defined by the $(n-t)^3$ characteristic constants $\gamma_{p_1 p_2 p_3}$ where

$$\sum_{p_3} (\gamma_{p_1 p_2 p_3} \gamma_{p_3 p_4 p_5} - \gamma_{p_1 p_3 p_5} \gamma_{p_2 p_4 p_3}) = 0 \quad (p = t+1, t+2, \cdots, n). \tag{9}$$

Furthermore, the complementary algebra of $F_1$ with respect to $E_1 (E_1 = F_1 + D_1)$ is defined by the $d_1^3$ characteristic constants $\gamma_{q_1 q_2 q_3}$ where

$$\sum_{q_3} (\gamma_{q_1 q_2 q_3} \gamma_{q_3 q_4 q_5} - \gamma_{q_1 q_3 q_5} \gamma_{q_2 q_4 q_3}) = 0 \qquad (q = 1, \cdots, d_1). \tag{10}$$

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Now it is seen in the same manner that the complementary algebra of $E_1'$ with respect to $E$ is defined by (10) and the complementary algebra of $F_1$ with repect to $E_1'$ is defined by (9), and therefore the normal complementary series defined by $E, E_1, F_1$ is identical, apart from the sequence, with the normal complementary series defined by $E, E_1', F_1$.

It is now necessary to show that the normal series $E, E_1, E_2 \cdots, E, E_1, F_1, F_2, \cdots$ define the same complementary series, and similarly for $E, E_1', E_2', \cdots, E, E_1', F_1, F_2, \cdots$. This amounts to proving the theorem for the algebras $E_1$ and $E_1'$, which involves a finite number of repetitions of the above proof.

## § 3.

Let $E, E_1, E_2, \cdots$ be a normal series of $E$, let $a_s$ be the order (number of units) of $E_s$ and let $l_s$ be the difference between $a_{s-1}$ and the maximal order of a subalgebra * of $E_{s-1}$ which contains $E_s (E_0 = E)$.

THEOREM IV. *The numbers $l_1, l_2, \cdots$, apart from their sequence, are independent of the choice of the normal series.*

This theorem is an immediate consequence of theorem 3, the integer $l_s$ being regarded as the difference between the order of the complementary algebra $K_s$ of $E_s$ with respect to $E_{s-1}$ and the order of a maximal subalgebra of $K_s (s = 1, 2, \cdots)$.

Similarly it follows also *that the series of integers $k_s = a_{s-1} - a_s$ is independent of the choice of the normal series.*

*Simple algebras.* An algebra which contains no invariant subalgebra is said to be simple.

THEOREM V. *The complementary algebras $K_1, K_2, \cdots$ defined by the normal series $E, E_1, E_2 \cdots$ are all simple.*†

In order to prove this, consider the algebra $A = B + A_1$ (where $B \frown A_1 = 0$) wherein $A_1$ is a maximal invariant subalgebra, and let $K$ be the complementary system of $A_1$ with respect to $A$. Suppose now that $K$ has an invariant subalgebra $K_1$, $K = K' + K_1$ (where $K' \frown K_1 = 0$). This says that

$$KK_1 \preceq K_1, \qquad K_1 K \preceq K_1. \tag{11}$$

From the correspondence between $K$ and $B$‡ it is seen that we can write $B = B' + B_1$ (where $B' \frown B_1 = 0$) where, by means of (11),

$$BB_1 \preceq B_1 + A_1, \qquad B_1 B \preceq B_1 + A_1. \tag{12}$$

Consider now the complex $B_1 + A_1$. We have, since $A_1$ is invariant in $A$, and by means of (12),

* i. e., the order of the subalgebra containing the largest number of independent units.

† This theorem is implicitly contained in MOLIEN's paper, loc. cit., p. 96.

‡ The $\gamma_{ijk}$ of $K$ are obtained by writing every $\gamma_{ijk} = 0$ in the $\gamma_{ijk}$ of $B$.

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$$A(B_1 + A_1) = (B + A_1)(B_1 + A_1) \leqq B_1 + A_1;$$

and similarly

$$(B_1 + A_1)A \leqq B + A_1.$$

Thus $B_1 + A_1$ is an invariant subalgebra of $A$ and since $B_1 \frown A_1 = 0$, it follows that $B_1 + A_1 > A_1$ which is impossible since $A_1$ is maximal by hypothesis. Therefore $K$ is simple.

*The chief series of an algebra.* The series $E, P_1, P_2, \cdots$ is said to be a chief or principal series (chief composition series) when $P_s$ is a maximal subalgebra of $P_{s-1}$ which is invariant in $E(P_0 = E)$. For the chief series it is shown in exactly the same manner as for the normal series that *the system of complementary algebras $C_1, C_2, C_3, \cdots$ is invariant, apart from the sequence ($C_s$ being complementary to $P_s$ with respect to $P_{s-1}$).*

We can now define a series of chief indices of composition analogous to the indices of composition above, and show that for a given algebra $E$, *the system of chief indices of composition is independent of the choice of the chief series, apart from the sequence.*

THE UNIVERSITY OF CHICAGO,
*December*, 1904.

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# ON A DEFINITION OF ABSTRACT GROUPS*

BY

ELIAKIM HASTINGS MOORE

In a paper entitled *A definition of abstract groups* (Transactions, vol. 3, pp. 485–492, October, 1902), my second definition (l. c., p. 490)

$$(M'') = (1, 2, 3', 3'', 3''_r, 4''_l)$$

involves postulates not mutually independent. I shall prove here (as stated in October, 1904, Transactions, vol. 5, p. 549) that $(3''_r)$ is redundant, that in the new definition

$$(\bar{M}''): (1, 2, 3'', 3''_l, 4''_l)$$

the postulates are mutually independent, and that this mutual independence remains even for the system

$$(M''_A): (1, 2, 3'', 3''_l, 4''_l, A)$$

defining an abelian group, obtained by adding to those of $(M'')$ the postulate $(A)$ that the multiplication or composition of two elements is commutative.

We have for consideration a set † or class $(K)$ of elements and a multiplication-table or rule of combination $(\circ)$ whereby to every two elements $a, b$ taken in the definite order $a, b$ there corresponds a definite so-called product, in notation $a \circ b$, or, when without confusion, more simply, $ab$; this product may or may not be an element of the class. The postulates in question are then the following:

(1) If $a$ and $b$ are elements, then $ab$ is an element of the class.

$(A)$ If $a$ and $b$ are elements such that $ab$ and $ba$ are elements, then $ab = ba$.

(2) If $a, b, c$ are elements such that $ab, bc, (ab)c, a(bc)$ are elements, then $(ab)c = a(bc)$.

$(3'')$ There exists an element $a$ such that $aa = a$.

$(3''_l)$ If $a$ and $b$ are elements and $aa = a$, then $ab = b$.

$(3''_r)$ If $a$ and $b$ are elements and $aa = a$, then $ba = b$.

$(4''_l)$ If $a$ and $b$ are elements and $aa = a$, then there exists an element $b_l^{(a)}$ such that $b_l^{(a)} b = a$.

* Presented to the Society December 30, 1904. Received for publication March 17, 1905.

† In the first definition $(M)$ there was the underlying understanding (l. c., p. 485, footnote ‡) or postulate (0) that the class contain at least one element, and this carried over by implication to $(M'')$, where however in view of $(3'')$ it was not needed; it is now omitted.



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If $a$ is an element such that $aa = a$ we designate those parts of $(3''_l)$, $(3''_r)$, $(4''_l)$ which refer to the element $a$ by $(3''^a_l)$, $(3''^a_r)$, $(4''^a_l)$ respectively. Then we prove that $(3''_r)$ is deducible from $(1, 2, 3'', 3''_l, 4''_l)$ in that we prove that $(3''^a_r)$ is deducible from $(1, 2, 3''^a_l, 4''^a_l)$. This fact is proved by a suitable modification of the method used p. 486, 7° loc. cit. If $a$ and $b$ are elements and $aa = a$, then we designate by $b'$, $b''$ elements, which by $(4''^a_l)$ surely exist, such that $b'b = a$, $b''b' = a$ and have in virtue of $(1, 2, 3''^a_l)$ the continued equality,

$$ba = bb'b = abb'b = b''b'bb'b = b''ab'b = b''b'b = ab = b,$$

that is, $ba = b$, which was to be proved. *

The proof that the postulates of $(M''_A)$ are independent covers the independence of the postulates of $(\bar{M}'')$ and appears from the following proof systems $(K, \circ)$:

For $(1)$. $K =$ all integers $0, \pm 1, \pm 2, \cdots$ except $\pm 1$. $\circ = +$.

For $(2)$. $K =$ an element $a$ and any class of (at least three) elements $x$ distinct from $a$. $a \circ a = a$. $a \circ x = x \circ a = x$. $x_1 \circ x_2 = a$ if $x_1 \neq x_2$; $x_1 \circ x_1 = x_3$ where, for every $x_1$, $x_3$ is any $x$ except $x_1$.

For $(3'')$. $K =$ all positive integers. $\circ = +$.

For $(3''_l)$. $K =$ a class of (at least two) elements $x$. $x_1 \circ x_2 = a$, where $a$ is a definite element, the same for all pairs $x_1$, $x_2$.

For $(4''_l)$. $K =$ all positive integers and $0$. $\circ = +$.

For $(A)$. $(K, \circ) =$ any non-abelian group.

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* [A second continued equality

$$ba = aba = b''b'ba = b''aa = b''a = b''b'b = ab = b$$

should be noticed.]

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# NOTE ON THE DEFINITIONS OF ABSTRACT GROUPS AND FIELDS BY SETS OF INDEPENDENT POSTULATES*

BY

EDWARD V. HUNTINGTON

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## *Introduction.*

The theory of groups † was first studied in connection with the algebraic solution of equations, the leaders in the early development being LAGRANGE (1770), RUFFINI (1799), CAUCHY (1814), ABEL (1824), and especially GALOIS (1831). The first discussion of the theory from an abstract point of view was by CAYLEY ‡ in 1854, and the earliest explicit sets of postulates for abstract groups were

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† The most recent bibliography is given by B. S. EASTON, *The constructive development of group-theory*, Philadelphia, 1902; and the most recent text-book is the *Éléments de la théorie des groupes abstraits*, by J. A. DE SÉGUIER, Paris, 1904.

Cf. also G. A. MILLER's two reports on recent progress in group-theory, Bulletin of the American Mathematical Society, vol. 5 (1898–1899), pp. 227–251, and vol. 7 (1900–1901), pp. 121–130.

‡ A. CAYLEY, Philosophical Magazine, vol. 7 (1854), p. 40; see also Proceedings of the London Mathematical Society, vol. 9 (1878), p. 126, or American Journal of Mathematics, vol. 1 (1878), p. 51. [Collected Papers, vol. 2, p. 123; vol. 10, p. 324, or p. 401.] Cf. W. Dyck, Mathematische Annalen, vol. 20 (1882), p. 1.



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given by KRONECKER * in 1870 and WEBER † in 1882. WEBER's definition, as finally formulated, ‡ is substantially as follows:

*A*. There is a rule of combination among the elements of the given set, such that every two elements, $a$ and $b$, determine uniquely a third element $c$, called the result of the composition; in symbols: $ab = c$.

*B*. The associative law holds throughout the set; that is, $(ab)c = a(bc)$.

*C*. 1) If $ab = ab'$, then $b = b'$.

2) If $ab = a'b$, then $a = a'$.

*D*. 1) Given $a$ and $c$, there is an element $b$ such that $ab = c$.

2) Given $b$ and $c$, there is an element $a$ such that $ab = c$.

*E*. In the case of abelian groups, the commutative law also holds; that is $ab = ba$.

WEBER notes that if the group is finite, $D$ is a consequence of $A$, $B$, and $C$.

This definition was somewhat simplified by BURNSIDE § in 1897, and, more explicitly, by Professor PIERPONT ‖ in 1900. PIERPONT replaced conditions $C$ and $D$ by the following:

*C′*. There exists a unique element 1, called the identity, such that $a1 = 1a = a$ for every element $a$.

*D′*. For every element $a$ there exists an element $a^{-1}$, called the inverse of $a$, such that $aa^{-1} = a^{-1}a = 1$.

The equivalence of the two definitions is readily established. PIERPONT's definition, like most of the earlier definitions, was stated only for the case of finite groups; but essentially the same definition had been used for groups in general by Professor MOORE in lectures ¶ in 1897.

The earliest discussion of the *independence* of the postulates for abstract groups was, as far as I know, that contained in a paper of my own ** in 1902. I noticed in the first place that WEBER's postulate $C$ was deducible from his $A$, $B$, and $D$, †† and further, that by a peculiar wording of the associative law,

* L. KRONECKER, Monatsberichte der königlich preussischen Akademie der Wissenschaften zu Berlin (1870), p. 882.

† H. WEBER, Mathematische Annalen, vol. 20 (1882), p. 302. Cf. also G. FROBENIUS in Crelle's Journal für die reine und angewandte Mathematik, vol. 100 (1887), p. 179, and in the Sitzungsberichte der königlich preussischen Akademie der Wissenschaften zu Berlin (1895), p. 163.

‡ H. WEBER, Mathematische Annalen, vol. 43 (1893), p. 521; or *Algebra*, second edition, vol. 2 (1899), p. 3.

§ W. BURNSIDE, *Theory of groups of finite order*, 1897, p. 11.

‖ J. PIERPONT, Annals of Mathematics, ser. 2, vol. 2 (1900–01), p. 47. (From a course of lectures delivered in 1896 at the Buffalo Colloquium of the American Mathematical Society.)

¶ Cf. E. H. MOORE, Transactions, vol. 3 (1902), p. 488, footnote.

** E. V. HUNTINGTON, Bulletin of the American Mathematical Society, vol. 8 (1901–02), pp. 296–300, revised in Transactions, vol. 4 (1903), p. 30. See also a second definition, in Bulletin, loc. cit., pp. 388–391, and two corresponding definitions for abelian groups, in Transactions, loc. cit., pp. 27–29.

†† For the case of Abelian groups, cf. § 4, below.

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postulate $A$ also could be made redundant. Since postulate $A$ is the fundamental postulate of the whole theory, the resulting definition (comprising only $D$ and a modified form of $B$) had obvious disadvantages. Revised forms of this definition, however, in which the rule of combination is replaced by a relation, as suggested by Professor BÔCHER* in his address at the St. Louis Congress of 1904, have considerable logical interest, and will be presented in §§ 5–6 of this note.

More convenient definitions by sets of independent postulates were given in the same year (1902) by Professor MOORE.† The first of these definitions is of the type published by PIERPONT, differing from WEBER's definition in the use which it makes of the identity and the inverse elements. A second definition given at the end of the paper is a modification of the first, embodying a further analysis of the identical element, and demanding, in particular, the existence of an "idempotent" element $i$ such that $ii = i$. A note by MOORE on this second definition appears in the present number of the Transactions.

Another modification of MOORE's first definition, reverting more nearly to PIERPONT's original form, has been recently given by Professor DICKSON, and will also be found in this number of the Transactions.

In § 1 of the present note I propose a further modification of the PIERPONT-MOORE type of definition, in the direction indicated by the latter part of MOORE's paper. Although the points of difference are so slight as to seem almost trivial, yet the introduction of postulate 4, demanding explicitly the uniqueness of the identical element, and the "weakened" forms in which postulates 5 and 6 are now stated, will be found very convenient when one has to test a given system for the group property.‡

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Closely connected with the theory of groups is the theory of fields,§ suggested by GALOIS, and due, in concrete form, to DEDEKIND‖ in 1871. The word *field* is the English equivalent for DEDEKIND's term *Körper*; KRONECKER's term *Rationalitätsbereich*,¶ which is often used as a synonym, had originally a some-

* M. BÔCHER, Bulletin of the American Mathematical Society, vol. 11 (1904–05), p. 126, footnote.

† E. H. MOORE, Transactions, vol. 3 (1902), pp. 485–492, revised in the present number of the Transactions, p. 179.

‡ Further analysis of the postulates, such as I have attempted in Transactions, vol. 6 (1905), pp. 34–36, does not seem likely to lead to practical advantage, except possibly in the case of the associative law.

§ See parts of *Linear Groups, with an exposition of the Galois Field theory*, by L. E. DICKSON, in the TEUBNER series of mathematical text-books, 1901; and parts of the *Éléments de la théorie des groups abstraits*, by J. A. DE SÉQUIER, Paris, 1904. [See review by DICKSON in the Bulletin of the American Mathematical Society, vol. 9 (1904–05), pp. 159–162.]

‖ P. G. L. DIRICHLET, *Vorlesungen über Zahlentheorie*, edited, with supplementary material, by R. DEDEKIND; 2d edition (1871), p. 424; 4th edition (1894), p. 452.

¶ L. KRONECKER, *Grundzüge einer arithmetischen Theorie der algebraischen Grössen*, 1882, Crelle, vol. 92 (1882), p. 1; Werke, vol. 2, p. 237. Cf. J. KÖNIG, *Einleitung in die allgemeine Theorie der algebraischen Grösszen*, 1903, p. 183.

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what different meaning. The earliest expositions of the theory from the general or abstract point of view were given independently by WEBER * and by MOORE, † in 1893, WEBER's definition of an abstract field being substantially as follows:

I. There are two rules of combination among the elements of the given set, the one called addition, and the other multiplication.

II. The set forms a group with respect to addition (see conditions $A$, $B$, $C$, $D$, above). The result of the composition of two elements is here called their sum $(a+b)$, the identity of the group being called zero (0), and the inverse operation subtraction.

III. If the identity of the additive group is excluded, the remaining elements form a group with respect to multiplication. The result of the composition of two elements is here called their product ($a \times b$, or $a \cdot b$, or $ab$), the identity being called unity (1), and the inverse operation division (the divisor being different from zero).

IV. Addition and multiplication are both commutative.

V. Addition is distributive with respect to multiplication; that is, $a(b+c) = (ab) + (ac)$.

VI. $a(-b) = -(ab)$.

As consequences of this definition, ‡ WEBER proves that $a \times 0 = 0 \times a = 0$ for every $a$, and that a product cannot be zero unless at least one of its factors is zero; but he makes no attempt to free the definition from redundant statements.

The earliest sets of *independent* postulates for abstract fields were given in 1903 by Professor DICKSON § and myself; ‖ all these sets were natural extensions of the sets of independent postulates that had already been given for groups.

In § 2 of the present note I propose another set of postulates for fields, based on the postulates for groups given in § 1, and possessing similar advantages.

Still other sets will be found in § 4, and in the paper by Professor DICKSON (already referred to) in the present number of the Transactions.

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* H. WEBER, Mathematische Annalen, vol. 43 (1893), p. 526.

† E. H. MOORE, Mathematical Papers read at the International Mathematical Congress at Chicago in 1893 (published in 1896), p. 210; abstract in the Bulletin of the New York Mathematical Society, vol. 3 (1893–94), p. 75. MOORE's theorem that every existent finite field is the abstract form of some Galois Field is established in this paper; but the term *Galois Field* as here used must not be confused with the term *Galois'scher Körper*, introduced by DEDEKIND (loc. cit., 2d edition, p. 455).

‡ The most familiar and important example of an infinite field is furnished by the rational numbers, under the operations of ordinary addition and multiplication. In fact, a field may be briefly described as a system in which the rational operations of algebra may all be performed (excluding division by zero). A field may be finite, provided the number of elements (called the order of the field) is a prime or a power of a prime.

§ L. E. DICKSON, Transactions, vol. 4 (1903), pp. 13–20.

‖ E. V. HUNTINGTON, Transactions vol. 4 (1903), pp. 31–37.

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## § 1. *A definition of groups and abelian groups.*

As already stated, the definition of abstract groups contained in this section is a modification of the definition given by Professor MOORE in 1902.

We consider a class $K$, and a rule of combination, subject to the following conditions (the result of combining $a$ and $b$ being denoted by $ab$):

POSTULATE 1. If $a$ and $b$ are elements of the class, then there is an element $c$ in the class such that $ab = c$; and this element $c$ is uniquely determined by $a$ and $b$.*

POSTULATE 2. The associative law holds throughout $K$; that is,

$$(ab)c = a(bc),$$

whenever $a$, $b$, $c$, $ab$, $bc$, $(ab)c$, and $a(bc)$ are elements of $K$.

POSTULATE 3. There is at least one element $i$ such that $ii = i$.

POSTULATE 4. There is not more than one element $i$ such that $ii = i$; that is, if $x$ and $y$ are elements such that $xx = x$ and $yy = y$, then $x = y$.

POSTULATE 5. If there is a unique element $i$ such that $ii = i$, then either $ia = a$ for every element $a$, or else $ai = a$ for every element $a$.

POSTULATE 6. If there is a unique element $i$ such that $ii = i$, then for every element $a$ there is either an element $a_r$ such that $aa_r = i$, or else an element $a_l$ such that $a_l a = i$.

From these postulates 1–6 the following theorems can be deduced, the proofs for which will be given in § 7:

*Theorem* I. If $i$ is the unique element described in postulates 3 and 4, then $ia = ai = a$ for every element $a$.

*Theorem* II. If $ab = ab'$, then $b = b'$; if $ab = a'b$, then $a = a'$.

From these theorems we have at once two further theorems, by the aid of postulate 6:

*Theorem* III. Every element $a$ determines uniquely an element $a^{-1}$ such that $aa^{-1} = a^{-1}a = i$, where $i$ is the element described in postulates 3 and 4.

*Theorem* IV. Every two elements $a$ and $b$ determine uniquely an element $x$,

* If $c$ were not uniquely determined by $a$ and $b$, then the expression $ab$ would have to be regarded as a multiple-valued function of $a$ and $b$, and the assertion: $ab = c$, would mean merely that *one of the values* of $ab$ is equal to $c$. Similarly, $(ab)c$ would mean the multiple-valued function obtained by combining $(ab)$, itself multiple valued, with the element $c$; and the equation $(ab)c = a(bc)$ would assert merely that *one* of the values of $(ab)c$ is equal to *one* of the value of $a(bc)$.

With this understanding of the notation, a system which satisfies all the postulates except the last half of (1) may be constructed as follows: Let $K =$ any finite or infinite class which contains a special element $X$ and at least one other element; and define the rule of combination so that if $a \neq b$, $ab$ may have any value within the class; while if $a = b$, $aa = X$.

Hence the second part of postulate 1 is essential to the definition. In fact, it is only the uniqueness of the symbol $ab$ which gives this notation any advantage over the relational notation employed in §§ 5–6.

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namely $x = a^{-1}b$, such that $ax = b$; and also an element $y$, namely $y = ba^{-1}$, such that $ya = b$.

Thus we see that any system which satisfies the postulates 1–6 will be a *group* with respect to the given rule of combination. The element $i$ is the *identity* of the group, and the element $a^{-1}$ is the *inverse of* $a$.

If we wish to make the group *abelian*, we must add another postulate, namely:

POSTULATE 7 (for abelian groups). The commutative law holds throughout $K$; that is,

$$ab = ba,$$

whenever $a$, $b$, $ab$, and $ba$ are elements of $K$.

These seven postulates are independent, as will be shown by the examples constructed in § 3; so that no one of them can be deduced from the remaining six.

If we wish to make the group *finite*, we must add the following postulate:

POSTULATE $N$ (for finite groups). The number of elements in the class is some positive integer, $n$.

After the introduction of this postulate $N$, postulates 3 and 6 become redundant (see § 7), so that *the five postulates*

$$1,\ 2,\ 4,\ 5,\ \textit{and}\ N$$

*are sufficient for finite groups.* The independence of these postulates, when $n > 2$, is established in § 3.

## § 2. *A definition of fields.*

The following set of postulates for abstract fields is suggested immediately by the postulates given in § 1 for groups.

We consider a class $K$, and two rules of combination, called addition and multiplication, subject to the following conditions ($A1 - A6$, $M1 - M4$, $M6$, $M7$, and $D$):*

POSTULATE $A1$. If $a$ and $b$ are elements of the class, then their "sum" $a + b$, is an element of the class, and is uniquely determined by $a$ and $b$.

POSTULATE $A2$. The associative law for addition holds throughout the class:

$$(a + b) + c = a + (b + c).$$

POSTULATE $A3$. There is at least one element $z$ such that $z + z = z$.

POSTULATE $A4$. There is not more than one element $z$ such that $z + z = z$.

POSTULATE $A5$. If there is a unique element $z$ such that $z + z = z$, then either $z + a = a$ for every element $a$, or else $a + z = a$ for every element $a$.

* The letters $A$ and $M$ in the designation of the postulates are intended to suggest addition and multiplication respectively, while $D$ indicates the distributive law for addition with respect to multiplication.

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POSTULATE $A6$. If there is a unique element $z$ such that $z + z = z$, then for every element $a$ there is either an element $a'_r$ such that $a + a'_r = z$, or else an element $a'_l$ such that $a'_l + a = z$.

From these postulates $A1$–$A6$ we have at once, by § 1, the following theorem:

*Theorem* 1. *The class $K$ is a group with respect to addition.*

The identity, $z$, of this group, is called the *zero-element* of the field, and is denoted by 0; while the inverse of an element $a$ is here called the *negative of* $a$, and is denoted by $-a$.

POSTULATE $M1$. If $a$ and $b$ are elements of the class, then their "product," $a \times b$ (or $a \cdot b$, or $ab$), is an element of the class, and is uniquely determined by $a$ and $b$.

POSTULATE $M2$. The associative law for multiplication holds throughout the class:

$$(a \times b) \times c = a \times (b \times c).$$

POSTULATE $M3$. There is at least one element $u$ such that $u \times u = u$ and $u + u \neq u$.

POSTULATE $M4$. There is not more than one element $u$ such that $u \times u = u$ and $u + u \neq u$.

*Lemma* $M5$. If there is a unique element $u$ such that $u \times u = u$ and $u + u \neq u$, then either $u \times a = a$ for every element $a$, or else $a \times u = a$ for every element $a$.

This lemma $M5$ is not included in the list of postulates, since it will prove to be deducible from $A1$–$A6$, $M1$–$M4$, $M6$, $M7$, and $D$ (see § 7); it is assumed for the moment, however, as are also lemmas $M7'$ and $D'$, below, in order to show exactly how much of the postulates $M7$ and $D$ is required for the proof of theorem 3.

POSTULATE $M6$. If there is a unique element $u$ such that $u \times u = u$ and $u + u \neq u$, then for every element $a$, provided $a + a \neq a$, there is either an element $a''_r$ such that $a \times a''_r = u$, or else an element $a''_l$ such that $a''_l \times a = u$.

*Lemma* $M7'$. If 0 is the zero-element of the system (theorem 1), and $u$ the element in postulates $M3$–$M4$, then $0 \times u = u \times 0$.

This lemma will appear as a special case of postulate $M7$.

*Lemma* $D'$. If 0 is the zero-element of the system (theorem 1), then either $0 \times a = 0$ for every element $a$, or else $a \times 0 = 0$ for every element $a$.

This lemma will prove to be an immediate consequence of postulate $D$, in view of $A3$–$A4$, taking $b = c = 0$.

The propositions $A1$–$A6$, $M1$–$M6$, $M7'$, and $D'$ are now sufficient (see § 7) to establish theorems 2–3, without requiring the more general postulates $M7$ and $D$; and until these postulates $M7$ and $D$ are introduced, the fourteen propositions just mentioned are all independent (see § 3).

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The theorems in question concern the group-property of the field with regard to multiplication, as follows:

*Theorem* 2. $a \times 0 = 0 \times a = 0$, *for every element* $a$.

*Theorem* 3. *If the zero-element is excluded, the remaining elements form a group with respect to multiplication.*

The identity, $u$, of this group is called the *unit-element* of the field and is denoted by 1; while the inverse of any element $a$ in this group is called the *reciprocal of* $a$ and is denoted by $1/a$.

The remaining postulates for a field are the following:

POSTULATE $M7$. The commutative law for multiplication holds throughout the class; that is,

$$a \times b = b \times a.$$

POSTULATE $D$. Either the left-hand or else the right-hand distributive law for multiplication with respect to addition holds throughout the class; that is, either

$$a \times (b + c) = (a \times b) + (a \times c),$$

or else

$$(b + c) \times a = (b \times a) + (c \times a).$$

From $M7$ and $D$, in view of $A1$ and $M1$, we have at once:

*Theorem* 4. *Multiplication is commutative and distributive.*

To show that $a \times (-b) = -(a \times b)$, we have only to put $b + c = 0$ in postulate $D$, using $M7$ and theorem 2.

Finally, by a method due to HILBERT (see § 7), we can deduce the commutative law for addition:

*Theorem* $A7$. *Throughout the class,* $a + b = b + a$.

Thus we see that any system $(K, +, \times)$ which satisfies the postulates $A1$–$A6$, $M1$–$M4$, $M6$, $M7$, and $D$, will be a field with respect to the rules of combination $+$ and $\times$.

All these postulates are independent, as will be shown by the examples constructed in § 3.

## § 3. *Proof of the independence of the postulates of* §§ 1–2.

In this section, the symbols $+$, $\times$, 0, and 1 are used only in their ordinary arithmetical meanings, the general operations of "addition" and "multiplication" being denoted by $\oplus$ and $\odot$, and the "zero-" and "unit-" elements by $z$ and $u$ respectively.*

The independence of the thirteen postulates of § 2, for abstract fields, is

* The symbols $\oplus$ and $\odot$ were first used in this connection in Transactions, vol. 4 (1903), p. 31; cf. vol. 5 (1904), p. 292, and vol. 6 (1905), pp. 19 and 22.

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established by the existence of the following systems $(K, \oplus, \odot)$, each of which satisfies all the other postulates, but not the one for which it is numbered.

The first six of these systems, together with any non-abelian group, serve also to show the independence of the seven postulates of § 1 for abstract groups.

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*For A*1. $K =$ all real numbers; $a \oplus b = a + b$ when $a$ or $b$ or $a + b$ is zero, otherwise $a \oplus b$ not in the class; $a \odot b = a \times b$.

*For A*2. $K =$ all real numbers; $a \oplus b = a + b$ except that $a \oplus a = 0$; $a \odot b = a \times b$.

*For A*3. $K =$ all positive real numbers; $a \oplus b = a + b$; $a \odot b = a \times b$.

*For A*4. $K =$ all real numbers, together with an extra element $\infty$; $a \oplus b = a + b$ and $a \odot b = a \times b$, understanding that $\infty \oplus \infty = \infty$, $\infty \oplus a = a \oplus \infty = \infty$, and $a \odot \infty = \infty$, $\infty \odot a = a \odot \infty = \infty$. Here $z = 0$ or $\infty$ and $u = 1$.

*For A*5. $K =$ all real numbers; $a \oplus b = 0$; $a \odot b = a \times b$.

*For A*6. $K =$ all positive real numbers with 0; $a \oplus b = a + b$; $a \odot b = a \times b$.

*For M*1. $K =$ all real numbers; $a \oplus b = a + b$; $a \odot b = a \times b$ when $a$ or $b$ is 1 or when $a \times b$ is 1 or 0, otherwise $a \odot b$ not in the class.

*For M*2.* $K =$ all complex numbers of the form $(\alpha, \beta, \gamma) = \alpha + \beta i + \gamma j$, where $\alpha$, $\beta$, and $\gamma$, are real numbers; $\oplus = +$; $\odot = \times$, the multiplication-table for the three principal units being the following:

| $\odot$ | $1$ | $i$ | $j$ |
|---|---|---|---|
| $1$ | $1$ | $i$ | $j$ |
| $i$ | $i$ | $-1$ | $1$ |
| $j$ | $j$ | $1$ | $-2$ |

Here $z = (0, 0, 0)$ and $u = (1, 0, 0)$. To find the reciprocal, $(X, Y, Z)$ of $(\alpha, \beta, \gamma)$, take $X = \alpha/\Delta$, $Y = -\beta/\Delta$, $Z = -\gamma/\Delta$, where $\Delta = \alpha^2 + (\beta - \gamma)^2 + \gamma^2$. The associative law fails, since $(ii)j \neq i(ij)$.

*For M*3. $K =$ all even integers (positive, negative, and zero); $\oplus = +$; $\odot = \times$.

*For M*4. $K =$ all complex numbers $(\alpha, \beta)$, where $\alpha$ and $\beta$ are real; $\oplus = +$; $(\alpha_1, \beta_1) \odot (\alpha_2, \beta_2) = (\alpha_1\alpha_2, \beta_1\alpha_2 + \alpha_1\beta_2 + \beta_1\beta_2)$.

Here $z = (0, 0)$, and $u = (1, 0)$ or $(0, 1)$ or $(1, -1)$.

*For M*6. $K =$ all integers; $\oplus = +$; $\odot = \times$. The following system may also be used: $K =$ all complex numbers $(\alpha, \beta)$, where $\alpha$ and $\beta$ are real; $\oplus = +$; $(\alpha_1, \beta_1) \odot (\alpha_2, \beta_2) = (\alpha_1\alpha_2, \beta_1\alpha_2 + \alpha_1\beta_2)$.

* This system was suggested to me by Professor DICKSON.

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*For M7.* $K =$ all quaternions, $\alpha + \beta i + \gamma j + \delta k$, where $\alpha$, $\beta$, $\gamma$, and $\delta$ are real numbers; $\oplus = +$; $\odot = \times$.

*For D.* $K =$ all integers; $a \oplus b = a + b$; $0 \odot a = a \odot 0 = 0$; when $a \neq 0$ and $b \neq 0$, $a \odot b$ is defined as follows: let $A = a$ or $a + 1$, according as $a$ is positive or negative, and let $B = b$ or $b + 1$, according as $b$ is positive or negative; then $a \odot b = A + B$ when $A + B$ is positive, and $a \odot b = A + B - 1$ when $A + B$ is negative or zero.

The independence of the postulates is thus established. Since each of the systems is infinite, the postulates will remain independent even when we add the demand that the group or field shall be infinite.

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In regard to the lemmas $M5$, and $M7'$, and $D'$, we notice that all the systems just given satisfy them; further, the following systems show that they cannot be proved without the aid of $M7$ or $D$.

*For M5 and D.* Let $K =$ all real numbers; $\oplus = +$; $a \odot b = 1$, except that $a \odot 0 = 0 \odot a = 0$.

*For M7′ and M7.* $K =$ a class of two elements, 0 and 1, with $a \oplus 0 = 0 \oplus a = a$, $1 \oplus 1 = 0$, and $a \odot b = a$; or the same system, with $a \odot b = b$. No system of this kind exists that has more than two elements.

*For D′ and D.* Let $K =$ all real numbers; $\oplus = +$; $a \odot b = a + b - 1$.

---

It remains to prove the independence of the postulates 1, 2, 4, 5, $N$, for finite groups* of order $n > 2$ (§1).† To show the independence for the first four cases, let $K =$ the class of $n$ integers from 0 to $n - 1$ inclusive, and define the rule of combination as follows:

*For* 1. $a \oplus b = 0$ when $a + b = n$, and $a \oplus 0 = 0 \oplus a = a$; otherwise $a \oplus b$ not in the class.

*For* 2. $a \oplus b = 0$ except that $a \oplus 0 = 0 \oplus a = a$.

*For* 4. $a \oplus b = 0$ except that $a \oplus a = a$.

*For* 5. $a \oplus b = 0$.

To show the independence of postulate $N$, consider any infinite group, or an empty class $K$.

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* I have not attempted to get a set of independent postulates for *finite fields*. All that has been done in this direction will be found in Professor DICKSON's paper in the present number of the Transactions.

† If $n = 1$, postulate 1 is sufficient; if $n = 2$, postulates 1, 2, 4, 5 are independent, the non-group systems for 1 and 2 being the following:

| | 0 | 1 |
|---|---|---|
| 0 | 1 | $X$ |
| 1 | $X$ | 0 |

and

| | 0 | 1 |
|---|---|---|
| 0 | 1 | 1 |
| 1 | 1 | 0 |

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In regard to the independence of the commutative law in the case of finite groups of order $n$, Professor DICKSON has just proved the following interesting theorem: Let $n = A \cdot B \cdot C \cdots$, where $A = a^\alpha$, $B = b^\beta$, $C = c^\gamma, \cdots$, the $a, b, c, \cdots$, being distinct primes; then

(1) if any one of the exponents $\alpha, \beta, \gamma, \cdots$ is greater than 2, there is a non-abelian group of order $n$; and

(2) if any one of the expressions $A-1, B-1, C-1, \cdots$ is divisible by any one of the primes $a, b, c, \cdots$, there is a non-abelian group of order $n$.

If neither of these conditions is satisfied, every group of order $n$ is abelian.

## § 4. *A shorter definition of a field.*

This section contains a shorter definition of a field, by means of ten postulates, the independence of which, however, is not completely established.*

POSTULATES 1–7. The same as $A1, A2, A7$; $M1, M2, M7$; $D$. (See § 2.)

POSTULATE 8. Given $a$ and $b$, there is an $x$ such that $a + x = b$.

POSTULATE 9. Given $a$ and $b$, and $a+a \neq a$, there is a $y$ such that $a \times y = b$.

POSTULATE 10. There are at least two elements in the class.

Five of these postulates, namely 1, 2, 3, 8, and 10, constitute a definition of abelian groups; but for the purpose of testing a given system, neither of these shorter definitions is so convenient as those given in § 1.

The following theorems, proved in § 7, show that any system which satisfies the postulates 1–10 is a field with respect to $+$ and $\times$:

*Theorem* I. There is a unique element 0 such that $0 + a = a + 0 = a$ for every element $a$. (From 1, 2, 3, 8, 10.)

*Theorem* II. If $a + b = a + b'$, then $b = b'$. (From 1, 2, 3, 8, 10.)

*Theorem* III. There is a unique element 1, different from 0, such that $1a = a1 = a$ for every element $a$.

*Theorem* IV. If $a \times b = a \times b'$, and $a \neq 0$, then $b = b'$.

*Theorem* V. If $a \neq 0$ and $b \neq 0$, then $a \times b \neq 0$.

The independence of all the postulates, except 1 and 4, is established by the following systems:

[*For* 1 *and* 8. $K =$ all real numbers; $a \oplus b = a + b$ when $a$ and $b$ are integers, otherwise $a \oplus b$ not in the class; $\odot = \times$.]

*For* 2. $K =$ all real numbers; $a \oplus b = 2(a + b)$; $\odot = \times$.

*For* 3. $K =$ all real numbers; $a \oplus b = b$; $\odot = \times$.

[*For* 4 *and* 9. $K =$ all real numbers; $\oplus = +$; $a \odot b = a \times b$ when $a$ and $b$ are rational, otherwise $a \odot b$ not in the class.]

*For* 5. $K =$ all complex numbers $(\alpha, \beta)$, where $\alpha$ and $\beta$ are real; $\oplus = +$; $(\alpha_1, \beta_1) \odot (\alpha_2, \beta_2) = (\alpha_1\alpha_2 - \beta_1\beta_2, -\alpha_1\beta_2 - \alpha_2\beta_1)$.

* Compare a similar set of thirteen postulates, all independent, in a paper of mine on complex algebra, in the present number of the Transactions.

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In the geometrical representation of this system, $\odot$ is the ordinary multiplication followed by reflection in the axis of reals.

*For* 6. $K =$ all real numbers; $\oplus = +$; $a \odot b = b$.

*For* 7. $K =$ all real numbers; $\oplus = +$; $\odot = +$.

*For* 8. $K =$ all positive real numbers; $\oplus = +$; $\odot = \times$.

*For* 9. $K =$ all integers; $\oplus = +$; $\odot = \times$.

*For* 10. $K =$ an empty class; or $K =$ a class containing a single element $a$, with $a \oplus a = a$ and $a \odot a = a$.

The independence of postulates 1 and 4 is still an open question, unless postulates 8 and 9 are artificially "weakened".

### §§ 5–6. *Definitions of groups in terms of a triadic relation, $R$.*

The sets of postulates for abstract groups given in §§ 5–6 are modifications of the writer's first definition of 1902 (loc. cit.), expressed in terms of the triadic relation suggested by Professor BÔCHER (loc. cit.).

The fundamental concepts are here a *class*, $K$, and a *relation*, $R$; instead of writing "$ab = c$," which means "the combination of $a$ and $b$ equals $c$," we write "$R(abc)$," which means "the three elements $a$, $b$, and $c$ satisfy the given relation $R$."

The use of this notation suggested the more explicit division of the first postulate of § 1 into its two component statements, namely postulates IV in § 5, and postulate IV′ in § 6.

### § 5.

One set of postulates for which, however, the proofs of independence are not complete, is the following ($a$, $b$, $c$, etc. denoting elements of $K$):

POSTULATE I. The class $K$ is not an empty class.

POSTULATE II. Given $b$ and $c$ there is an $a$ such that $R(abc)$.

POSTULATE III. Given $a$ and $c$ there is a $b$ such that $R(abc)$.

POSTULATE IV. Given $a$ and $b$ there is a $c$ such that $R(abc)$.

POSTULATE V. If $R(abp)$, $R(pcM)$, $R(bcq)$, and $R(aqN)$, then $M = N$.

From these postulates I–V the following theorems are deduced (see § 7):

*Theorem* 1. If $R(abc)$ and $R(a'bc)$, then $a = a'$.

*Theorem* 2. If $R(abc)$ and $R(ab'c)$, then $b = b'$.

*Theorem* 3. If $R(abc)$ and $R(abc')$, then $c = c'$.

If now we write $ab = c$ in place of $R(abc)$, postulate V gives us the associative law: $(ab)c = (p)c = M$; $a(bc) = a(q) = N$; and all the conditions in WEBER's definition of a group are clearly satisfied.

The independence of all the postulates except IV is established by the systems used in § 6; but the question of the independence or deducibility of postulate IV is undecided.

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§ 6.

Another set of postulates, less symmetrical than that given in § 5, but admitting complete proofs of independence, is the following:

POSTULATES I–III. The same as in § 5.

POSTULATE IV′. If $R(abc)$ and $R(abc')$, then $c = c'$.

POSTULATE V′. Either (1): If $R(abp)$, $R(bcq)$, and $R(pcM)$, then $R(aqM)$; or else (2): If $R(abp)$, $R(bcq)$, and $R(aqN)$, then $R(pcN)$.

From these postulates we can prove (see § 7) the theorems 1–2 of § 5, and also

*Theorem* 3′. Given $a$ and $b$ there is a $c$ such that $R(abc)$.

Hence the two definitions are clearly equivalent.

The proofs of independence are as follows:

*For* I. $K =$ an empty class.

*For* II–III. $K =$ any class of more than two elements; for II, let $R(abc)$ mean $b = c$; for III, let $R(abc)$ mean $a = c$.

*For* IV′. $K =$ any class of more than two elements, with $R(abc)$ true for all values of $a$, $b$, and $c$.

*For* V′. $K =$ all real (or all rational) numbers, with $R(abc)$ signifying $a + b = 2c$; or again, $K =$ the $n$ positive integers $(n > 2)$ from 0 to $n - 1$ inclusive, with $R(abc)$ holding whenever $c = n - s$, where $s$ is congruent to $a + b$ modulo $n$. In this last system, $(11)2 \neq 1(12)$.

It may be noticed that the postulates I–III, IV, and V′ are independent,* but not sufficient to define a group, as witness the systems in which $R(abc)$ is always true.

The postulates I–III, IV′, and V are likewise independent, but their sufficiency is still an open question.

§ 7. *Proofs of theorems in the preceding sections.*

To avoid interruption in reading, a number of theorems in the preceding sections have been stated without proof; the requisite demonstrations are here supplied.

*In* § 1: *Proof of theorem* I. If $i$ is the element described in postulates 3 and 4, we have the following lemmas:

*Lemma* 1. If $ab = i$, then $ba = i$. (By 1–5.) For, if $ab = i$, then $(ba)(ba) = b(ab)a = bia = ba$; whence $ba = i$, by 4.

*Lemma* 2. For every element $a$ there is at least one element $\alpha$ such that $a\alpha = \alpha a = i$. (By postulate 6 and lemma 1.)

The proof of the main theorem is then as follows: Suppose $ia = a$ for every element $a$, and take $\alpha$ so that $a\alpha = \alpha a = i$; then

$$ai = i(ai) = i[a(\alpha a)] = i[(a\alpha)a] = i(ia) = (ii)a = ia = a.$$

*To show the independence of IV in this set, let $R(abc)$ signify $a = b$.

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Similarly, if we suppose $ai = a$ for every element $a$, then

$$ia = (ia)i = [(a\alpha)a]\,i = [a(\alpha a)]\,i = (ai)i = a(ii) = ai = a.$$

*In* § 1: *Proof of theorem* II. By lemma 2, take $\alpha$ so that $\alpha a = i$, and $\beta$ so that $b\beta = i$. Then if $ab = ab'$,

$$b = ib = (\alpha a)b = \alpha(ab) = \alpha(ab') = (\alpha a)b' = ib' = b';$$

and similarly, if $ab = a'b$,

$$a = ai = a(b\beta) = (ab)\beta = (a'b)\beta = a'(b\beta) = a'i = a'.$$

*In* § 1: *Proof of postulates* 3 *and* 6 *for finite groups.* (The proof of 3 depends merely on 1, 2, and $N$; that for 6 requires 1, 2, 4, 5, and $N$.)

Let $a$ be any element, and form the sequence of elements

$$a_1 = aa,\ a_2 = a_1a_1,\ a_3 = a_2a_2,\ \cdots,\ a_{k+1} = a_ka_k.$$

Since the class is finite, this sequence must eventually contain repetitions; that is, there must be indices $p$ and $p+q$ such that $a_p = a_{p+q}$. The element

$$x = a_pa_{p+1}a_{p+2}\cdots a_{p+q-1}$$

will then be such that $xx = x$, which establishes postulate 3.*

Now this element $x$ is simply the combination of a certain number, say $m$,† of the $a$'s:

$$aaa\cdots a = i;$$

hence the combination of $m-1$ of the $a$'s will be an element $a'$ such that $aa' = i$, which establishes postulate 6.

*In* § 2: *Proof*‡ *of lemma* $M5$.

$a$) $a \times 0 = 0$. (From $A1$, $A3$–$A4$, $M1$, $M7$, $D$.) For,

$$a \times 0 = a \times (0+0) = (a \times 0) + (a \times 0),$$

whence $a \times 0 = 0$, by $A4$.

$b$) If $u \times x = 0$, then $x = 0$. (From $A1$, $A2$–$A4$, $M1$–$M4$, $M6$–$M7$, $D$.) For suppose $x \neq 0$, and take $x''$ by $M6$ so that $x'' \times x = u$. Then by $a$), $u = u \times u = u \times (x \times x'') = (u \times x) \times x'' = 0 \times x'' = 0$, which is impossible.

$c$) $a \times (b - c) = (a \times b) - (a \times c)$. (From $A1$–$A6$, $M1$–$M4$, $M6$–$M7$, $D$.) For, $a \times (-c) + a \times c = a \times (-c + c) = a \times 0 = 0$, whence

$$a \times (-c) = -(a \times c).$$

* This proof for 3, which suggested the proof for 6, is taken from the paper by E. H. MOORE, loc. cit. (1902), p. 490.

† Here $m = 2^p(2^q - 1)$.

‡ I am indebted to Mr. G. D. BIRKHOFF for this demonstration.

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*d*) If $u \times a = x$, then $u \times x = x$. (From $M1$–$M4$.) For,

$$u \times x = u \times (u \times a) = (u \times u) \times a = u \times a = x.$$

From *c*) and *d*) we can now prove the lemma, as follows: Let $u \times a = x$; then $u \times (a - x) = (u \times a) - (u \times x) = x - x = 0$, whence $a - x = 0$, or $x = a$.

*In* § 2: *Proof of theorem* 2. (The proof depends on $A3$–$A4$, $M1$–$M5$, $M7'$ and $D'$, but not on $M6$.)

If the first part of $D'$ is true, $0 \times a = 0$ for every element $a$. Then $(a \times 0) \times (a \times 0) = a \times (0 \times a) \times 0 = a \times 0 \times 0 = a \times 0$, whence, by $M4$, $a \times 0 =$ either 0 or $u$.

But the second alternative, $a \times 0 = u$, must be rejected, in view of $M7'$. For, suppose $a \times 0 = u$ for any particular value of $a$. Then $u \times c = (a \times 0) \times c = a \times (0 \times c) = a \times 0 = u$, for every element $c$, so that first half of $M5$ cannot be true. Therefore, by the second half of $M5$, $c \times u = c$ for every element $c$. Then, for every $c$, $c \times c = (c \times u) \times c = c \times (u \times c) = c \times u = c$, whence, by $M4$, every $c$ must be either 0 or $u$. But if 0 and $u$ are the only elements in the class, then $u \times 0 = 0 \times u$, by $M7'$, or $u \times 0 = 0$.

Therefore $a \times 0 = 0$ for all values of $a$.

If the second part of $D'$ is true, instead of the first part, the theorem is proved in a similar way.

*In* § 2: *Proof of theorem* 3.

(1°) If $a \times b = u$, then $b \times a = u$. For, if $a \times b = u$, then $(b \times a) \times (b \times a) = b \times (a \times b) \times a = b \times u \times a = b \times a$, whence, by $M4$, $b \times a$ is either $u$ or 0. But it cannot be 0, since if it were we should have $u \times u = (a \times b) \times (a \times b) = a \times (b \times a) \times b = a \times 0 \times b = 0$ (by theorem 2), which is impossible by $M5$.

(2°) For every element $a$, provided $a \neq 0$, there is at least one element $\alpha$ such that $a \times \alpha = \alpha \times a = u$ [by (1°) and $M6$]; and any such element $\alpha$ will be $\neq 0$ when $a \neq 0$ [by theorem 2, since $0 \neq u$].

(3°) If $a \neq 0$, then $u \times a = a \times u = a$. [This follows from (1°) and (2°), as in the proof of theorem I in § 1.]

(4°) If $a \neq 0$ and $b \neq 0$, then $a \times b \neq 0$. For, suppose $a \times b = 0$ and $a \neq 0$; then taking $\alpha$ so that $\alpha \times a = u$ we have

$$b = u \times b = (\alpha \times a) \times b = \alpha \times (a \times b) = \alpha \times 0 = 0.$$

Thus all the conditions for a group (see § 1) are satisfied.

*In* § 2: *Proof* of theorem* $A7$. (The proof depends on $A1$–$A6$, $M1$–$M6$,

*D. HILBERT, *Über den Zahlbegriff*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 8 (1899–1900), p. 183.



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and both parts of $D$). Since the system is a group with respect to addition, we know that from $a + b = a + b'$ follows $b = b'$, and from $a + b = a' + b$ follows $a = a'$ (theorem 2, § 1). By $D$, in view of $M7$, we have

$$(a + b) \times (1 + 1) = a \times (1 + 1) + b \times (1 + 1) = a + a + b + b,$$
$$= (a + b) \times 1 + (a + b) \times 1 = a + b + a + b.$$

Therefore $a + a + b = a + b + a$, and hence $a + b = b + a$.

*In* § 4: *Proof of theorems* I–V.

I. Let $c$ be any fixed element and $a$ any other element. Take $z$ so that $c + z = c$, and $x$ so that $x + c = a$. Then $a+z = (x+c) + z = x+(c + z) = a$. Further, if $a + z_1 = a$ for every $a$, and also $a + z_2 = a$ for every $a$, then $z_2 + z_1 = z_2$ and $z_1 + z_2 = z_1$, whence by postulate 3, $z_1 = z_2 = 0$.

II. Take $x$ so that $x+a=0$; then $b=0+b=x+a+b=x+a+b'=0+b'=b'$.

III. Let $c$ be any fixed element not $0$, and $a$ any element not $0$. Take $u$ so that $cu = c$ and $y$ so that $yc = a$. Then $au = (yc)u = y(cu) = yc = a$.

IV. Take $y$ so that $ya = u$; then $b = ub = yab = yab' = yb' = b'$.

V. Suppose $ab = 0$ and $a \neq 0$, and take $y$ so that $ya = 1$. Then $b = 1b = yab = y0 = 0$.

*In* § 5: *Proof of theorem* 1. Given $R(abc)$ and $R(a'bc)$.

Take $z$ and $x$ so that $R(za'a)$ and $R(zcx)$; and $y$ so that $R(cya')$. Then from $R(za'a)$, $R(abc)$, $R(a'bc)$, $R(zcx)$, follows $c = x$; and from $R(zcc)$, $R(cya')$, $R(cya')$, $R(za'a)$, follows $a' = a$.

*In* § 5: *Proof of theorem* 2. Given $R(abc)$ and $R(ab'c)$.

Take $z$ and $x$ so that $R(b'zb)$ and $R(czx)$; and $y$ so that $R(ycb')$. Then from $R(ab'c)$, $R(czx)$, $R(b'zb)$, $R(abc)$, follows $c = x$; and from $R(ycb')$, $R(b'zb)$, $R(czc)$, $R(ycb')$, follows $b = b'$.

*In* § 5: *Proofs of theorem* 3. Given $R(abc)$ and $R(abc')$.

First proof (using only II, IV, and V): Take $x$, $\beta$, and $y$ so that $R(xab)$, $R(ax\beta)$, and $R(\beta ay)$. Then from $R(ax\beta)$, $R(\beta ay)$, $R(xab)$, $\left\{\begin{smallmatrix} R(abc) \\ R(abc') \end{smallmatrix}\right\}$, follows $\left\{\begin{smallmatrix} y=c \\ y=c' \end{smallmatrix}\right\}$.

Second proof (using only III, IV, and V): Take $x$, $\alpha$, and $y$ so that $R(bxa)$, $R(xb\alpha)$, and $R(b\alpha y)$. Then from $R(bxa)$, $\left\{\begin{smallmatrix} R(abc) \\ R(abc') \end{smallmatrix}\right\}$, $R(xb\alpha)$, $R(b\alpha y)$, follows $\left\{\begin{smallmatrix} y=c \\ y=c' \end{smallmatrix}\right\}$. In either case, $c = c'$.

*In* § 6: *Proof of theorems* 1–2. In order to show that either half of postulate $V'$ is sufficient, the proof is divided into four steps, ($a$–$d$):

$a$) Proof of theorem 1, using $V'(1)$: Given $R(abc)$ and $R(a'bc)$; to prove, $a = a'$. Take $z$ so that $R(za'a)$ and $y$ so that $R(cya')$. From $R(za'a)$,

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$R(a'bc)$, $R(abc)$, follows $R(zcc)$; and from $R(zcc)$, $R(cya')$, $R(cya')$, follows $R(za'a')$. Hence, from $R(za'a)$ and $R(za'a')$, we have $a = a'$.

*b*) Proof of $V'(2)$ from $V'(1)$ and theorem 1: Given, $R(abp)$, $R(bcq)$, and $R(aqN)$. Take $P$ so that $R(PcN)$ and $A$ so that $R(AbP)$. Then from $R(AbP)$, $R(bcq)$, $R(PcN)$, follows $R(AqN)$. From $R(AqN)$ and $R(aqN)$, we have $A = a$; and from $R(abP)$ and $R(abp)$ we have $P = p$. Hence $R(pcN)$.

*c*) Proof of theorem 2, using $V'(2)$: Given, $R(abc)$ and $R(ab'c)$; to prove $b = b'$. Take $z$ so that $R(bzb')$ and $x$ so that $R(xcb)$. From $R(abc)$, $R(bzb')$, $R(ab'c)$, follows $R(czc)$; and from $R(xcb)$, $R(czc)$, $R(xcb)$, follows $R(bzb)$. Hence, from $R(bzb)$ and $R(bzb')$, we have $b = b'$.

*d*) Proof of $V'(1)$ from $V'(2)$ and theorem 2: Given, $R(abp)$, $R(bcq)$, and $R(pcM)$. Take $Q$ so that $R(aQM)$, and $C$ so that $R(bCQ)$. Then from $R(abp)$, $R(bCQ)$, $R(aQM)$, follows $R(pCM)$. From $R(pCM)$ and $R(pcM)$ we have $C = c$; and from $R(bcQ)$ and $R(bcq)$ we have $Q = q$. Hence $R(aqM)$.

Thus from either half of postulate $V'$ we can prove the other half, and hence both the theorems 1–2.

*In* § 6: *Proof of theorem* 3′. Given $a$ and $b$; take $z$ so that $R(aza)$, $b'$ so that $R(bb'z)$, and $c$ so that $R(cb'a)$. Then $c$ is the required element such that $R(abc)$. For, take $\beta$ so that $R(a\beta c)$ and $\beta'$ so that $R(\beta\beta'z)$. Then from $R(a\beta c)$, $R(\beta\beta'z)$, $R(aza)$, follows $R(c\beta'a)$, by $V'(2)$. From $R(c\beta'a)$ and $R(cb'a)$ we have $\beta' = b'$, by theorem 2; and from $R(\beta b'z)$ and $R(bb'z)$ we have $\beta = b$, by theorem 1. Hence $R(abc)$.

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# DEFINITIONS OF A GROUP AND A FIELD BY INDEPENDENT POSTULATES*

BY

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## *Introduction.*

1. The simple definition here given for a general abstract group relates as to origin and character to Professor MOORE's two definitions, Transactions, vol. 3 (1902), pp. 485–492. A few days before the appearance of the addition, Transactions, vol. 5 (1904), p. 549, to his paper, Professor MOORE remarked to me that one of his postulates relating to an inverse was redundant, meaning postulate $(3''_r)$ of his second definition. I thought the reference was to his first definition and attempted to reconstruct the proof of a redundancy, there absent. This attempt led me to alter his postulate $(4_l)$ to read $aa'_r = i_r$ instead of $a'_l a = i_r$ and to note that postulate $(3_l)$ becomes redundant in the altered set, thus obtaining the present definition. Subsequently I learned that Professor MOORE, in his proof of the redundancy of $(3''_r)$ in his second definition, had obtained relations ‡ sufficient to establish the present definition but had not applied them to set up the definition itself.

The present postulates for a general group possess the desirable property that they remain independent within sets of postulates for special classes of groups, the specialization being either in the direction of the number $(n > 1)$ of elements or their commutativity (§§ 3–5).

The definition of a field (§ 6), based on the present definition of a group, has evident advantages over the earlier definitions. §

The postulates for a field remain independent under an assumption that the set is finite, or forms an enumerable infinitude, or a non-enumerable infinitude.

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‡ MOORE, this number of Transactions, p. 179.

§ HUNTINGTON and DICKSON, Transactions, vol. 4 (1903), p. 31, p. 13. For the new definitions by HUNTINGTON, with a bibliography, see this number of Transactions, p. 181.



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*Definition of a group*, §§ 2–5.

2. Given a function $a \circ b$ of two arguments and a set of elements, we will say that the elements form a group with respect to $\circ$ when the following postulates hold:*

($1_1$) For every two † equal or distinct elements $a$ and $b$ of the set, there is a determination of $a \circ b$.

($1_2$) For $a$ and $b$ in the set, there is at most one determination of $a \circ b$.

($1_3$) If, for $a$ and $b$ in the set, there is at least one determination of $a \circ b$, one determination is an element of the set.

(2) $(a \circ b) \circ c = a \circ (b \circ c)$ whenever $a, b, c$, and all the determinations of $a \circ b$, $b \circ c$, $(a \circ b) \circ c$, and $a \circ (b \circ c)$ occur in the set.

(3) There occurs in the set an element $i$ such that, for every element $a$ of the set, $a \circ i$ has the determination $a$.

(4) If such elements $i$ occur, then for a particular $i$, and for every ‡ $a$ in the set there occurs in the set an element $a'$ such that $a \circ a'$ has the determination $i$.

Postulates ($1_1$), ($1_2$), ($1_3$) may be combined into the triple statement:

(1) For every two equal or distinct elements $a$ and $b$ of the set $a \circ b$ is uniquely determined as an element of the set.

In view of (3) and (4), to any element $a$ there correspond elements $a'$ and $a''$ such that $a \circ a' = i$, $a' \circ a'' = i$, where $i$ is a fixed element such that $e \circ i = e$ for every element $e$. Applying also (1) and (2), we have

$$a = a \circ i = a \circ (a' \circ a'') = (a \circ a') \circ a'' = i \circ a'',$$

$$a' \circ a = a' \circ (i \circ a'') = (a' \circ i) \circ a'' = a' \circ a'' = i.$$

Hence $a' \circ a = i$. By this theorem, $a'' \circ a' = i$. Hence

$$i \circ a = (i \circ i) \circ a = [i \circ (a'' \circ a')] \circ a = (i \circ a'') \circ (a' \circ a) = a \circ i = a.$$

Since $a \circ i = a = i \circ a$ for every $a$, $i$ is called an identity element. If also $i_1$ is an identity element, then $i_1 = i_1 \circ i = i$. *Hence there is an unique identity element.* Since $a \circ a' = i = a' \circ a$, $a'$ is called an inverse of $a$. If also $a_1'$ is an inverse of $a$, then

$$a_1' = a_1' \circ i = a_1' \circ (a \circ a') = (a_1' \circ a) \circ a' = i \circ a' = a'.$$

*Hence there is an unique inverse of each element.*

* In (2), (6), and (7), we mean by $A = B$ that one of the determinations of $A$ equals one of the determinations of $B$.

† The assumption need not be made for $a \circ i$ or $a \circ a'$, in view of (3), (4).

‡ The assumption need not be made for $a = i$, in view of (3).

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3. We prove that *the postulates* $(1_1)$, $(1_2)$, $(1_3)$, $(2)$, $(3)$, $(4)$, $(5_k)$, $k = 1, 2,$ *or* $3$, *are consistent and independent*, where

$(5_1)$ The number of distinct elements is a fixed integer $n$, $n > 1$;*

$(5_2)$ The distinct elements of the set form an enumerable infinitude;

$(5_3)$ The distinct elements form a non-enumerable infinitude.

Their consistency follows from the existence of the group of the elements $0, 1, \cdots, n-1$ with $a \circ b = a + b \pmod{n}$, and the group of all rational (or real) numbers with $a \circ b = a + b$.

Let $I_1$ be a set containing $i$ and exactly $n-1$ further elements $b$; $I_2$ or $I_3$ a set containing $i$ and further elements $b$ forming an enumerable or a non-enumerable infinitude, respectively.

To prove the independence of a postulate $(j)$, we exhibit a set $[j]$ in which postulate $(j)$ fails while each of the remaining postulates hold. In the sets, $a \circ b$ is understood to have a unique determination unless the contrary is stated.

$[1_1]$ Set $I_k$, $i \circ i = i$, $b \circ b = i$, $b \circ i = b$, no determination of $i \circ b$ or of $b \circ b'$ $(b \neq b')$.

$[1_2]$ Set $I_k$ forming a group under $\odot$; $a \circ c$ with the determinations $a \odot c$ and $i$.

$[1_3]$ Set $I_k$; $i \circ i = i$, $b \circ i = b$, $b \circ b' = i$, $i \circ b$ not in the set.

$[2]$ Set $I_k$; $i \circ i = i$, $b \circ i = b$, $b \circ b' = i$, $i \circ b = i$.

$[3]$ Set $I_k$; $i \circ i = i$, $b \circ i = i$, $b \circ b' = i$, $i \circ b = i$.

$[4]$ Set $I_k$; $i \circ i = i$, $b \circ i = b$, $i \circ b = b$, $b \circ b' = b_1$, $b_1$ a fixed $b$.

4. We next examine the effect of adding the commutative law:

$(6)$ $a \circ b = b \circ a$ whenever $a$, $b$, and all the determinations of $a \circ b$ and $b \circ a$ occur in the set.

*The postulates* $(1_1)$, $(1_2)$, $(1_3)$, $(2)$, $(3)$, $(4)$, $(5_k)$, $(6)$, *for* $k = 0, 2,$ *or* $3$, *are consistent and independent.* Here the new postulate is

$(5_0)$ The number of distinct elements is finite but undetermined.

The proof follows from the sets $[1_1]$, $[1_2]$, $[1_3]$, $[3]$, $[4]$ above, and

$[2]'$ Set $I_k$, $n > 2$; $i \circ i = i$, $i \circ b = b \circ i = b$, $b \circ b = i$, $b \circ b' = b_1 (b \neq b')$,

Then, for $b \neq b_1$, $(b \circ b_1) \circ b_1 = b_1 \circ b_1 = i$, $b \circ (b_1 \circ b_1) = b \circ i = b$.

$[6]$ There exist finite and infinite non-commutative groups.†

The question of the independence of the postulates for $k = 1$ is answered by § 5 in connection with the sets $[1_1]$, $[1_2]$, $[1_3]$, $[2]'$, $[3]$, $[4]$.

5. THEOREM.‡ *Let* $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_v^{\alpha_v}$, *where* $p_1, \cdots, p_v$ *are distinct primes, and each* $\alpha_j > 0$. *The necessary and sufficient conditions that all existing*

* For $n = 1$, the group may be defined by the independent postulates $(1_1)$, $(1_2)$, $(1_3)$; also by the independent postulates $(1_2)$, $(3)$.

† E. g., that of the transformations $x' = \pm x + b$, where $b$ ranges over all integers, or all real numbers, or the residues modulo $m$, $m > 2$.

‡ Addition of February 2, 1905.

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*groups of order $n$ shall be abelian are: (i) each $\alpha_j \leqq 2$; (ii) no $p_j^{\alpha_j} - 1$ is divisible by one of the primes $p_1, \cdots, p_v$.*

We first prove that the conditions are necessary. If $\alpha_1 > 2$, a non-abelian $G_n$ is given by the direct product of the cyclic groups $C_{p_k^{\alpha_k}} (k = 2, \cdots, v)$ and a non-abelian $G_{p_1^{\alpha_1}}$, say of the type generated by $P$ and $Q$ where

$$P^{p_1^{\alpha_1 - 1}} = I, \qquad Q^{p_1} = I, \qquad Q^{-1} P Q = P^{1 + p_1^{\alpha_1 - 2}} \qquad (\alpha_1 > 2).$$

If each $\alpha_j \leqq 2$ and $p_1^{\alpha_1} - 1$ is divisible by $p_2$, there exists a non-abelian group of order $p_1^{\alpha_1} p_2^{\alpha_2}$ and hence, as before, one of order $n$. Indeed, if $\alpha_1 = \alpha_2 = 1$, we use the group generated by $P_1$ and $P_2$ where

$$P_1^{p_1} = I, \qquad P_2^{p_2} = I, \qquad P_2^{-1} P_1 P_2 = P_1^{\pi} \qquad [p_1 \equiv 1 \pmod{p_2}],$$

where $\pi$ is an existing primitive root of $x^{p_2} \equiv 1 \pmod{p_1}$. If $\alpha_1 = 1$, $\alpha_2 = 2$, we use the direct product of the preceding group by a cyclic $C_{p_2}$. If $\alpha_1 = 2$, $\alpha_2 = 1$, the case in which $p_1 - 1$ is divisible by $p_2$ is disposed of as before, while for $p_1 + 1$ divisible by $p_2$, $p_2 > 2$, we use the group* generated by $S$, $T_1$, $T_2$, with

$$T_1^{p_1} = T_2^{p_1} = S^{p_2} = I, \quad T_1 T_2 = T_2 T_1, \quad S^{-1} T_1 S = T_2, \quad S^{-1} T_2 S = T_1^{-1} T_2^{b},$$

where $b$ is an (existing) integer such that $x^2 + bx + 1 \equiv 0 \pmod{p_1}$ is irreducible, viz., $b = i + i^{p_1}$, $i$ a mark of the $GF[p_1^2]$ belonging to the exponent $p_2$. Finally, if $\alpha_1 = \alpha_2 = 2$, we use the direct product of one of the preceding groups by $C_{p_2}$.

It remains to prove that the conditions (*i*) and (*ii*) are sufficient to make every $G_n$ abelian. To proceed by induction, we assume that this statement is true for every $n' = p_1^{\beta_1} \cdots p_v^{\beta_v}$, $\beta_1 \leqq \alpha_1, \cdots, \beta_v \leqq \alpha_v$, $n' < n$. Hence every subgroup of $G_n$ is abelian, so that† either $G_n$ is abelian or else $n$ is divisible by at most two distinct primes. But for $n = p$, $n = p^2$, or $n = p_1^{\alpha_1} p_2^{\alpha_2}$, $0 < \alpha_1 \leqq 2$, $0 < \alpha_2 \leqq 2$, $p_1^{\alpha_1} \not\equiv 1 \pmod{p_2}$, $p_2^{\alpha_2} \not\equiv 1 \pmod{p_1}$, $G_n$ is immediately seen to be abelian. Hence the induction is complete.

## *Definition of a field*, §§ 6–9.

6. We employ a set of elements and two functions $a \oplus b$ and $a \otimes b$. For $\circ = \oplus$, postulate ($j$) of §§ 2, 4, with $j = 1, 2, 3, 4$, or $6$, is designated $j^+$. An element $i$ satisfying $3^+$ is designated $i_+$. For $\circ = \otimes$, ($j$) is designated $j^\times$; but $4^\times$ is assumed only for elements $a$ distinct from each $i_+$. An element $i$ satisfying $3^\times$ is designated $i_\times$. As the distributive law we take‡

* HÖLDER, Mathematische Annalen, vol. 43 (1893), pp. 345–357. We may use the form by COLE and GLOVER, American Journal of Mathematics, vol. 15 (1893), pp. 207–8.

† MILLER and MORENO, Transactions, vol. 4 (1903), p. 398.

‡ If we alter (7) to read $a \otimes (b \oplus c) = (a \otimes c) \oplus (a \otimes b)$, we deduce $6^+$ by taking $a = i_\times$ and applying $6^\times$. Then also (7) follows.

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(7) $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$ whenever $a, b, c$, and all the determinations of the functions involved occur in the set.

In view of the theorems in §§ 7, 8, we make the definition:* A set of elements forms a field with respect to $\oplus$ and $\otimes$ if postulates $1^+$, $2^+$, $3^+$, $4^+$, $1^\times$, $2^\times$, $3^\times$, $4^\times$, $6^\times$, and 7 hold.

7. THEOREM.† *From* $1^+$, $2^+$, $3^+$, $4^+$, $1^\times$, $3^\times$, $6^\times$, 7, *follows* $6^+$.

We substitute $\beta \oplus \gamma$ for $a$ in (7), apply $1^+$, $1^\times$, $6^\times$, 7 and get

$$(\beta \oplus \gamma) \otimes (b \oplus c) = [(b \otimes \beta) \oplus (b \otimes \gamma)] \oplus [(c \otimes \beta) \oplus (c \otimes \gamma)],$$

for every $\beta, \gamma, b, c$ in the set. Interchanging $b$ with $\beta$, $c$ with $\gamma$, and noting that the first member is unaltered in view of $6^\times$, we get by $6^\times$ and $2^+$,

$$(b \otimes \beta) \oplus (c \otimes \beta) \oplus (b \otimes \gamma) \oplus (c \otimes \gamma) = (b \otimes \beta) \oplus (b \otimes \gamma) \oplus (c \otimes \beta) \oplus (c \otimes \gamma),$$

Since $1^+$, $2^+$, $3^+$, $4^+$ imply an inverse under $\oplus$, we get

$$(c \otimes \beta) \oplus (b \otimes \gamma) = (b \otimes \gamma) \oplus (c \otimes \beta).$$

Taking $\beta = \gamma = i_\times$, we get $c \oplus b = b \oplus c$.

8. THEOREM. *Let* $i_+$ *be chosen so that* $4^+$, *as well as* $3^+$, *is satisfied. Then* $a \otimes i_+ = i_+ \otimes a = i_+$ *for every element* $a$. *If* $a \otimes b = i_+$ *and* $b \neq i_+$, *then* $a = i_+$. *Hence* $i_+$ *has the ordinary properties of zero under* $\oplus$ *and* $\otimes$.

By 7 for $b = c = i_+$, $e = e \oplus e$, where $e = a \otimes i_+$. Hence (§ 2), $a \otimes i_+ = i_+$ for every $a$. By $1^\times$, $6^\times$, $i_+ \otimes a = i_+$.

By $2^\times$, from $a \otimes b = i_+$ follows $i_+ = a \otimes (b \otimes c)$ for every $c$. Let $i_\times$ be chosen so that $4^\times$, as well as $3^\times$, holds. Taking $c$ such that $b \otimes c = i_\times$, we get $i_+ = a \otimes i_\times = a$.

9. THEOREM. *Postulates* $1^+$, $2^+$, $3^+$, $4^+$, $1^\times$, $2^\times$, $3^\times$, $4^\times$, $6^\times$, 7, $5_k$ ($k = 0$, 2, *or* 3) *are independent.*

For $k = 0$, i. e., for a finite number of elements, we employ the sets:

[$1^+$] Elements $0, 1, -1$; $a \oplus b = a + b$, $a \otimes b = a \times b$.

[$2^+$] $0, 1, -1$; $0 \oplus a = a \oplus 0 = a$, $a \oplus b = 0 (a \neq 0, b \neq 0)$, $\otimes = \times$.

[$3^+$] $0, 1, \cdots, n-1 (n > 1)$; $a \oplus b = b$, $a \otimes b = a + b \pmod{n}$.

---

* If the set is finite, we may omit $3^+$ and insert $6^+$. Then if $3^+$ fails, all the elements form a group with respect to $\otimes$. It cannot contain an element $a$ of period $\alpha$, $\alpha > 1$. Indeed, if we set $y = i_\times \oplus a \oplus a^2 \oplus \cdots \oplus a^{\alpha-1}$, we get $a \otimes y = y$, whence $a = i_\times$. The single element in the set necessarily satisfies $3^+$.

† Some months after devising this proof I learned that a similar proof had been given by HILBERT, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 8 (1899-1900), p. 183. But earlier writers have noted the essential point in the proof; viz., that the uniqueness of the expansion of $(\beta + \gamma)(b + c)$ depends upon the validity of the commutative law for addition.

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[$4^+$]

| $\oplus$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 1 |

| $\otimes$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

[$1^\times$] $0, 1$; $a \oplus b = a + b \pmod 2$, $0 \otimes a = 0$, $1 \otimes 1 = 1$, $1 \otimes 0$ not in the set.

[$2^+$] Eight* elements $(\xi, \eta, \zeta)$, $\xi, \eta, \zeta$ taken modulo 2;

$$(\xi, \eta, \zeta) \oplus (x, y, z) = (\xi + x, \eta + y, \zeta + z),$$

$$(\xi, \eta, \zeta) \otimes (x, y, z) = (z\xi + x\zeta, (x + y)\xi + (x + z)\eta + y\zeta, y\xi + x\eta + z\zeta).$$

The latter equals $i_\times = (0, 0, 1)$ if $D \equiv \xi + \zeta + \xi\zeta \not\equiv 0 \pmod 2$ and

$$x \equiv \xi(1 + \zeta)/D, y \equiv (\xi + \xi\eta + \eta\zeta)/D, z \equiv \zeta(1 + \xi)/D.$$

But $D \equiv 0$ only when $\xi \equiv \zeta \equiv 0$; while if $\xi \equiv \zeta \equiv 0$, $\eta \equiv 1$, we have

$$(0, 1, 0) \otimes (x, y, z) = (0, x + z, x) = i_\times \text{ if } x \equiv z \equiv 1.$$

Finally, $2^\times$ fails for $a = (0, 1, 0)$, $b = (0, 1, 0)$, $c = (1, 1, 1)$.

[$3^\times$] $0, 1$; $a \oplus b = a + b \pmod 2$, $a \otimes b = 0$.

[$4^\times$] $0, 1, \cdots, n - 1, n$ composite; $a \oplus b = a + b \pmod n$, $a \otimes b = a \times b \pmod n$

[$6^\times$] Nine elements $a + bj$, $a, b \equiv 0, 1, -1 \pmod 3$;

$$(a + bj) \oplus (c + dj) = a + c + (b + d)j,$$

$$(a + bj) \otimes (c + dj) = ac - (-1)^{ab}bd + \{bc + (-1)^{ab}ad\}j,$$

where the exponent $ab$ is taken in the form $0, 1$, or $-1$. Then $j \otimes j = -1$,

$$j \otimes (1 + j) = -1 + j, \qquad (1 + j) \otimes j = 1 - j,$$

so that $6^\times$ and the right-hand distributive law fail. For $4^\times$,

$$(a + bj) \otimes \left(\frac{a - (-1)^{ab}bj}{a^2 + b^2}\right) = 1 = i_\times,$$

since $a^2 + b^2 \equiv 0$ only when $a \equiv b \equiv 0 \pmod 3$. Computation shows that $2^\times$ holds.

[7] Any finite group of order $> 1$; $a \oplus b = a \otimes b = a \circ b$.

[$5_0$] All rational numbers; $\oplus = +$, $\otimes = \times$.

* A much simpler statement of $\Sigma_7$, Transactions, vol. 4 (1903), p. 20.

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For $k = 2\,(k = 3)$, *i. e.*, for an enumerable (a non-enumerable) infinitude of elements, we employ the following sets $[j]$, $R$ denoting the set of all rational (real) numbers, $R_+$ that of all positive rational (positive real) numbers:

$[1^+]$ $R$; $\otimes = \times$, $a \oplus 0 = 0 \oplus a = a$, $a \oplus a = 0$, $a \oplus b$ not in $R$ if $a \neq b$, $a \neq 0$ or $b \neq 0$.

$[2^+]$ $R$; $\otimes = \times$, $a \oplus a = 0$, $a \oplus b = a + b\,(a \neq b)$.

$[3^+]$ $R_+$; $\oplus = +$, $\otimes = \times$.

$[4^+]$ $R_+$ and zero; $\oplus = +$, $\otimes = \times$.

$[1^\times]$ $R$; $\oplus \equiv +$, $a \otimes b = ab$ (if $a = 1$, $b = 1$, or $ab = 1$), otherwise $a \otimes b$ not in $R$.

$[2^\times]$ Hypercomplex numbers $\alpha + \beta i + \gamma j$, $\alpha, \beta, \gamma$ arbitrary in $R$;

| $\otimes$ | $1$ | $i$ | $j$ |
|---|---|---|---|
| $1$ | $1$ | $i$ | $j$ |
| $i$ | $i$ | $-1$ | $1$ |
| $j$ | $j$ | $1$ | $-2$ |

Then $(ii)j = -j$, $i(ij) = i$; $(\alpha + \beta i + \gamma)(x + yi + zj) = 1$ if

$$x = \alpha/\Delta, \quad y = -\beta/\Delta, \quad z = -\gamma/\Delta, \quad \Delta = \alpha^2 + \gamma^2 + (\beta - \gamma)^2.$$

$[3^\times]$ $R$; $a \oplus b = a + b$, $a \otimes b = 0$.

$[4^\times]$ Positive and negative integers and zero; $\oplus = +$, $\otimes = \times$. (All $a_1 e_1 + a_2 e_2$ with $a_1$ and $a_2$ real, $e_1^2 = 0$, $e_1 e_2 = e_2 e_1 = e_1$, $e_2^2 = e_2$; $\oplus = +$, $\otimes = \times$; then $e_1 \otimes y = i_\times = e_2$ is impossible since $e_1(y_1 e_1 + y_2 e_2) = y_2 e_1$.)

$[6^\times]$ All quaternions $ai + bj + ck + d$, $a, b, c, d$ in $R$; $\oplus = +$, $\otimes = \times$.

$[7]$ $R_+$; $\oplus = \otimes = \times$.

$[5_k, k = 2 \text{ or } 3]$ Any finite field, *e. g.*, the classes of residues modulo $p$.

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# ON SEMI-GROUPS AND THE GENERAL ISOMORPHISM BETWEEN INFINITE GROUPS*

BY

LEONARD EUGENE DICKSON†

1. When there exists a correspondence between the elements of two finite groups such that the product of two elements of one corresponds to the product of the corresponding elements of the other, then the elements of either which correspond to the identity of the other group form themselves a group. It is somewhat surprising that this familiar theorem fails in general for infinite groups, as shown by definite examples in §§ 7–8. Nevertheless, DE SÉGUIER ‡ has attempted to establish the theorem for any groups; the error § in his argument is quite subtile. The correct theorem involves the concept ‖ *semi-group*, which reduces to a group when there is a finite number of elements, but not in general for an infinitude of elements.

2. Given a function $a \circ b$ of two arguments and a set of elements, we say that the elements form a semi-group with respect to $\circ$ when the following postulates hold:

(1) For every two elements $a$ and $b$ of the set, $a \circ b$ is uniquely determined as an element of the set.

(2) $(a \circ b) \circ c = a \circ (b \circ c)$, in the sense of Transactions, p. 199.

(3) If $a, x, x'$ occur in the set, and if there are equal determinations in the set of $a \circ x$, $a \circ x'$, then $x = x'$.

(4) If there are equal determinations of $x \circ a$, $x' \circ a$, then $x = x'$.

We may replace the triple statement (1) by the three postulates $(1_1)$, $(1_2)$, $(1_3)$ of Transactions, p. 199.

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‡ *Éléments de la théorie des groupes abstraits* (1904), p. 66. In this text a group is any finite or infinite group unless a limitation is explicitly given. The theorem in question is expressly applied to infinite groups in § 66; cf. top of p. 68.

§ *Ibid.*, p. 66, l. 9: "tous les éléments ··· répondent d'ailleurs à $a_i^{-1}$."

‖ Otherwise defined by DE SÉGUIER, *ibid.*, p. 8. Cf. the writer's review in the Bulletin of the American Mathematical Society, vol. 11 (1904), pp. 159–164. The postulates (3) and (4) there given by the writer are here replaced by the equivalent but more desirable postulates (3) and (4). The proof of the independence of the new postulates applies as well to the former, whose independence was stated without proof.



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Semi-groups fall naturally into three classes, according as

($5_1$) The number of elements is a fixed integer $n$;

($5_2$) The elements of the set form an enumerable infinitude;

($5_3$) The elements of the set form a non-enumerable infinitude.

To establish the independence of the postulates of the system

$$S_k = (1, 2, 3, 4, 5_k) \qquad (k = 1, 2, \text{ or } 3),$$

with $n > 2$ in ($5_1$), we exhibit, for $j = 1, 2, 3, 4, 5_k$, a set $\Sigma_j$ of elements and a rule of combination $\circ$ such that the $j$th postulate fails while the remaining four hold. According as $k = 1, 2$, or 3, we employ as

$\Sigma_1$: $N$, $R$ with 2 omitted, $P$ with 2 omitted, with $a \circ b = a + b$;

$\Sigma_2$: $C$; $R$, with $a \circ b = a^{-1} \times b^{-1}$; $P$, with $a \circ b = a^{-1} \times b^{-1}$;

$\Sigma_3$: $N$, $R$, $P$, with $a \circ b = a$;

$\Sigma_4$: $N$, $R$, $P$, with $a \circ b = b$;

$\Sigma_{5k}$: $R$, $P$, $R$, with $a \circ b = a \times b$;

where $N$ is the set of the first $n$ positive integers, $R$ the set of all positive rational numbers, $P$ the set of all positive real numbers; and where $C$ is any group of order $n$ modified by interchanging two rows of its multiplication-table; for example, having the elements $0, 1, \cdots, n-1$, such that for any element $b$,

$$0 \circ b = (b+1), \qquad 1 \circ b = b, \qquad a \circ b = (a+b) \text{ if } a > 1,$$

$(d)$ denoting the least positive residue of $d$ modulo $n$. Then

$$(0 \circ 0) \circ 0 = 1 \circ 0 = 0, \qquad 0 \circ (0 \circ 0) = 0 \circ 1 = 2.$$

A finite semi-group is a group. Indeed, the system* $S_1$ is precisely the definition of a finite group given by WEBER.† The corresponding definition of a finite commutative group was given earlier by KRONECKER.‡ Every infinite group is a semi-group, but not inversely.

3. THEOREM. *A set of elements belonging to a group and such that the product of any two occurs in the set forms a semi-group.*

* Not until I had written out the independence proof for $S_1$ did I notice that a similar proof had been given by HUNTINGTON at the end of his paper in the Bulletin of the American Mathematical Society, vol. 8 (1902), pp. 296–300.

† Mathematische Annalen, vol. 20 (1882), p. 302; *Lehrbuch der Algebra*, 2d ed., vol. 2, pp. 3–4.

‡ KRONECKER, Berliner Monatsbericht, 1870, pp. 882–883.

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Indeed, postulates (1)–(4) hold for the set. For example, the elements $a^n (n = 1, 2, \cdots)$ of the infinite cyclic group $\{a, a^{-1}\}$ form a semi-group.

4. THEOREM. *In a semi-group any right-hand identity is a left-hand identity, and conversely; there is at most one identity.*

If $i$ is a right-hand identity, then for every $a$ and $b$,

$$a \circ (i \circ b) = (a \circ i) \circ b = a \circ b,$$

whence $i \circ b = b$ by (3), so that $i$ is a left-hand identity. The converse follows from $(b \circ i) \circ a = b \circ (i \circ a) = b \circ a$ and (4). Finally, if $i$ and $i_1$ are identities, $i = i \circ i_1 = i_1$.

5. THEOREM. *In a semi-group containing the identity, any right-hand inverse of $a$ is a left-hand inverse of $a$, and conversely; there is at most one inverse of $a$.*

If $a \circ a' = i$, then

$$a \circ (a' \circ a) = (a \circ a') \circ a = i \circ a = a = a \circ i,$$

whence $a' \circ a = i$ by (3). The converse follows similarly from (4).

6. A correspondence between the elements of two sets $A$ and $B$ is called *mutual* when, if $a$ is one of the elements corresponding to $b$, $b$ is one of the elements corresponding to $a$. We write $a \sim b$.

Two groups $A$ and $B$ are called *isomorphic* if there exists a mutual correspondence between their elements such that to each element of $A$ corresponds one or more elements of $B$ and to each element of $B$ corresponds one or more elements of $A$, and such that $a_i a_j \sim b_i b_j$ if $a_i \sim b_i$, $a_j \sim b_j$.

Let $A'$ denote the set of elements of $A$ corresponding to the identity $I_B$ of $B$, and $B'$ the set of elements of $B$ corresponding to $I_A$.

If $a_i' \sim I_B$, and $a_j' \sim I_B$, then $a_i' a_j' \sim I_B$. From § 3 follows the

THEOREM. *Each of the sets $A'$ and $B'$ is a semi-group.*

7. As an example in which neither of the semi-groups $A'$ and $B'$ is a group, consider the two infinite cyclic groups

$$(6) \qquad A = \{a, a^{-1}\}, \qquad B = \{b, b^{-1}\}.$$

To $a^i$ we make correspond $b^{-i+j}$ $(j = 0, 1, 2, \cdots)$, for every integer $i$ positive, negative, or zero. Then to $a^i$ $a^l$ correspond

$$b^{-(i+l)+r} = b^{-i+j} b^{-l+k} \qquad (r = j + k = 0, 1, 2, \cdots).$$

Moreover for any integer $i$, it follows that to $b^i$ correspond $a^{-i+j}$ $(j = 0, 1, 2, \cdots)$ and no further elements of $A$. Thus

$$A' = (I, a, a^2, \cdots), \qquad B' = (I, b, b^2, \cdots),$$

neither being a group. But the correspondence obeys the definition in § 6.

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8. As an example in which neither $A'$ nor $B'$ contains the identity, consider groups (6) and the correspondences

$$I_A \sim b^{1+j}, a \sim b^j, a^{2i} \sim b^{-i+j}, a^{2i+1} \sim b^{-i+j}, a^{-i} \sim b^{i+j} \quad (j=0, 1, 2, \cdots),$$

$i$ being any positive integer. Then, inversely,

$$I_B \sim a^{1+j}, b^i \sim a^{-i+j}, b^{-i} \sim a^{2i+j} \qquad (j=0, 1, 2, \ldots),$$

while any element of $B$ corresponds only to the indicated elements of $A$. The correspondence obeys the definition in § 6.

9. THEOREM. *If one of the semi-groups $A'$ and $B'$ is a group, the other is.*

Suppose that $B'$ is a group. Let $a'$ be any element of $A'$, and let $b$ be one of the elements of $B$ which correspond to $a'^{-1}$. Since $I_B \sim a'$, then $bI_B \sim a'^{-1}a' = I_A$. Hence $b$ is in $B'$. Then $b^{-1}$ lies in the group $B'$. Hence

$$b^{-1} \sim I_A, \; I_B = b^{-1}\, b \sim I_A\, a'^{-1} = a'^{-1},$$

so that $a'^{-1}$ lies in $A'$. Hence $A'$ is a group.

COROLLARY I. If $A'$ or $B'$ includes only a finite number of elements, $A'$ and $B'$ are groups.

COROLLARY II. If the groups $A$ and $B$ are isormorphic in the usual special sense so that there exists a mutual correspondence between their elements such that to each element of $A$ corresponds an unique element of $B$ and to each element of $B$ corresponds one or more elements of $A$, and such that $a_i a_j \sim b_i b_j$ if $a_i \sim b_i$, $a_j \sim b_j$, then the elements of $A$ which correspond to the identity of $B$ form a group.

THE UNIVERSITY OF CHICAGO,
*November 30, 1904.*

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# A SET OF POSTULATES FOR ORDINARY COMPLEX ALGEBRA*

BY

EDWARD V. HUNTINGTON

## *Introduction.*

The well-known algebra which forms one of the main branches of elementary mathematics is a body of propositions expressible in terms of five fundamental concepts—(the class of complex numbers, with the operations of addition and of multiplication, and the subclass of real numbers, with the relation of order)—and deducible from a small number of fundamental propositions, or hypotheses.

The object of the present paper is to analyze these fundamental propositions, as far as may be, into their simplest component statements, and to present a list which shall not only be free from redundancies, and sufficient to determine the algebra uniquely, but shall also bring out clearly the relative importance of the several fundamental concepts in the logical structure of the algebra.

A more precise statement of the problem is the following: we consider two undefined classes, $K$ and $C$; two undefined operations, which we may denote by $\oplus$ and $\odot$; and an undefined relation, which we may denote by $\circledcirc$; and we impose upon these five (undefined) fundamental concepts certain arbitrary conditions, or postulates, to serve as the fundamental propositions of an abstract deductive theory (the other propositions of the theory being all the propositions which are deducible from the fundamental propositions by purely logical processes); the problem then is, to choose these fundamental propositions so that all the theorems of algebra, regarded as formal or abstract propositions, shall be deducible from them—the class $K$ and $C$ corresponding to the classes of complex and real numbers, respectively, and the symbols $\oplus$, $\odot$, and $\circledcirc$ to the ordinary $+$, $\times$, and $<$.

Furthermore, the set of postulates, to be satisfactory, must determine the algebra *uniquely;* in other words, the set of postulates adopted must be such that any two systems $(K, C, \oplus, \odot, \circledcirc)$ which satisfy them all shall be *simply isomorphic* with respect to the fundamental concepts—that is, shall be capable of being brought into one-to-one correspondence in such a way that correspond-

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ing operations performed on corresponding elements shall lead to corresponding results (see the detailed theorem in § 8).*

The ordinary algebra will appear, then, as one among many *equivalent* concrete interpretations of the abstract theory — all the possible interpretations being called *equivalent, with respect to the fundamental concepts* $K$, $C$, $\oplus$, $\odot$, *and* $\circledcirc$, because they are not distinguishable by any properties which can be stated in terms of these symbols.† (Illustrations of such equivalent, or abstractly identical, systems will be given below in § 10.)

Finally, for the sake of elegance, the postulates should be *independent;* that is, no one of them should be deducible from the rest.

A problem of this kind can be solved, no doubt, in a great variety of ways; in fact, one has considerable freedom, not only in the choice of the postulates themselves, but also in the choice of the fundamental concepts in terms of which the postulates are stated. For example, in the present paper, the class $C$ might easily be defined in terms of $K$, $\oplus$, $\odot$, and $\circledcirc$, instead of being introduced as a fundamental concept.‡ Or again, it would be possible, I believe, to develop the whole algebra on the basis of a single fundamental operation, in terms of which $C$, $\oplus$, $\odot$, and $\circledcirc$ could all be defined.§

For the present purpose, however, I have not attempted to reduce the number of fundamental concepts to a minimum, but have sought to give a set of postulates which shall conform as closely as possible to familiar forms of presentation. With this end in view, I have omitted the circles from the symbols $\oplus$, $\odot$ and $\circledcirc$, wherever it is possible to do so without confusion with the ordinary $+$, $\times$, and $<$ of arithmetic; it must be constantly borne in mind, however, that the

* A set of postulates having this property has been called a *categorical* set, as distinguished from a *disjunctive* set; see O. VEBLEN, Transactions, vol. 5 (1904), p. 346. The notion of equivalence, which had long been familiar in the case of two isomorphic groups, became in the hands of G. CANTOR the fundamental notion of his theory of classes (Mengenlehre, théorie des ensembles).

† In the case of any categorical set of postulates one is tempted to assert the theorem that if any proposition can be stated in terms of the fundamental concepts, either it is itself deducible from the postulates, or else its contradictory is so deducible; it must be admitted, however, that our mastery of the processes of logical deduction is not yet, and possibly never can be, sufficiently complete to justify this assertion. My statement in the last footnote on page 17 of the present volume of the Transactions must therefore be taken, as Mr. H. N. DAVIS first pointed out to me, with some qualification. Compare in this connection remarks by D. HILBERT in his address on the problems of mathematics at the Paris congress of 1900, translated in the Bulletin of the American Mathematical Society, ser. 2, vol. 8 (1901–02), especially pp. 444–445; also his "Axiom of Completeness" for real numbers in the Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 8 (1899), p. 183.

‡ To do this, we have merely to demand the existence of a class $C$ having all the properties mentioned in the postulates of groups II–V, below. The additional postulate required for this purpose would be thus in a high degree a compound statement.

§ The operation I have in mind is the operation of taking "the absolute value of the difference of."

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symbols $+$, $\times$, and $<$, when used in this more general sense, are simply arbitrary signs, having no properties not expressly stated in the postulates.

Furthermore, in selecting the postulates, I have chosen only such statements as do not involve the assumption of the existence of any kind of numbers.

It should be said, in conclusion, that no attempt is here made to give a metaphysical analysis of the concepts *class*, *operation*, and *relation*, on which the algebra is based, or of the *laws of deductive logic* by which its propositions are deduced — the discussion of these more fundamental notions, here assumed as familiar, being matter for the trained student of philosophy. I hope, however, that a paper like the present may be indirectly of service on the philosophical side of the subject, by enabling one to formulate very precisely the problems involved in the question: *What is algebra?*

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I have made use, in the course of the work, of results obtained in my previous paper on real algebra, which will be found in the present volume of the Transactions.* To this paper, and to the *Theoretische Arithmetik* of STOLZ and GMEINER (1901–), as well as the *Principles of Mathematics*, by BERTRAND RUSSELL (vol. 1, 1903), the reader is referred for bibliographical references. Compare also the addresses by JOSIAH ROYCE † and MAXIME BÔCHER ‡ at the Congress of Arts and Science, St. Louis, September, 1904, and a series of articles by LOUIS COUTURAT in a French review.§

## PART I. THE POSTULATES FOR COMPLEX ALGEBRA, AND DEDUCTIONS FROM THEM.

We consider *two classes*, $K$ and $C$, *two operations*, $+$ and $\times$, and a *relation*, $<$, subject to the conditions prescribed in the twenty-eight postulates of this section. ‖ The symbols $a + b$ and $a \times b$ (or $ab$) may be read as the "sum" and the "product" of the elements $a$ and $b$, by analogy with the ordinary language of arithmetic, while the relation $a < b$, or $b > a$, may be read "$a$ pre-

* E. V. HUNTINGTON, *A set of postulates for real algebra, comprising postulates for a one-dimensional continuum and for the theory of groups*, Transactions, vol. 6 (1905), pp. 17–41. — The statement on page 21, that the postulates $R1$–$R8$ form a categorical set, is clearly erroneous, and is corrected in § 4 below. Since this statement was merely parenthetical, the correction does not affect the rest of the paper. — In postulate $R6$, on pages 20 and 32, the element $x$ in 2°) must be noted as "different from $X$," in order to make the proof of independence for $R3$ conclusive.

† J. ROYCE, *The sciences of the ideal*, published in Science for October 7, 1904 (new series, vol 20, pp. 449–462).

‡ M. BÔCHER, *The fundamental conceptions and methods of mathematics*, published in the Bulletin of the American Mathematical Society, vol. 11 (1904–05), pp. 115–135.

§ L. COUTURAT, *Les principes des mathématiques*, in the Revue de Métaphysique et de Morale, vol. 12, 1904.

‖ A summary of these postulates will be given below, in § 9.



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cedes $b$," or "$b$ follows $a$."* As already noted, however, these interpretations of the symbols are by no means the only possible ones (compare the examples below, in § 10).

The postulates are arranged in five groups, numbered I–V.

## § 1. *The class $K$ with regard to $+$ and $\times$.*

The following thirteen postulates, I:1–13, make the class $K$ a *field*† with respect to the operations $+$ and $\times$.

The first seven postulates state the general laws of operation in the field, and are to be understood as holding only in so far as the elements, sums, and products involved are elements of $K$. The remaining six postulates give the requisite "existence-theorems."

POSTULATE I,1. $a + b = b + a$. (Commutative law for addition.)

POSTULATE I,2. $(a + b) + c = a + (b + c)$. (Associative law for addition.)

POSTULATE I,3. *If* $a + b = a + b'$, *then* $b = b'$.

This may be called the (left-hand) law of cancelation for addition.

POSTULATE I,4. $ab = ba$. (Commutative law for multiplication.)

POSTULATE I,5. $(ab)c = a(bc)$. (Associative law for multiplication.)

POSTULATE I,6. *If* $ab = ab'$, *and* $a + a \neq a$, *then* $b = b'$.

This may be called the (left-hand) law of cancelation for multiplication.

POSTULATE I,7. $a(b + c) = (ab) + (ac)$.

This is the (left-hand) distributive law for multiplication with respect to addition.

POSTULATE I,8. *If $a$ and $b$ are elements of $K$, then their "sum," $a + b$, is an element of $K$*; that is, there is an element $s$, uniquely determined by $a$ and $b$, such that $a + b = s$.

POSTULATE I,9. *There is an element $x$ in $K$, such that $x + x = x$.*

From postulates I:1,2,3,8, it follows that there cannot be more than one element $z$ such that $z + z = z$, and by postulate I,9 there must be at least one such element. This unique element $z$ is called the *zero-element* of the system, and will be denoted by $0$, whenever there is no danger of confusion with the ordinary $0$ of arithmetic.

From the same postulates we have: $0 + a = a + 0 = a$, for every element $a$. (For proof of this and the preceding statement see p. 23.)

* The expressions "before" and "after," etc., are preferable to the expressions "greater" and "less," etc., in this connection, since the notion of *size* is not involved in the notion of *order*.

† Cf. E. V. HUNTINGTON, *Note on the definitions of abstract groups and fields*, with bibliography, in the present number of the Transactions.

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POSTULATE I,10. *If there is a unique zero-element* $0$, *then for every element* $a$ *there is an element* $a'$ *such that* $a + a' = 0$.

From postulates I:1,3,10, it follows that $a'$ is uniquely determined by $a$, and that $a + a' = a' + a = 0$. This element $a'$ is called the *negative of* $a$, and is denoted by $-a$, so that

$$a + (-a) = (-a) + a = 0.$$

Furthermore, every two elements $a$ and $b$ determine uniquely an element $x$ such that $a + x = b$; this element $x$, namely $x = (-a) + b$, is called their *difference*, $b$ *minus* $a$, and is denoted by $b - a$, so that

$$a + (b - a) = b.$$

(The proof depends on postulates I:1,2,3,8,9,10.) The usual properties of subtraction follow readily from this definition.*

POSTULATE I,11. *If* $a$ *and* $b$ *are elements of* $K$, *then their "product,"* $ab$, *is an element of* $K$; that is, there is an element $p$, uniquely determined by $a$ and $b$, such that $ab = p$.

POSTULATE I,12. *If there is a unique zero-element* $0$ *in* $K$, *then there is an element* $y$ *in* $K$ *different from* $0$, *and such that* $yy = y$.

From postulates I:4,5,6,11, we see that there cannot be more than one element $u$, different from $0$, such that $uu = u$, and by postulate I,12 there must be at least one such element. This unique element $u$ is called the unit-element of the system, and will be denoted by $1$ whenever there is no danger of confusion with the ordinary $1$ of arithmetic.

From the same postulates we have: $1 \times a = a \times 1 = a$, for every element $a$. (Proof: if $uu = u$, then $u(ua) = (uu)a = ua$, whence $ua = a$, by I,6.)

POSUTLATE I,13. *If there is a unique zero-element* $0$, *and a unique unit-element* $1$, *different from* $0$, *then for every element* $a$, *provided* $a$ *is not* $0$, *there is an element* $a''$ *such that* $aa'' = 1$.

From postulates I:4,6,13, it follows that $a''$ is uniquely determined by $a$, and that $aa'' = a''a = 1$ (where $a \neq 0$). This element $a''$ is called the *reciprocal of* $a$, and is denoted by $\frac{1}{a}$, or $1/a$, so that

$$a(1/a) = (1/a)a = 1 \text{ if } a \neq 0.$$

Furthermore, every two elements $a$ and $b$, provided $a \neq 0$, determine uniquely an element $y$ such that $ay = b$; this element $y$, namely $y = (1/a)b$, is called

* It would probably be preferable, however, on both pedagogical and logical grounds, not to introduce the operation of subtraction at all, but to regard $b - a$ always as the sum of the two elements $b$ and $-a$.

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their *quotient*, *b divided by a*, and is denoted by $\frac{b}{a}$, or $b/a$, so that

$$a(b/a) = (b/a)a = b \text{ if } a \neq 0.$$

(The proof depends on postulates I:4,5,6,11,12,13.) The usual properties of division follow readily from this definition.*

Now from the distributive law (postulate I,7), which we have not yet utilized, we see that

$$0 \times a = a \times 0 = 0,$$

for every element $a$. Then if $a$ is different from $0$, its reciprocal, $1/a$, will also be different from $0$, since otherwise we should have $1 = 0$. Hence the important theorem that *if a and b are both different from* $0$, *their product is also different from* $0$; or in other words, a product $ab$ cannot be zero unless at least one of its factors is zero. [Proof: if $ab = 0$ and $a \neq 0$, then $b = 1 \times b = (1/a)a \times b = (1/a) \times ab = (1/a) \times 0 = 0$.]

These theorems are sufficient to show that any system $(K, +, \times)$ which satisfies the postulates I:1–13 has all the properties of a *field* with respect to the operations $+$ and $\times$. To summarize:

*The elements of $K$ form an abelian group with respect to $+$; the elements excluding the zero of that group form an abelian group with respect to $\times$; for every element $a$, $a \times 0 = 0 \times a = 0$; and the operation $\times$ is distributive with respect to the operation $+$.*

We now consider the multiples, submultiples, and rational fractions of any element of the field.

Assuming the system of the ordinal, or natural, numbers, the characteristic properties of which are summarized in PEANO's five postulates (see page 27), we define the *mth multiple*, $ma$, *of any elememt* $a$ by the usual recurrent formulæ

$$1a = a,\ 2a = 1a + a,\ 3a = 2a + a,\ \cdots,\ (k+1)a = ka + a,$$

where $1, 2, 3, \cdots, k, \cdots, m$ denote ordinal numbers, and $k + 1$ the arithmetical sum of $k$ and $1$.

The *mth submultiple of* $a$, denoted by $a/m$, is then defined by the equation $a/m = a/(mu)$,† from which we have $m(a/m) = a$.

From these definitions it follows at once that $p(a/q) = (pa)/q$, so that we may denote either member of this equation by $pa/q$.

* It would probably be preferable, however, on both pedagogical and logical grounds, not to introduce the operation of division at all, but to regard $b/a$ always as the product of the two elements $b$ and $1/a$.

† The unit-element of the field is here denoted by $u$, instead of by $1$, to avoid confusion with the number $1$.

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Any element of the form $pa/q$ (where $p$ and $q$ are ordinal numbers) is called a *rational fraction of* $a$. The following theorems on rational fractions in which $m, n, p, q$ denote ordinal numbers, are readily deduced from the properties of a field:

$$(a)\quad \frac{mpa}{mq} = \frac{pa}{q}; \qquad (b)\quad \frac{pa}{q} + \frac{ma}{n} = \frac{(np+mq)a}{qn}; \qquad (c)\quad \frac{pa}{q} \times \frac{ma}{n} = \frac{pma}{qn}.$$

It should be noticed that we cannot yet infer from $ma = na$ that $m = n$, since, as far as postulates I:1–13 are concerned, the field may be finite, that is, contain only a finite number of elements.* (See below, under postulate IV, 2.)

## § 2. *The subclass $C$, with regard to $+$ and $\times$.*

The following seven postulates, II:1–7, make $C$ a subclass in $K$, which shall be, like $K$, a field with respect to the operations $+$ and $\times$.

POSTULATE II,1. *If $a$ is an element of $C$, then $a$ is an element of $K$.*

POSTULATE II,2. *The class $C$ contains at least one element.*

POSTULATE II,3. *If $a$ is an element of $C$, then there is an element $b$ in $C$, such that $a \neq b$.*

These three postulates tell us that $C$ is a subclass in $K$, containing at least two elements.

POSTULATE II,4. *If $a$ and $b$ are elements of $C$, then their sum, $a + b$, if it exists at all in $K$, is an element of $C$.*

POSTULATE II,5. *If $a$ is an element of $C$, then its negative, $-a$, if it exists at all in $K$, is an element of $C$.*

Hence the zero element, $0$, of $K$ is an element of $C$.

POSTULATE II,6. *If $a$ and $b$ are elements of $C$, then their product, $ab$, if it exists at all in $K$, is an element of $C$.*

POSTULATE II,7. *If $a$ is an element of $C$, then its reciprocal, $1/a$, if it exists at all in $K$, is an element of $C$.*

Hence the unit element, $1$, of $K$ is an element of $C$.

Now by virtue of postulate II,1, all the general laws of operation, I:1–7, hold as well for the subclass $C$ as for the whole class $K$; hence the postulates of I and II make the subclass $C$, like $K$, a *field* with respect to $+$ and $\times$.

It should be noticed that the subclass $C$ may be identical with $K$, as far as the postulates of I and II are concerned. (See § 6.)

## § 3. *The subclass $C$, with regard to $<$.*

In the following five postulates, III:1–5, we impose further restrictions on the subclass $C$, which, together with the postulates II:2–3, make the class $C$ a *one-*

* As is well known, the number of elements in a finite field must be a prime number, or a power of a prime.

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*dimensional continuum* (in the sense defined by DEDEKIND) with respect to the relation $<$. (On the use of the word continuum, see remarks in §4.) The fifth postulate, III,5, however, proves to be a consequence of the other four, when the properties of the field (I and II), and the postulate IV:1–2 are utilized; it should therefore not be included in the final list.

POSTULATE III,1. *If $a$ and $b$ are elements of $C$, and $a \neq b$, then either $a < b$ or else $a > b$.*

POSTULATE III,2. *If $a < b$, then $a \neq b$.*

That is, the relation $<$ is *non-reflexive* throughout $C$.

POSTULATE III,3. *If $a$, $b$ and $c$ are elements of $C$, and if $a < b$ and $b < c$, then $a < c$.*

That is, the relation $<$ is *transitive* throughout $C$.

From III:2–3 we see that the relations $a < b$ and $a > b$ cannot both be true;* that is, the relation $<$ is *non-symmetric* for every pair of elements in $C$. Hence, *if $a$ and $b$ are elements of $C$, we must have either*

$$a = b, \text{ or } a < b, \text{ or } a > b;$$

*and these three relations are mutually exclusive.*

POSTULATE III,4. *If $\Gamma$ is a non-empty subclass in $C$, and if there is an element $b$ in $C$ such that every element of $\Gamma$ precedes $b$, then there is an element $X$ in $C$ having the following two properties with regard to the subclass $\Gamma$:*

1°) *every element of $\Gamma$ precedes $X$ or is equal to $X$; while*

2°) *if $x'$ is any element of $C$ which precedes $X$, then there is at least one element of $\Gamma$ which follows $x'$.*

This is the postulate of *continuity* in the form essentially due to DEDEKIND.†

The element $X$, which is readily seen to be uniquely determined by the subclass $\Gamma$, is called the "upper limit" of $\Gamma$, or sometimes its "least upper bound." From III:1–4 can be deduced, as on page 20, the following theorem, which is DEDEKIND's original form of the postulate of continuity:

*If the elements of $C$ are divided into two non-empty classes, $\Gamma$ and $\Gamma'$, such that every element of $C$ belongs either to $\Gamma$ or to $\Gamma'$, and if every element of $\Gamma$ precedes every element of $\Gamma'$, then there is an element $X$ in $C$ which has the following properties:*

1°) *every element of $C$ which precedes $X$ belongs to $\Gamma$; and*

2°) *every element of $C$ which follows $X$ belongs to $\Gamma'$.*

It should be noticed in regard to this postulate of continuity, so-called, that all the postulates III:1–4 might be satisfied by a *discrete* assemblage; in order to introduce the property of *density*, we add the following postulate, III,5:

* In my previous paper (§1) this proposition, at least for the case $a \neq b$, was used as a postulate, the other postulates being "weakened" to allow of its independence. It has not seemed worth while to carry the analysis so far in the present article.

† Loc. cit., below.

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*Postulate* III,5. If $a$ and $b$ are elements of $C$, and $a < b$, then there is at least one element $x$ in $C$, such that $a < x$ and $x < b$. Such elements $x$ are said to lie *between* $a$ and $b$.

These five postulates, III:1–5, together with postulates II:2–3 (which merely tell us that the class $C$ contains at least two elements), make the class $C$ a *one-dimensional continuum*, in DEDEKIND's sense, with respect to the relation $<$ (see § 1 in my previous paper).

The postulate III,5 is not printed in italics, because, as already stated, when we utilize the postulates of I, II and IV, as well as those of III, it can be shown to be redundant. The proof will be given in § 5.

## § 4. *Note on the use of the word "continuum."*

The term *one-dimensional* (or *linear*) *continuum* with respect to the relation of order ($<$), as used in this and my previous paper, must be understood not in the exact sense defined by CANTOR* in 1895, but in the older and wider meaning in which the word was used by DEDEKIND† and WEBER.‡

DEDEKIND would call any system $(K, <)$ which has at least two elements and satisfies the postulates III:1–5, a continuum.

CANTOR restricts the term to systems which possess not only all these properties, but also two further properties, namely: 1) the class $K$ shall have a first and a last element; and 2) the class $K$ *shall contain a denumerable subclass, $R$, such that between every two distinct elements of $K$ there shall be at least one element of $R$.*§ (A denumerable class means an infinite class which can be placed in one-to-one correspondence with the class of natural numbers.)‖ He

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*G. CANTOR, *Beiträge zur Begrundung der transfiniten Mengenlehre*, Mathematische Annalen, vol. 46 (1895), p. 510; French translation by F. MAROTTE, *Sur les fondements de la théorie des ensembles transfinis*, 1899, p. 42. Cf. B. RUSSELL, loc. cit., chap. 36, and L. COUTURAT, loc. cit., pp. 664–675.

† R. DEDEKIND, *Stetigkeit und irrationalen Zahlen*, 1872; English translation by BEMAN.

‡ H. WEBER, *Algebra*, vol. 1 (1898), pp. 4–12.

§ Compare also A. N. WHITEHEAD, in a footnote on p. 299 of B. RUSSELL's *Principles of Mathematics*, vol. 1 (1903).

‖ The term *abzählbar*, introduced by G. CANTOR in 1879 [Mathematische Annalen, vol. 15 (1879), p. 4; vol. 20 (1882), p. 116; vol. 21 (1883), p. 550], and uniformly translated in French by *dénombrable* [Acta Mathematica, vol. 2 (1883), p. 365; Formulaire de Mathématiques, vol. 4 (1903), p. 130], has been variously rendered in English. The Century Dictionary gave *enumerable* in 1889; but the editor in charge of the mathematical vocabulary, Mr. C. S. PEIRCE, has since used *numerable*, and finally *denumerable*, in his own writings [see Monist, vol. 2, p. 541, 1892; and especially vol. 7, p. 214, 1897]. *Enumerable* is also the form proposed by Professor OSGOOD [American Journal of Mathematics, vol. 19, p. 161, 1897], and by Professor BEMAN [Bulletin of the American Mathematical Society, vol. 3, p. 174, 1897]. At least one writer, W. H. YOUNG, uses *countable* [Proceedings of the London Mathematical Society, vol. 34, p. 285, 1902, and later papers]; while the Encyclopedia Brittanica prefers *numerable* (article "Number," vol. 31, p. 282, 1902; G. B. MAT-

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then shows* that any two systems which satisfy this definition of the linear continuum are *ordinally similar*, and equivalent to the class of real numbers from 0 to 1 inclusive, as far as the relation $<$ is concerned. His postulates therefore form a *categorical set*, with regard to the relation $<$.

DEDEKIND's definition, on the other hand, is not categorical, but admits various non-equivalent types of order. This possibility is due not merely to the ambiguity in regard to the presence or absence of end points, but rather to the omission of CANTOR's second requirement, concerning the existence of a denumerable subclass within the system. This remarkable postulate is independent of all the other properties, as may be shown by the following example: †

Let $K =$ the class of all complex numbers $(\alpha, \beta)$ in which $\alpha$ and $\beta$ are real numbers such that $0 \leqq \alpha \leqq 1$ and $0 \leqq \beta \leqq 1$; with $(\alpha_1, \beta_1) \circledcirc (\alpha_2, \beta_2)$ whenever $\alpha_1 < \alpha_2$; and if $\alpha_1 = \alpha_2$, then whenever $\beta_1 < \beta_2$.

This system (which can be stated, if one prefers, in geometric terms) satisfies all of DEDEKIND's postulates, and has a first element, $(0, 0)$, and a last element, $(1, 1)$; but it is not a continuum under CANTOR's definition, since it is not possible to find within it a denumerable subclass of the kind required.‡

CANTOR's definition is thus clearly superior to DEDEKIND's for the purposes of a purely ordinal discussion of the continuum; if, however, as in algebra, the elements of the system considered are not only connected by the relation of order, but are also combined by the operation of addition,§ then DEDEKIND's definition may be, as it is in this paper, amply sufficient. It is therefore perhaps not undesirable that both definitions should remain in current use.

A further analysis of CANTOR's last postulate, with a general discussion of geometric continuity, will be found in a paper by O. VEBLEN in the present number of the Transactions.

## § 5. *Further postulates for subclass C.*

The following two postulates, IV:1–2, serve to connect the operations $+$ and $\times$ and the relation $<$, within the subclass $C$, and make it equivalent to the class of all real numbers with respect to $+$, $\times$, and $<$.

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THEWS]. The form *denumerable* is the form adopted by RUSSELL and WHITEHEAD in their *Principles of Mathematics*, and seems to me preferable to the other terms because it avoids a suggestion of finitude (cf. PEIRCE, loc. cit., 1897). I am also following RUSSELL and WHITEHEAD, and many Italian writers, in translating *Menge* (Mannigfaltigkeit, ensemble, insieme, aggregato) by *class* instead of by manifold, mass, set, ensemble, collection, assemblage, or aggregate.

* Loc. cit.

† Cf. H. WEBER, loc. cit., p. 11.

‡ For, there would have to be elements of the subclass for every value of $\alpha$ between 0 and 1, and these values of $\alpha$ form an infinite class which is not denumerable.

§ WEBER, loc. cit., speaks of such systems as systems which admit *measurement* (messbare Mannigfaltigkeiten).

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POSTULATE IV,1. *Within the class $C$, if $x < y$, then $a + x < a + y$, whenever* $a + x \neq a + y$.*

From this postulate we have at once: if $a > 0$ and $b > 0$, then $a + b > a$; and if $a < 0$ and $b < 0$, then $a + b < a$ (compare postulates $RA1$ and $RA2$ in my previous paper, page 25). Hence, if we define *positive and negative elements* as those elements of $C$ which are $> 0$ and $< 0$ respectively, we can show that if $a < b$ then there is a positive $x$ such that $a + x = b$, and conversely, if $x$ is positive in $a + x = b$, then $a < b$. [Proof as for theorem 14 on page 26.]

POSTULATE IV,2. *Within the class $C$, if $a > 0$ and $b > 0$, then $ab > 0$.*

By the aid of this postulate it is easy to show that the product, $ab$, of two elements of $C$ will be positive when $a$ and $b$ are both positive or both negative; and negative if $a$ and $b$ are of opposite signs (compare theorem 33 on page 40). Hence the unit-element 1, or $u$, with all its multiples and submultiples, will be positive;† and therefore all the multiples and submultiples of any positive element $a$ will also be positive.

We are now in a position to prove the theorem of *density*, which was stated provisionally in § 3 as postulate III,5. The proof, which involves all the postulates I–IV, is as follows: If $a < b$, $b - a$ will be positive, and if we take $c = (b - a)/2$, $c$ will also be positive; if then we take $x = a + c$ (from which follows $x + c = b$), we find at once that $a < x$ and $x < b$.

It thus appears that all the postulates of the set for real algebra given in the appendix of my previous paper (page 39) are satisfied by the class $C$; hence, in any system $(K, C, +, \times, <)$ which satisfies the twenty-six postulates of groups I–IV, the subclass $C$ will be equivalent to the class of all real numbers with respect to $+$, $\times$, and $<$.

## § 6. *Note on real algebra.*

As we have just seen, any class $C$ which is a field with respect to $+$ and $\times$, and possesses DEDEKIND's property of continuity with respect to $<$, and satisfies the postulates IV:1–2, will be equivalent to the class of all real numbers with respect to $+$, $\times$, and $<$.

Hence the twenty-six postulates of groups I–IV, with an additional postulate demanding that the class $K$ shall contain no elements which do not belong to $C$, would form a categorical set of postulates for real algebra.

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* This proviso makes possible the proof of the independence of postulate I,3, which would otherwise be deducible from IV,1, with the aid of III,2.

† For, if we suppose $u < 0$, then $-u > 0$, whence $u \times (-u) < 0$, or $-u < 0$, and therefore $u > 0$. The multiples of $u$ will form an ascending sequence: $u < 2u < 3u < \cdots$; whence: if $mu = nu$, then $m = n$; or more generally, if $ma = na$, and $a \neq 0$, then $m = n$, where $m$ and $n$ are ordinal numbers.

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On the other hand, these same twenty-six postulates, with the addition of the two postulates of group V, below, form a categorical set for complex algebra.

We may say, then, that the postulates of groups I–IV contain the properties common to both real and complex algebra.

## § 7. *Postulates peculiar to complex algebra.*

The two following postulates, V:1–2, which complete the list of postulates for complex algebra, concern the existence of elements of $K$ which do not belong to the subclass $C$.

POSTULATE V,1. *If $K$ is a field with respect to $+$ and $\times$, then there is an element $j$ in $K$ such that $jj = -1$ (where $-1$ is the negative of the unit-element of the field).*

If $i$ is one element which has this property (namely, $ii = -1$), then $-i$ will have the same property; and this will be true of no other element besides these two. That is, the element $j$ whose existence is postulated in V.1 is not uniquely determined, but may have two values, one the negative of the other. Which of these values we shall denote by $i$ and which by $-i$ is a matter of arbitrary choice.

Neither $i$ nor $-i$ can be an element of $C$, since the square of every element of $C$ is positive or zero; hence, if $x$ and $y$ are elements of $C$, the product $iy$ and the sum $x + iy$ will be elements of $K$ which do not belong to $C$ unless $y = 0$. This gives us the important theorem that if $x + iy = x' + iy'$, where $x, y, x', y'$, belong to $C$, then $x = x'$ and $y = y'$.*

The next, and last, postulate demands that the class $K$ shall contain no further elements, that is, no elements not expressible in the form $x + iy$, where $x$ and $y$ are elements of $C$.

POSTULATE V,2. *If $K$ and also $C$ are fields with respect to $+$ and $\times$, and if there is an element $i$ such that $ii = -1$ (see postulate V,1), then for every element $a$ in $K$ there are elements $x$ and $y$ in $C$ such that $x + iy = a$.*

We are now in a position to establish the categorical character of the whole set of postulates.

## § 8. *Proof of the equivalence of all systems that satisfy the postulates* I–V.

The following theorem for complex algebra is analogous to the theorem numbered 37 in my paper on real algebra (see page 40), and is proved in a similar way.

*If $(K, C, +, \times, <)$ and $(K', C', +, \times, <)$ are any two systems satisfying the twenty-eight postulates of groups I–V, then these systems are* EQUIV-

* Thus, from $x + iy = x' + iy'$ follows $x - x' = i(y' - y)$; hence $y' - y = 0$, and therefore $x - x' = 0$.

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ALENT, *or* ABSTRACTLY IDENTICAL, *with respect to* $K$, $C$, $+$, $\times$, *and* $<$; *that is, the classes* $K$ *and* $K'$ *can be brought into one-to-one correspondence in such a way that if* $a'$, $b'$, *etc., are the elements of* $K'$ *which correspond to the elements* $a$, $b$, *etc., in* $K$, *then we shall have :*

1°) *the subclass* $C$ *corresponds to the subclass* $C'$ ;

2°) *if* $a + b = c$, *then* $a' + b' = c'$, *and conversely ;*

3°) *if* $ab = c$, *then* $a'b' = c'$, *and conversely ;*

4°) *within the subclasses* $C$ *and* $C'$, *if* $a < b$, *then* $a' < b'$, *and conversely.*

In other words, the twenty-eight postulates form a categorical set.*

The proof of the theorem consists simply in bringing the elements of $C$ in $K$ into correspondence with the elements of $C'$ in $K'$, as in the proof of the analogous theorem for real algebra, and then making the element $i$ in $K$ correspond to the element $i'$ in $K'$. Then every element $x + iy$ in $K$ will correspond to an element $x' + iy'$ in $K'$, where $x'$ and $y'$ are the elements in $C'$ which correspond to the elements $x$ and $y$ in $C$.

It should be noticed that this correspondence between two equivalent systems can be set up in two, and only two, ways; the only ambiguity resulting from the arbitrary choice of the element $i$ (see postulate V,2)†.

## PART II. SUMMARY OF THE POSTULATES, AND PROOF OF THEIR CONSISTENCY AND INDEPENDENCE.

In this part, for convenience of reference, I give a list of the twenty-eight postulates of groups I–V, using the general symbols $\oplus$, $\odot$, $\circledcirc$, $z$, and $u$, instead of the symbols $+$, $\times$, $<$, 0, and 1; these latter symbols are used, in this part, only in their ordinary arithmetic meanings.

Next, I give some examples of systems ($K$, $C$, $\oplus$, $\odot$, $\circledcirc$) which satisfy all the twenty-eight postulates. These systems, as has just been shown, are all equivalent, or *abstractly identical*, with respect to $\oplus$, $\odot$, and $\circledcirc$; the existence of any one of them proves the *consistency* of the postulates.

Finally, I give, for each of the twenty-eight postulates, an example of a system ($K$, $C$, $\oplus$, $\odot$, $\circledcirc$) which satisfies all the other postulates, but not the one in question, thus establishing in the usual way the *independence* of all the postulates.

In constructing these systems, the existence of the ordinary systems of real and complex numbers, as derived by the "genetic" method from the system of the natural numbers, is assumed, as well as the usual geometric representations of these numbers by points of the plane or the sphere.

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* See the first two footnotes in the introduction, above.

† Speaking in geometric terms, the complex plane has two *aspects:* the first plane may be rotated through 180° about the axis of reals before being applied to the second plane.

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## § 9. *List of the postulates.*

Before giving the postulates themselves, it will be convenient to repeat the more important definitions which have been used in the course of the work. All these definitions are simply abbreviations, introduced in order to avoid tedious circumlocution.

DEFINITION 1. If there is a uniquely determined element $z$ such that $z \oplus z = z$, then $z$ is called the *zero-element*, or zero.

DEFINITION 2. If there is a unique zero-element $z$ (see definition 1), and if there is a uniquely determined element $u$, different from zero, and such that $u \odot u = u$, then $u$ is called the *unit-element*, or unity.

DEFINITION 3. If there is a unique zero-element $z$ (see definition 1), and if a given element $a$ determines uniquely an element $a'$ such that $a \oplus a' = z$, then $a'$ is called the *negative of* $a$, and is denoted by $-a$.

DEFINITION 4. If there is a unique zero-element $z$ and a unique unit-element $u$ (see definitions 1 and 2), and if a given element $a$, different from $z$, determines uniquely an element $a''$ such that $a \odot a'' = u$, then $a''$ is called the *reciprocal of* $a$, and is denoted by $1/a$.

---

The first seven postulates, giving the general laws of operation in the system, are to be understood to hold only in so far as the elements, sums, and products involved are elements of $K$.

POSTULATE I,1. $a \oplus b = b \oplus a$.

POSTULATE I,2. $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

POSTULATE I,3. If $a \oplus b = a \oplus b'$, then $b = b'$.

POSTULATE I,4. $a \odot b = b \odot a$.

POSTULATE I,5. $(a \odot b) \odot c = a \odot (b \odot c)$.

POSTULATE I,6. If $a \odot b = a \odot b'$, and $a \oplus a \neq a$, then $b = b'$.

POSTULATE I,7. $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$.

POSTULATE I,8. If $a$ and $b$ are elements of $K$, then $a \oplus b$ is an element of $K$.

POSTULATE I, 9. There is an element $x$ in $K$ such that $x \oplus x = x$.

POSTULATE I,10. If there is a unique zero-element $z$ in $K$ (see definition 1), then for every element $a$ in $K$ there is an element $a'$ in $K$, such that $a \oplus a' = z$.

POSTULATE I,11. If $a$ and $b$ are elements of $K$, then $a \odot b$ is an element of $K$.

POSTULATE I,12. If there is a unique zero-element, $z$, in $K$ (see definition 1), then there is an element $y$ in $K$, different from $z$, and such that $y \odot y = y$.

POSTULATE I,13. If there is a unique zero-element, $z$, and a unique unit-element, $u$, different from $z$, in $K$ (see definitions 1 and 2), then for every element $a$ in $K$, provided $a \neq z$, there is an element $a''$ in $K$ such that $a \odot a'' = u$.

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*The postulates* I:1–13 *make the class* $K$ *a field with respect to* $\oplus$ *and* $\odot$.

POSTULATE II,1. If $a$ is an element of $C$, then $a$ is an element of $K$.

POSTULATE II,2. The class $C$ contains at least one element.

POSTULATE II,3. If $a$ is an element of $C$, then there is an element $b$ in $C$, such that $a \neq b$.

POSTULATE II,4. If a and $b$ are elements of $C$, then $a \oplus b$, if it exists in $K$ at all, is an element of $C$.

POSTULATE II,5. If $a$ is an element of $C$, then its negative, $-a$ (see definition 3), if it exists in $K$ at all, is an element of $C$.

POSTULATE II,6. If $a$ and $b$ are elements of $C$, then $a \odot b$, if it exists in $K$ at all, is an element of $C$.

POSTULATE II,7. If $a$ is an element of $C$, then its reciprocal $1/a$ (see definition 4), if it exists in $K$ at all, is an element of $C$.

*The postulates* II:1–7, *taken with the postulates* I:1–13, *make the sub-class* $C$, *like the class* $K$, *a field with respect to* $\oplus$ *and* $\odot$.

POSTULATE III,1. If $a$ and $b$ are elements of $C$, and $a \neq b$, then either $a \lessdot b$ or else $a \gtrdot b$.

POSTULATE III,2. If $a \lessdot b$, then $a \neq b$.

POSTULATE III,3. If $a$, $b$, and $c$ are elements of $C$, and if $a \lessdot b$ and $b \lessdot c$, then $a \lessdot c$.

POSTULATE III,4. If $\Gamma$ is a non-empty subclass in $C$, and if there is an element $b$ in $C$ such that $\alpha \lessdot b$ for every element $\alpha$ of $\Gamma$, then there is an element $X$ in $C$ having the following two properties with regard to the sub-class $\Gamma$:

1°) if $\alpha$ is an element of $\Gamma$, then $\alpha \lessdot X$ or $\alpha = X$; while

2°) if $x'$ is any element of $C$ such that $x' \lessdot X$, there is an element $\xi$ in $\Gamma$ such that $\xi \gtrdot x'$.

*The postulates* III:1–4 *and* II:2–3, *taken with the redundant postulate* III,5 (*which is here omitted*), *make the sub-class* $C$ *a one-dimensional continuum with respect to* $\lessdot$, *in the sense defined by* DEDEKIND.

POSTULATE IV,1. If $a, x, y, a \oplus x$, and $a \oplus y$ are elements of $C$, and $x \lessdot y$, then $a \oplus x \lessdot a \oplus y$, whenever * $a \oplus x \neq a \oplus y$.

POSTULATE IV,2. If $a$, $b$, and $a \odot b$ are elements of $C$, and $a \gtrdot z$ and $b \gtrdot z$, then $a \odot b \gtrdot z$ (where $z$ is the zero-element of definition 1).

*The twenty-six postulates of groups* I–IV *make the sub-class* $C$ *equivalent to the class of all real numbers with respect to* $\oplus$, $\odot$, *and* $\lessdot$.

POSTULATE V,1. If $K$ is a field with respect to $\oplus$ and $\odot$, then there is an element $j$ in $K$ such that $j \odot j = -u$, where $-u$ is the negative of the unit-element of the field (see definitions 2 and 3).

* Cf. footnote under postulate IV,1, above.

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POSTULATE V,2. If $K$ and also $C$ are fields with respect to $\oplus$ and $\odot$, and if there is an element $i$ such that $i \odot i = -u$ (see postulate V,1), then for every element $a$ in $K$ there are elements $x$ and $y$ in $C$ such that $x \oplus (i \odot y) = a$.

*These twenty-eight postulates make the class $K$ equivalent to the class of all (ordinary) complex numbers with respect to $\oplus$, $\odot$, and $\circledcirc$.* (See the theorem in § 8.)

## § 10. *The consistency of the postulates.*

To prove that these postulates are consistent—that is, to prove that contradictory propositions can never be deduced from them by the processes of formal logic—it is sufficient to show the existence of any system $(K, C, \oplus, \odot, \circledcirc)$ in which all the postulates are satisfied; for then the postulates themselves and all their logical consequences express properties of this system, and must therefore be free from contradiction (since no really existent system can have contradictory properties).*

The following systems, all of which are *equivalent*, or abstractly identical, with respect to $K$, $C$, $\oplus$, $\odot$, and $\circledcirc$, are examples of systems which satisfy all the postulates:

1) $K =$ the class of ordinary complex numbers (that is, the class of all couples of the form $(x, y)$, where $x$ and $y$ are real numbers), with $\oplus$ and $\odot$ defined as the ordinary $+$ and $\times$; $C =$ the class of real numbers (that is, the class of DEDEKIND'S "cuts," or the class of CANTOR'S "fundamental sequences"), with $\circledcirc$ defined as the ordinary $<$. Here $z = 0$, $u = 1$, and $j = \pm\sqrt{-1}$.

2) $K =$ the class of all the points in a plane (the ordinary "complex plane"), with $\oplus$ and $\odot$ defined in the usual geometric manner;† $C =$ the points of a fixed straight line in the plane (the "axis of reals"), with $\circledcirc$ defined as "on the left of." Here $z$ is the point $(0, 0)$, $u$ the point $(1, 0)$, and $j$ the point $(0, 1)$ or $(0, -1)$.

* This statement, of course, makes large assumptions in regard to the objective validity of our processes of logical deduction. In this connection see the second of the problems proposed by D. HILBERT at the Paris congress of 1900 (loc. cit.) and an article by A. PADOA in L'Enseignement Mathématique, vol. 5 (1903), pp. 85-91.

† Thus, the point $a + b$ is the point reached by starting from $a$ and travelling a path equal in length and direction to the path from $O$ to $b$; while the point $a \times b$ is the point whose "angle" (from $OX$) equals the *sum* of the angles of $a$ and $b$, and whose "distance" (from $O$) equals the *product* of the distances of $a$ and $b$. Here the product, $x$, of the two distances, $\alpha$ and $\beta$, may be defined geometrically by the proportion $x : \alpha = \beta : 1$; or it may be defined in terms of measurement, as follows: let

$$\alpha = \lim_{n=\infty}\left(\frac{p_n}{q_n}\right) \quad \text{and} \quad \beta = \lim_{n=\infty}\left(\frac{r_n}{s_n}\right),$$

where the $p_n/q_n$ and $r_n/s_n$ are successive rational approximations to the lengths of $\alpha$ and $\beta$; then the sequence of products $p_n r_n / q_n s_n$ will have a limit, and this limit will be the required product of $\alpha$ and $\beta$. [Cf. E. V. HUNTINGTON, Strassburg dissertation, *Die Grund-Operationen an absoluten und complexen Grössen in geometrischer Behandlung*, Braunschweig, 1901.]

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System (1) is an "arithmetical" system, of which (2) is the "geometrical representation."

In the following systems, the "complex numbers" employed may be interpreted in either way, the geometric phraseology being perhaps the most convenient. All these systems are obtained from (1) or (2) by a projective transformation of the plane, as explained below.

3) $K =$ the class of ordinary complex numbers; $a \oplus b = a + b$; $a \odot b = kab$, where $k$ is any real number not zero; $C =$ the class of real numbers, with $\circledcirc$ defined as $<$ or as $>$, according as $k$ is positive or negative.

Here $z = 0$, $u = 1/k$, and $j = \pm (1/k)\sqrt{-1}$.

4) $K =$ the class of ordinary complex numbers; $a \oplus b = ab/(a + b)$, except that $a \oplus b = a + b$ whenever $a$ or $b$ or $a + b$ is zero; $a \odot b = ab$; $C =$ the class of real numbers, with $\circledcirc$ defined as the ordinary $<$, except that $(a \circledcirc b) = (a > b)$ whenever $a$ and $b$ are both positive or both negative.

Here $z = 0$, $u = 1$, and $j = \pm \sqrt{-1}$.

It will be noticed that the ordinary meaning of addition is preserved in system (3), and that of multiplication in system (4).

5) $K =$ the class of ordinary complex numbers; $a \oplus b = a + b - h$, and $a \odot b = kab - hk(a + b) + h(1 + hk)$, where $h$ and $k$ are any real or complex numbers, provided $k \neq 0$;* $C =$ the class of complex numbers whose corresponding points, in the complex plane, lie on the straight line through the two points $h$ and $(1/k) + h$; with $\circledcirc$ defined as the relation of order along this line, the forward direction being so chosen that $h \circledcirc (1/k) + h$.

Here $z = h$, $u = (1/k) + h$, $-u = -(1/k) + h$, and $j = \pm(\sqrt{-1}/k) + h$.

6) Let $\alpha$, $\beta$, $\gamma$, $\delta$ be any real or complex numbers such that $\alpha\delta - \beta\gamma \neq 0$; and let $F$, for the moment, stand for the class comprising all the ordinary complex numbers together with an extra element to be denoted by $\omega$. (Geometrically speaking, $F$ is the class of all the points on the complex sphere, including the "North Pole," $\omega$.)† We then define $K$, $\oplus$, $\odot$, $C$ and $\circledcirc$ as follows:

$K =$ all the elements of $F$, *excluding one point*, $P$, where if $\gamma \neq 0$, $P = \alpha/\gamma$, and if $\gamma = 0$, $P = \omega$.

If $a \neq \omega$ and $b \neq \omega$, then

$$a \oplus b = \frac{(2\alpha\gamma\delta - \beta\gamma^2)ab - \alpha^2\delta(a + b) + \alpha^2\beta}{\gamma^2\delta ab - \beta\gamma^2(a + b) + 2\alpha\beta\gamma - \alpha^2\delta},$$

except when the denominator is zero, in which case $a \oplus b = \omega$.

If $a \neq \omega$ and $b \neq \omega$, then

* For an especially simple case, take $h = -1$ and $k = 1$.

† The complex sphere, omitting the North Pole, is the stereographic projection of the complex plane.

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$$a \odot b = \frac{(\beta\gamma^2 + \alpha\delta^2)ab - (\alpha\beta\gamma + \alpha\beta\delta)(a+b) + (\alpha^2\beta + \alpha\beta^2)}{(\gamma\delta^2 + \gamma^2\delta)ab - (\beta\gamma\delta + \alpha\gamma\delta)(a+b) + (\beta^2\gamma + \alpha^2\delta)},$$

except when the denominator is zero, in which case $a \odot b = \omega$.

If $a = \omega$, or $b = \omega$, or both, then $a \oplus b$ and $a \odot b$ are defined as the limits of the expressions above, when $\omega$ is replaced by $1/x$ and $x$ approaches 0; it being understood that (on the sphere)

$$\lim_{x=0}\left[\frac{1}{x}\right] = \omega.$$

With these definitions of $\oplus$ and $\odot$, the class $K$ proves to be a field* in which $z$ and $u$ have the following values:

$$\text{if } \delta \neq 0,\ z = \frac{\beta}{\delta}; \qquad \text{if } \delta = 0,\ z = \omega;$$

$$\text{if } \gamma + \delta \neq 0,\ u = \frac{\alpha + \beta}{\gamma + \delta}; \qquad \text{if } \gamma + \delta = 0,\ u = \omega.$$

We then take $C =$ the class of all the points on the circumference of a circle drawn on the sphere through the three points $P$, $z$, and $u$, excluding the point $P$;† the relation $\circledcirc$ being defined as the relation of order along this curve in the direction $P - z - u - P$.

[This system (6), which includes the preceding systems as special cases, was suggested to me by Professor C. L. Bouton, who had noticed that the general formulæ

$$a \oplus b = f[f^{-1}(a) + f^{-1}(b)],$$

and

$$a \odot b = f[f^{-1}(a) \times f^{-1}(b)],$$

in which $f$ and its inverse $f^{-1}$ are single-valued functions, provide a pair of operations which satisfy, in general, the postulates for a field with respect to $\oplus$ and $\odot$. In the present example,

$$f(x) = \frac{\alpha x + \beta}{\gamma x + \delta} \qquad (\alpha\delta - \beta\gamma \neq 0).$$

This transformation (which is the general projective transformation) carries the system (1) into the system (6), the points 0, 1, and the excluded point $\omega$ in system (1) going over into the points $z$, $u$, and the excluded point $P$ in system (6).

To obtain the system (3), use $f(x) = x/k$. To obtain (4), take $f(x) = 1/x$, and afterwards denote the point $\omega$ by the symbol 0 (the excluded point being here $P = 0$). To obtain (5), use the transformation $f(x) = (x/k) + h$.]

* It is especially important to notice that the values of $a \oplus b$ and $a \odot b$, as given by the definitions, are determinate, and belong to the class $K$, whenever $a$ and $b$ are elements of $K$.

† It should be noticed that under the given conditions no two of these three points can coincide.

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## § 11. *The independence of the postulates.**

The independence of the postulates is proved by the following twenty-eight systems, in each of which $K$, $\oplus$, $\odot$, $C$, and $\circledcirc$ are so defined as to satisfy the twenty-seven other postulates, but not the one in question. No one of the postulates, then, can be deducible from the others; for if it were, every system which had the other properties would have this property also, which is not the case.†

*For* I,1. $K =$ all positive real numbers; $a \oplus b = b$; $a \odot b = ab$; $C = K$; $\circledcirc = <$.

*For* I,2. $K =$ all real numbers; $a \oplus b = (a + b)/3$; $a \odot b = ab$; $C = K$; $\circledcirc = <$.

*For* I,3. $K =$ all real numbers; $a \oplus b = 0$; $a \odot b = ab$; $C = K$; $\circledcirc = <$.

*For* I,4. $K =$ all real numbers; $a \oplus b = a + b$; $a \odot b = b$; $C = K$; $\circledcirc = <$.

*For* I,5. $K =$ the class of all couples $(\alpha, \beta)$ in which $\alpha$ and $\beta$ are real numbers; with

$$(\alpha_1, \beta_1) \oplus (\alpha_2, \beta_2) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2)$$

and

$$(\alpha_1, \beta_1) \odot (\alpha_2, \beta_2) = (\alpha_1\alpha_2 - \beta_1\beta_2, -\alpha_1\beta_2 - \alpha_2\beta_1);$$

$C =$ the class of real numbers, with $\circledcirc = <$. (If we represent these couples as points $\alpha + i\beta$ in the complex plane, $\oplus$ will be the ordinary addition of complex numbers, and $\odot$ will be the ordinary multiplication followed by reflection in the axis of $\alpha$.)

*For* I,6. The same system as for I,5, except that here $(\alpha_1, \beta_1) \odot (\alpha_2, \beta_2) = (\alpha_1\alpha_2, \beta_1\alpha_2 + \alpha_1\beta_2 + \beta_1\beta_2)$.

*For* I,7. $K =$ all real numbers; $a \oplus b = a + b$; $a \odot b = a + b + 1$; $C = K$; $\circledcirc = <$.

*For* I:8–13. In constructing the proof-systems for these postulates, $\oplus$, $\odot$, and $\circledcirc$ are defined as the ordinary $+$, $\times$, and $<$ (except as to $\odot$ in the case of I,11), and the subclass $C$ is taken identical with $K$; class $K$ itself being defined appropriately for each case, as follows:

8) $K =$ the class of the three real numbers $-1$, $0$, and $1$.

9) $K =$ all positive real numbers.

10) $K =$ all positive real numbers with $0$.

11) $K =$ all real numbers; $a \odot b = ab$ when $ab = 0$ or $1$; otherwise $a \odot b$ not in the class.

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*I am greatly indebted to Mr. G. D. BIRKHOFF, of the Harvard Graduate School, who has read this section in manuscript, and checked the correctness of the proofs of independence.

†This is the method of proving independence which has become familiar during the last decade, especially through the work of PEANO, PADOA, PIERI, and HILBERT; but the remark made in the first footnote in § 10, concerning the proof of consistency, applies also here.



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12) $K =$ all even integers.

13) $K =$ all integers.

*For* II,1. $K =$ all complex numbers of the form $(x, y)$, where $x$ and $y$ are rational; $C =$ all real uumbers; with $\oplus$, $\odot$, and $\circledcirc$ defined as the ordinary $+$, $\times$, and $<$.

*For* II:2–7. In constructing the proof-systems for these postulates we take $K =$ all complex numbers, and define $\oplus$ and $\odot$ as the ordinary $+$ and $\times$; the subclass $C$ and the relation $\circledcirc$ are then chosen appropriately for each postulate, as follows:

2) $C =$ an empty class.

3) $C =$ the class containing the single element 0; with $0 \circledcirc 0$ false.

4) $C =$ the class of all real numbers $x$ such that $-1 \leqq x \leqq 1$; with $\circledcirc = <$.

5) $C =$ all positive real numbers, with or without zero; $\circledcirc = <$.

6) $C =$ all pure imaginary numbers, with 0; $\circledcirc$ defined so that $a \circledcirc b$ is true whenever $a/i < b/i$ (where $i = \sqrt{-1}$).

7) $C =$ all integers, or all even integers; $\circledcirc = <$.

*For* III:1–3. As the proof-systems for these postulates, take $K =$ all complex numbers, and $C =$ all real numbers, with $\oplus$ and $\odot$ defined as the ordinary $+$ and $\times$, and with $\circledcirc$ defined appropriately for each case, as follows:

1) $a \circledcirc b$ always false.

2) $a \circledcirc b$ always true within $C$.

3) $a \circledcirc b$ always true within $C$ when $a \neq b$; $a \circledcirc a$ false.

*For* III,4. $K =$ all complex numbers of the form $(x, y)$, where $x$ and $y$ are rational; $C =$ all rational real numbers; and $\oplus$, $\odot$, and $\circledcirc$ defined as the ordinary $+$, $\times$, and $<$.

*For* IV,1. $K =$ all complex numbers; $\oplus$ and $\odot$ defined as the ordinary $+$ and $\times$; $C =$ all real numbers, with $\circledcirc$ defined as the ordinary $<$, except that $(a \circledcirc b) = (a > b)$ whenever $a$ and $b$ are both positive.

*For* IV,2. $K =$ all complex numbers; $a \oplus b = a + b$; $a \odot b = -ab$; $C =$ all real numbers, with $\circledcirc$ defined as the ordinary $<$.

*For* V,1. $K =$ all real numbers; $C = K$; $\oplus$, $\odot$, and $\circledcirc$ defined as the ordinary $+$, $\times$ and $<$.

*For* V,2.* $K =$ the class of all expressions $T$ of the form

$$T = A_m t^m + A_{m+1} t^{m+1} + A_{m+2} t^{m+2} + \cdots,$$

where $t$ is a parameter, and $m$ any integer (positive, negative or zero), while the $A$'s are ordinary complex numbers. The operations $\oplus$ and $\odot$ are defined as the ordinary $+$ and $\times$ for such (finite or infinite) expressions. The class $C$ is

* The systems for V,2 were suggested by the non-archimedean number-system given by D. HILBERT in his *Grundlagen der Geometrie*, 1899, § 33. In this connection compare also H. HANKEL, *Theorie der complexen Zahlensysteme*, 1867, § 31.

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the class of all those elements $T$ in which all the coefficients are zero except $A_0$, and $A_0$ is real; that is, $C =$ the class of real numbers. Within the class $C$ the relation $\odot$ is defined as the ordinary $<$.

This system contains the system of ordinary complex numbers as a part of itself, just as the system of the ordinary complex numbers contains the system of real numbers. A still more inclusive system, still satisfying all the postulates except V,2, may be constructed by replacing the $A$'s in the expression above by expressions $S_m$, $S_{m+1}$, $S_{m+2}$, $\cdots$, of the form

$$S = B_n s^n + B_{n+1} s^{n+1} + B_{n+2} s^{n+2} + \cdots,$$

where $s$ is another parameter, and $n$ any integer, while the $B$'s are ordinary complex numbers.

The list of proofs of independence is thus complete.

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# ON IMPRIMITIVE LINEAR HOMOGENEOUS GROUPS*

BY

H. F. BLICHFELDT

1. The present paper is devoted first to the proof of a theorem fundamental in the construction of imprimitive linear homogeneous groups † in a given number of variables. Then, by means of this and earlier theorems given by the author on the subject of linear groups, † JORDAN's theorem, ‡ to the effect that the order of a linear homogeneous group $G$ in $n$ variables is of the form $\lambda f$, where $f$ is the order of an abelian self-conjugate subgroup of $G$, and $\lambda$ is less than a fixed number depending only upon $n$, is proved for imprimitive groups, a number being found that $\lambda$ must divide. Finally, the principal imprimitive collineation-groups in 4 variables are found and their generating substitutions given.

THEOREM. *Either an imprimitive linear homogeneous group $G$ can be written in monomial form,* § *or the $n$ variables of the group can be so selected that they fall into $k$ sets of imprimitivity of $m$ variables each* ($n = km$), *permuted according to a permutation-group $K$ in $k$ letters, which group is transitive* (*in the sense of transitivity of permutation-groups*). *The subgroup* ($G'$) *of $G$, corresponding to the subgroup* ($K'$) *of $K$ which leaves one letter unchanged, is primitive* (*in the sense used in linear homogeneous groups*) *in the $m$ variables of the set corresponding to the letter that $K'$ leaves unchanged.*

In order that $G$ may be transitive (as a linear homogeneous group, i. e., "irreducible"), it is plainly necessary that its sets of imprimitivity contain the same number of variables, and that the permutation-group $K$, permuting these sets, is transitive (as a permutation-group). We shall prove that, if the

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† See articles, cited below as *Linear groups* I and II, by the author in these Transactions, vol. 4 (1903), p. 387, and vol. 5 (1904), p. 310, for definitions of terms and phrases used and for theorems employed.

‡ Journal für Mathematik, vol. 84 (1878), p. 89. JORDAN does not find a superior limit to $\lambda$. Such a limit is given for *primitive groups* by the author in *Linear groups* II. Dr. J. SCHUR has given a limit for $\lambda f$ for such groups in $n$ variables, the sum of the multipliers of the substitutions of which belong to a given algebraical field (Berliner Sitzungsberichte, January, 1905).

§ This term is used by MASCHKE in American Journal of Mathematics, vol. 17 (1895), p. 168. The author used the word "semi-canonical" in *Linear groups* II, p. 313.



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subgroup $G'$ of $G$, corresponding to the subgroup $K'$ of $K$ leaving one letter fixed, is not primitive in the variables of the set corresponding to the letter that $K'$ leaves fixed, then will $G$ be imprimitive in a greater number of sets than $k$.

Let the $k$ sets of $G$ be $x_{11}, x_{12} \cdots x_{1m}$; $x_{21}, \cdots, x_{2m}$; $\cdots$; $x_{k1}, \cdots, x_{km}$, and let the corresponding letters of $K$ be denoted by $y_1, y_2, \cdots, y_k$. Let $K_i$ be the subgroup of $K$, leaving $y_i$ fixed, and let $G_i$ be the corresponding subgroup of $G$. The group $K$, being transitive, must contain a substitution $(A_2')$ changing $y_1$ into $y_2$, one $(A_3')$ changing $y_1$ into $y_3$, etc. The group $G$ is plainly generated by $G_1$ and $k-1$ substitutions $(A_2, A_3, \cdots)$ found among those of $G$ corresponding to $A_2', A_3', \cdots$, respectively. The group obtained by erasing in $G_i$ all the variables except $x_{i1}, x_{i2}, \cdots, x_{im}$ will be denoted by $X_i$.

It may readily be seen that, no matter how the group $X_1$ be written, the variables $x_{i1}, x_{i2}, \cdots, x_{im}$ $(i = 2, 3, \cdots, k)$ may be so selected that the substitutions $A_2, A_3, \cdots$, have the forms

$$\begin{aligned} A_2: \quad & x_{11}' = x_{21},\ x_{12}' = x_{22},\ \cdots,\ x_{1m}' = x_{2m}; \\ & x_{21}' = \alpha_1 x_{11} + \alpha_2 x_{12} + \cdots + \alpha_m x_{km}, \text{ etc.}; \\ A_3: \quad & x_{11}' = x_{31},\ x_{12}' = x_{32},\ \cdots,\ x_{1m}' = x_{3m};\ \text{etc., etc.,} \\ & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

Let $P'$ be any substitution of $K$, and $P$ any substitution of $G$ taken from those corresponding to $P'$. If $P'$ replaces $y_i$ by $y_j$, the variables $x_{i1}, \cdots, x_{ik}$ are transformed by $P$ in the following manner:

$$P: \begin{cases} x_{i1}' = p_{11} x_{j1} + p_{12} x_{j2} + \cdots + p_{1m} x_{jm}, \\ x_{i2}' = p_{21} x_{j1} + p_{22} x_{j2} + \cdots + p_{2m} x_{jm}, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_{im}' = p_{m1} x_{j1} + p_{m2} x_{j2} + \cdots + p_{mm} x_{jm}. \end{cases}$$

Now, if the group $X_1$ is not primitive, the variables $x_{11}, x_{12}, \cdots, x_{1m}$ may be supposed to have been selected so that they fall into $k_1 > 1$ sets $x_{11}, \cdots, x_{1\alpha}$; $x_{1\alpha+1}, \cdots, x_{1\beta}$; $\cdots$, all the variables of each set being by $X_1$ transformed into linear functions of the variables of the same set, or all into the variables of another set. Then, by building the substitution $A_i P A_j^{-1}$,* belonging to $G_1$:

$$\begin{aligned} x_{11}' &= p_{11} x_{11} + p_{12} x_{12} + \cdots + p_{1m} x_{1m}, \\ x_{12}' &= p_{21} x_{11} + p_{22} x_{12} + \cdots + p_{2m} x_{1m}, \\ & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_{1m}' &= p_{m1} x_{11} + p_{m2} x_{12} + \cdots + p_{mm} x_{1m}, \\ & \text{etc.; etc.,} \end{aligned}$$

* For $A_1$ we may take the identical substitution of $G$.

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and bearing in mind the assumption made with regard to $X_1$, we see that the $km$ variables of $G$ fall into $kk' > k$ sets

$$x_{i1}, \cdots, x_{i\alpha}; \; x_{i\alpha+1}, \cdots, x_{i\beta}; \; \cdots; \qquad (i=1, 2, \cdots, k),$$

which are mutually permuted by $P$. The variables of $G$ are therefore broken up into a greater number of sets of imprimitivity than $k$. Starting with these $kk'_0$ sets and proceeding as above, we conclude that we must arrive ultimately at a selection of imprimitive sets for which the groups $X_i$ are primitive or reduce to groups in one variable each, in which case $G$ is written in monomial form. The theorem stated above is therefore proved.

It may be remarked that the writer's theorem 9, in *Linear groups* II, p. 313, follows immediately after it has been proved that any group, whose order is the power of a prime, is not primitive.

2. We shall now prove JORDAN's theorem for imprimitive groups. Let us consider such a group $G$ of order $g$ in $n = km$ variables, the group $X_i$ being primitive in the variables $x_{i1}, x_{i2}, \cdots, x_{im}$, if $m > k$. By § 12 of *Linear groups* II, the order of such a group $X_i$ in $m$ variables is of the form $\lambda_i f_i$, where $f_i$ is the order of a self-conjugate subgroup of $X_i$ composed of similarity substitutions, and where $\lambda_i$ is a factor of a certain number that can always be calculated when $m$ is given. Let us call this number $\phi(m)$.

The subgroup $H$ of $G$ corresponding to the identical substitution of $K$ has for order $h$, an integral multiple of $g/k!$. This group has a subgroup $F$ composed of substitutions which are similarity-substitutions for each of the groups $X_1, X_2, \cdots, X_k$, and whose order is an integral multiple of $h/\{\phi(m)\}^k$; i. e. an integral multiple of

$$\frac{g}{k!\,\{\phi(m)\}^k}.$$

The group $F$ of order $f$ is abelian, and is evidently invariant within $G$, and the order of the latter is of the form $\lambda f$, where $\lambda$ is a factor of $k!\{\phi(m)\}^k$.

3. We now pass on to the construction of the principal imprimitive collineation-groups in four variables. According to the theorem stated above, unless such a group can be written in monomial form, it must possess two sets of imprimitivity, say $(x, y)$ and $(z, u)$. Only the latter class of groups shall be considered here.

Let $G$ be such a group. It is generated by an intransitive group $G'$:

$$G': \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix},$$

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where $X_1$ and $X_2$ are primitive groups in the variables $(x, y)$ and $(z, u)$ respectively, and a substitution $R$ of the form

$$R: \quad x' = a_1 z + b_1 u, \quad y' = c_1 z + d_1 u, \quad z' = a_2 x + b_2 y, \quad u' = c_2 x + d_2 y.$$

Exhibiting $R$ in the form

$$R: \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix} \equiv \begin{bmatrix} 0 & 0 & a_1 & b_1 \\ 0 & 0 & c_1 & d_1 \\ a_2 & b_2 & 0 & 0 \\ c_2 & d_2 & 0 & 0 \end{bmatrix},$$

we find that $RG'R^{-1}$ takes the form

$$RG'R^{-1}: \begin{pmatrix} PX_2P^{-1} & 0 \\ 0 & QX_1Q^{-1} \end{pmatrix}.$$

It follows that $X_1$ and $X_2$ (as collineation-groups in two variables) are transformable one into the other by the matrices $P$ and $Q$. Also, if $A_1$ and $A_2$ are corresponding substitutions of $X_1$ and $X_2$ respectively, so are $PA_2P^{-1}$ and $QA_1Q^{-1}$. Moreover, we may replace $R$ by the substitution

$$R' = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \cdot R = \begin{pmatrix} 0 & A_1P \\ A_2Q & 0 \end{pmatrix} = \begin{pmatrix} 0 & P' \\ Q' & 0 \end{pmatrix}.$$

Thus, if $P$ is a matrix belonging to $X_1$, we may assume that the matrix of $P'$ is of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Bearing these things in mind, we can construct the required groups without any theoretical difficulties, though the process will involve some labor, especially in reducing the different types obtained to certain standard forms.

We begin by determining all the groups $G'$ possible. The groups $X_1$ (or $X_2$) are the well known tetrahedral, octahedral, and icosahedral groups, and are given in WEBER'S *Algebra*, II, 2d edition, pp. 272–287.

The types sought are generated by the following substitutions ($\rho$ being a factor of proportionality):

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| | $\rho x'$ | $\rho y'$ | $\rho z'$ | $\rho u'$ | |
|---|---|---|---|---|---|
| $S_1$ | $x$ | $-y$ | $z$ | $-u$ | |
| $S_2$ | $y$ | $x$ | $u$ | $z$ | |
| $S_3^{(k)}$ | $x+y$ | $i(-x+y)$ | $k(z+u)$ | $ik(-z+u)$ | $k^{3n}=1.$ |
| $S_4^{(n)}$ | $x$ | $y$ | $\alpha z$ | $\alpha u$ | $\alpha$ a primitive root of $\alpha^n=1.$ |
| $S_5$ | $ix$ | $-iy$ | $z$ | $u$ | |
| $S_6$ | $y$ | $-x$ | $z$ | $u$ | |
| $S_7$ | $x$ | $y$ | $iz$ | $-iu$ | |
| $S_8$ | $x$ | $y$ | $u$ | $-z$ | |
| $S_9^{(k)}$ | $x+y$ | $i(-x+y)$ | $(1+i)kz$ | $(1+i)ku$ | $k^{3n}=1.$ |
| $S_{10}^{(l)}$ | $(1+i)lx$ | $(1+i)ly$ | $z+u$ | $i(-z+u)$ | $l^{3n}=1.$ |
| $S_{11}^{(n)}$ | $x$ | $iy$ | $\beta z$ | $i\beta u$ | $\beta^n=1.$ |
| $S_{12}$ | $x$ | $iy$ | $\gamma z$ | $\gamma u$ | $\gamma^{4n}=(-1)^n.$ |
| $S_{13}$ | $\epsilon x$ | $\epsilon^4 y$ | $\epsilon z$ | $\epsilon^4 u$ | $\epsilon$ a primitive root of $\epsilon^5=1.$ |
| $S_{14}$ | $\epsilon(\omega x+y)$ | $x-\omega y$ | $\epsilon(\omega z+u)$ | $z-\omega u$ | $\omega=\epsilon+\epsilon^4.$ |
| $S_{15}$ | $\epsilon x$ | $\epsilon^4 y$ | $\epsilon^2 z$ | $\epsilon^3 u$ | |
| $S_{16}$ | $\epsilon(\omega x+y)$ | $x-\omega y$ | $\epsilon^4(z-\omega u)_1$ | $-\epsilon^2(\omega z+u)$ | |
| $S_{17}$ | $\epsilon x$ | $\epsilon^4 y$ | $z$ | $u$ | |

and are as follows:

| Group. | Order. | Generating Substitutions. |
|---|---|---|
| $a$ | $12n$ | $S_1,\ S_2,\ S_3^{(k)},\ S_4^{(n)}.$ |
| $b$ | $4.12.2n$ | $S_3^{(k)},\ S_4^{(2n)},\ S_5,\ S_6,\ S_7,\ S_8.$ |
| $c$ | $12.12.2n$ | $S_4^{(2n)}, S_5,\ S_6,\ S_7,\ S_8,\ S_9^{(k)},\ S_{10}^{(l)}.$ |
| $d$ | $24n$ | $S_1,\ S_2,\ S_3^{(l)},\ S_4^{(n)},\ S_{11}^{(2n)}.$ |
| $e$ | $4.24.2n$ | $S_3^{(l)},\ S_4^{(2n)},\ S_5,\ S_6,\ S_7,\ S_8,\ S_{11}^{(4n)}.$ |
| $f$ | $12.24.2n$ | $S_4^{(2n)},\ S_5,\ S_6,\ S_7,\ S_8,\ S_9^{(l)},\ S_{10}^{(l)},\ S_{11}^{(4n)}.$ |
| $g$ | $24.24.2n$ | $S_4^{(2n)},\ S_5,\ S_6,\ S_7,\ S_8,\ S_9^{(l)},\ S_{10}^{(l)},\ S_{11}^{(4n)},\ S_{12}.$ |
| $h$ | $60n$ | $S_4^{(n)},\ S_{13},\ S_{14}.$ |
| $h''$ | $60n$ | $S_4^{(n)},\ S_{15},\ S_{16}.$ |
| $j$ | $60.60.2n$ | $S_4^{(n)},\ S_{13},\ S_{14},\ S_{17}.$ |

4. In all of these cases, the groups $X_1$ and $X_2$ (considered as collineation-groups in 2 variables) are identical. Noting what was stated at the beginning of § 3 concerning the substitution $R$, we see then that the matrices $P$ and $Q$ must in the cases $a, b, c$ leave invariant a given tetrahedral group, and

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must therefore belong to the octahedral group containing this group self-conjugately. Then, by replacing $R$ by $R' = A \cdot R$, where $A$ is some one of the substitutions of $G'$, we find readily that $P$ may be assumed to be of the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ or of the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right)$. Under these assumptions, there will be no trouble in determining $Q$. In all the remaining cases we may take $P$ in the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. After the resulting groups have been reduced to certain standard types, the different substitutions $R$ will be as follows:

$$R_1: \quad \rho x' = z, \quad \rho y' = u, \quad \rho z' = x, \quad \rho u' = y;$$

$$R_2: \quad \rho x' = z, \quad \rho y' = u, \quad \rho z' = x, \quad \rho u' = iy;$$

$$R_3: \quad \rho x' = z, \quad \rho y' = iu, \quad \rho z' = x, \quad \rho u' = iy;$$

$$R_4: \quad \rho x' = z, \quad \rho y' = u, \quad \rho z' = y, \quad \rho u' = -x.$$

We obtain the following types of imprimitive collineation-groups in 4 variables that cannot be written in monomial form:

| Group. | Order. | Generators. |
|---|---|---|
| 1° | $2.12n$ | $R_1$ and $a$ |
| 2° | $2.12n$ | $R_3$ and $a$ $(k=1)$ |
| 3° | $2.4.12.2n$ | $R_1$ and $b$ $(k=1)$ |
| 4° | $2.4.12.2n$ | $R_3$ and $b$ |
| 5° | $2.12.12.2n$ | $R_1$ and $c$ $(k=l=1)$ |
| 6° | $2.12.12.2n$ | $R_3$ and $c$ |
| 7° | $2.24.n$ | $R_1$ and $d$ |
| 8° | $2.4.24.2n$ | $R_1$ and $e$ |
| 9° | $2.12.24.2n$ | $R_1$ and $f$ |
| 10° | $2.12.24.2n$ | $R_2$ and $f$ |
| 11° | $2.24.24.2n$ | $R_1$ and $g$ |
| 12° | $2.60n$ | $R_1$ and $h'$ |
| 13° | $2.60n$ | $R_4$ and $h''$ |
| 14° | $2.60.60.2n$ | $R_1$ and $j$ |

The groups 1° and 12° are intransitive if $n=1$. The groups 2° and 7° are intransitive if $n=1$, $k=1$ and $\beta=1$, and 2° is of type 7° if $n=2$ and $k=1$. This is readily seen if new variables $x_1 = x+z$, $y_1 = y+u$, $z_1 = x-z$, $u_1 = y-u$ be chosen. The four groups 1°, 2°, 7°, 12° have the invariant $xu - yz = 0$ and occur therefore among those determined by GOURSAT in Annales scientifiques de l'École Normale supérieure, ser. 3, vol. 6 (1889), pp. 9–102. The corresponding types are there numbered XIV, XVI, XVIII, XV, XIX.

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A classification of the imprimitive ("decomposable") groups which are "regular," i. e., possess the bilinear invariant $x_1y_2 - x_2y_1 - z_1u_2 + z_2u_1$, has been given by AUTONNE in Journal de Mathématiques, ser. 5, vol. 7 (1901), p. 351–394, where he gives also an extensive discussion of some of their geometrical properties.

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# SUR LES LIGNES GÉODÉSIQUES DES SURFACES CONVEXES*

PAR

HENRI POINCARÉ

## §1. *Introduction.*

Dans mes *Méthodes nouvelles de la Mécanique céleste* j'ai étudié les particularités des solutions du problème des trois corps et en particulier des solutions périodiques et asymptotiques. Il suffit de se reporter à ce que j'ai écrit à ce sujet pour comprendre l'extrême complexité de ce problème; à côté de la difficulté principale, de celle qui tient au fond même des choses, il y a une foule de difficultés secondaires qui viennent compliquer encore la tâche du chercheur. Il y aurait donc intérêt à étudier d'abord un problème où on rencontrerait cette difficulté principale, mais où on serait affranchi de toutes les difficultés secondaires. Ce problème est tout trouvé, c'est celui des lignes géodésiques d'une surface; c'est encore un problème de dynamique, de sorte que la difficulté principale subsiste; mais c'est le plus simple de tous les problèmes de dynamique; d'abord il n'y a que deux degrés de liberté, et puis si l'on prend une surface sans point singulier, on n'a rien de comparable avec la difficulté que l'on rencontre dans les problèmes de dynamique aux points où la vitesse est nulle; dans le problème des lignes géodésiques en effet, la vitesse est constante et peut être regardée comme une des données de la question.

M. HADAMARD l'a bien compris, et c'est ce qui l'a déterminé à étudier les lignes géodésiques des surfaces à courbures opposées; il a donné une solution complète de ce problème dans un memoire du plus haut intérêt. Mais ce n'est pas aux géodésiques des surfaces à courbures opposées que les trajectoires du problème des trois corps sont comparables, c'est au contraire aux géodésiques des surfaces convexes.

J'ai donc abordé l'étude des lignes géodésiques des surfaces convexes; malheureusement le problème est beaucoup plus difficile que celui qui a été résolu par M. HADAMARD. J'ai donc dû me borner à quelques resultats partiels, relatifs surtout aux géodésiques fermées qui jouent ici le rôle des solutions périodiques du problème des trois corps.

* Presented to the Society at the St. Louis meeting, September 17, 1904. Received for publication January 4, 1905.



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## § 2. *Foyers et caustiques.*

Soit $S$ une surface *convexe, analytique;* supposons que ses deux rayons de courbure principaux restent constamment compris entre deux limites $L_1$ et $L_2$ et par conséquent sa courbure totale entre $L_1^2$ et $L_2^2$. Nous voyons d'abord que le rayon de courbure de l'une quelconque de ses géodésiques restera toujours compris entre $L_1$ et $L_2$. Je commence par rappeler les propriétés essentielles des *foyers* et des *caustiques.*

Soit $O$ un point fixe de la surface $S$. Envisageons une géodésique $OM$ passant par le point $O$, soit $OH$ une autre géodésique fixe passant par ce même point $O$, soit $v$ l'angle sous lequel ces deux géodésiques se coupent en $O$, soit $u$ l'arc $OM$ compté sur le géodésique. Ces deux quantités $u$ et $v$ peuvent être regardées comme des coordonnées définissant la position du point $M$ sur la surface; ce sont, en quelque sorte, des coordonnées polaires, le point $O$ jouant le rôle du pôle, et la géodésique $OH$ celui de l'axe polaire. Le carré de l'élément d'arc sera de la forme

$$ds^2 = du^2 + \lambda^2 dv^2,$$

où $\lambda$ est une fonction de $u$ et de $v$.

Il est à remarquer qu'un point quelconque $M$ de la surface correspond à une infinité de couples de valeurs des coordonnées $u$ et $v$, et en effet il y a une infinité de manières d'aller de $O$ en $M$ en suivant une géodésique. Par exemple si $S$ se réduit à une sphère de rayon 1, on a

$$ds^2 = du^2 + \sin^2 u . dv^2$$

et les couples de valeurs

$$(u, v),\ (u + 2\pi, v),\ (u + 4\pi, v), \cdots,$$

correspondent à un même point $M$.

Considérons deux géodésiques issues du point $O$ et infiniment peu différentes, et supposons qu'elles se coupent de nouveau en un point $M$. Nous aurons pour le point d'intersection deux couples de valeurs des coordonnées et l'on obtiendra l'un ou l'autre suivant qu'on atteindra ce point $M$ en suivant la première géodésique, ou bien en suivant la seconde qui en diffère fort peu. Comme ces deux géodésiques sont infiniment peu différentes, les deux couples de valeurs seront très peu différents, soit $(u, v)$, et $(u + du, v + dv)$. Comme le point $M$ représenté par ces deux couples est le même, on aura

$$ds^2 = du^2 + \lambda^2 dv^2 = 0,$$

soit

$$du = 0,\ \lambda = 0.$$

Si donc nous envisageons deux géodésiques infiniment voisines issues du point $O$, elles se recouperont successivement en une infinité de points d'après un

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théorème de M. HADAMARD et les points d'intersection successifs pourront être désignés par $F_1, F_2, \cdots, F_n$. Le point $F_n$ est ordinairement appelé le $n^{\text{ème}}$ *foyer* du point $O$; on voit que $F_p$ est le $(p-q)^{\text{ème}}$ foyer de $F_q$.

On appelle *caustique* l'enveloppe de toutes les géodésiques issues du point $O$; et on voit que la caustique est le lieu des foyers du point $O$. Cette caustique d'après ce qui précède a pour équation

$$\lambda = 0.$$

Considérons un instant $v$ comme une constante, de façon à suivre constamment une des géodésiques issues de $O$, et désignons par $\lambda'$, $\lambda''$, etc., les dérivées successives de $\lambda$ par rapport à $u$. On sait que

$$-\frac{\lambda''}{\lambda}$$

représente la courbure totale de sorte que l'on a

$$\frac{1}{L_1^2} < -\frac{\lambda''}{\lambda} < \frac{1}{L_2^2}. \tag{1}$$

On conclut aisément de ces inéqualités:

1°. que $\lambda$, $\lambda'$ et $\lambda''$ sont toujours finis.

2°. que les racines de l'équation $\lambda = 0$ sont séparées par celles de l'équation $\lambda' = 0$ et réciproquement.

3°. que si $u_0$ est une racine de l'equation $\lambda = 0$ et $u_1$ une racine de l'equation $\lambda' = 0$; si entre $u_0$ et $u_1$ il n'y a aucune racine ni de $\lambda = 0$, ni de $\lambda' = 0$, on aura les inégalités

$$\frac{\pi}{2} L_2 < |u_1 - u_0| < \frac{\pi}{2} L_1, \tag{2}$$

Si donc nous considérons $u$ et $v$ comme les coordonnées *polaires* d'un point dans un plan, et que nous construisions dans ce plan les courbes $\lambda = 0$, $\lambda' = 0$, nous voyons que ces courbes se composent d'une série d'ovales fermées s'enveloppant mutuellement et enveloppant le pôle, de sorte que si l'on s'éloigne du pôle en suivant un rayon vecteur on rencontre alternativement une ovale appartenant à $\lambda = 0$ et une ovale de $\lambda' = 0$; la distance de deux ovales consécutives est comprise entre $\pi/2L_2$ et $\pi/2L_1$. On a d'ailleurs pour $u = 0$,

$$\lambda = 0, \ \lambda' = 1.$$

de sorte que l'on peut considérer le pôle comme l'une des ovales de la courbe $\lambda = 0$, cette ovale se réduisant à un seul point.

Les $1^{\text{ers}}$ foyers se trouvent sur la $1^{\text{ère}}$ ovale, les $2^{\text{ds}}$ foyers sur la $2^{\text{de}}$, et ainsi de suite. On voit donc que les foyers des différents ordres sont entièrement séparés les uns des autres. C'est ce qui nous permet de définir la $n^{\text{ème}}$ *caustique* comme le lieu des $n^{\text{èmes}}$ *foyers*.

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Nous appellerons $P$ le plan sur lequel nous représentons ainsi les points de $S$, en prenant pour coordonnées polaires $u$ et $v$. Dans ce plan, nous appellerons $C_1, C_2, \cdots$ les ovales successives de la courbe $\lambda = 0$, ($C_0$ se réduisant au pôle); nous appellerons $C'_1, C'_2, \cdots$, les ovales successives de la courbe $\lambda' = 0$. Dans le cas ou la surface $S$ serait de révolution, le point $O$ étant le pôle, $\lambda$ ne dépendrait que de $u$ et les ovales $C$ et $C'$ se réduiraient à des cercles concentriques. Soient $x, y, z$ les coordonnées du point $M$ et

$$F(x, y, z) = 0$$

l'équation de la surface $S$. Les équations d'une ligne géodésique peuvent s'écrire

$$(3) \qquad \frac{d^2x}{du^2} = \mu \frac{dF}{dx}, \qquad \frac{d^2y}{du^2} = \mu \frac{dF}{dy}, \qquad \frac{d^2z}{du^2} = \mu \frac{dF}{dz},$$

$\mu$ étant déterminé par l'équation

$$(4) \qquad \Sigma \frac{d^2F}{dx^2}\left(\frac{dx}{du}\right)^2 + 2\Sigma \frac{d^2F}{dxdy}\frac{dx}{du}\frac{dy}{du} + \mu\Sigma\left(\frac{dF}{dx}\right)^2 = 0.$$

Les valeurs initiales de $x, y, z$ seront les coordonnées du point $O$; quant aux valeurs initiales de $dx/du$, $dy/du$, $dz/du$, elles dépendront linéairement de $\cos v$ et $\sin v$.

Les équations différentielles (3), (4) ne peuvent admettre de point singulier réel; elles n'en admettraient en effet que si $\mu$ devenait infini, c'est à dire si $\Sigma(dF/dx)^2$ s'annulait, c'est à dire si l'on avait à la fois

$$\frac{dF}{dx} = \frac{dF}{dy} = \frac{dF}{dz} = 0,$$

c'est à dire si la surface $S$ avait un point singulier, ce qui n'est pas. Nous sommes donc certain que la solution envisagée de nos équations différentielles n'ira passer par aucun point singulier. Je puis donc lui appliquer le théorème que j'ai démontré aux Nos. 23 à 28 du Tome 1er de *Méthodes nouvelles de la Mécanique céleste.* Il en résulte que $x$, $y$ et $z$ sont des fonctions holomorphes de $u$ et des valeurs initiales, et par conséquent de $u$ et de $v$; je veux dire que $x, y, z$ sont développables suivant les puissances de $u - u_0$, $v - v_0$, pourvu que $|u - u_0|$ et $|v - v_0|$ soient assez petits et cela quelles que soient les valeurs de $u_0$ et $v_0$.

Revenons à nos caustiques; soient $Q_1$ et $Q_2$ deux points d'une caustique entre lesquels nous supposerons qu'il n'y ait aucun point singulier. Quelle sera (sur le surface $S$) l'arc de caustique compris entre ces deux points? Si $u_1$ et $u_2$ sont les valeurs de $u$ correspondant à ces deux points; cet arc est

$$|u_1 - u_2|.$$

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On le verrait par un raisonnement tout pareil à celui qui donne l'arc de la développée d'une courbe plane.

L'élément d'arc de caustique est donc égal à $du$.

Reprenons les deux géodésiques infiniment voisines issues du point $O$ et correspondant aux angles $v$ et $v + dv$. La distance d'un point de la 1ère géodésique à la 2de sera évidemment $\lambda dv$; quand on approchera d'un foyer, cette distance tendra vers 0 puisque $\lambda$ tendra vers 0, les deux géodésiques se recouperont sous un angle $\lambda' dv$. L'angle de contingence géodésique de la caustique est donc $\lambda' dv$. La courbure géodésique de la caustique est donc

$$\frac{\left(\frac{d\lambda}{du}\right)^2}{\left(\frac{d\gamma}{dv}\right)}.$$

Cela posé reprenons l'équation $\lambda = 0$; en aucun point on ne pourra avoir $\lambda' = 0$: les deux courbes $\lambda = 0$, $\lambda' = 0$ ne peuvent se couper, puisque la distance des deux ovales $C$ et $C'$ est au moins égale à $\pi/2L_2$; donc l'équation $\lambda = 0$ peut être résolue par rapport à $u$, et $u$ est une fonction holomorphe de $v$. Comme $x$, $y$, $z$ sont fonctions holomorphes de $u$ et de $v$, nous voyons que *dans l'équation de la caustique les coordonnées $x$, $y$, $z$ sont des fonctions holomorphes de $v$.*

Toutes les fois que $d\lambda/dv$ n'est pas nul, l'équation $\lambda = 0$ peut être résolue par rapport à $v$, et $v$ est fonction holomorphe de $u$. *Les coordonnées d'un point de la caustique sont donc des fonctions holomorphes de l'arc $u$.* Tous les points singuliers de notre caustique nous seront donc donnés par l'équation $d\lambda/dv = 0$. J'ajoute que la courbure géodésique de la caustique ne peut pas s'annuler ni changer de signe, puisque $d\lambda/du$ ne peut s'annuler. Il n'y a donc rien qui corresponde à une sorte de point d'inflexion.

Les points singuliers de la caustique correspondent donc aux racines de l'équation $d\lambda/dv = 0$, c'est à dire aux minima de $u$ quand on décrit l'une des ovales $C$ dans le plan $P$.

Pour nous rendre compte de la nature de ces points singuliers prenons un instant pour origine le point singulier, pour plan des $xy$ le plan tangent, pour axe des $x$ la tangente à la courbe, et choisissons l'origine de l'angle $v$ de façon que $v = 0$ au point singulier. Alors $x$ et $y$ sont développables suivant les puissances de $v$, le 1er terme du développement est en $v^m$ pour $x$, en $v^n$ pour $y$; et les entiers $m$ et $n$ satisfont à l'inégalite

$$m < n.$$

Si $\lambda$ s'annule ainsi que ses $p$ premières dérivées par rapport à $v$ et que la

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$(p+1)^{\text{ème}}$ ne s'annule pas, comme d'ailleurs $d\lambda/du$ ne s'annule pas; on voit que le développement de $u$ commence par un terme en $v^{p+1}$. On aura donc

$$m = p + 1.$$

D'autre part $d\lambda/dv$ commence par un terme en $v^p$ de sorte que pour un point très voisin du point singulier, la courbure géodésique est de l'ordre de $v^{-p}$.

Or cette courbure a pour expression

$$\frac{\dfrac{dx}{dv}\dfrac{d^2y}{dv^2} - \dfrac{dy}{dv}\dfrac{d^2x}{dv^2}}{\left[\left(\dfrac{dx}{dv}\right)^2 + \left(\dfrac{dy}{dv}\right)^2\right]^{\frac{3}{2}}};$$

elle est donc de l'ordre de

$$m + n - 3 - 3(m-1) = n - 2m.$$

On a donc

$$n - 2m = -p,$$

d'où finalement

$$m = p + 1, \qquad n = p + 2.$$

Le cas le plus simple est celui de

$$p = 1, \qquad m = 2, \qquad n = 3;$$

le point singulier est alors un point de rebroussement ordinaire.

Ces points de rebroussement correspondent aux maxima et aux minima de $u$, mais les deux cas doivent être distingués. S'il s'agit d'un minimum, et si $R$ est le point de rebroussement en question, les deux branches de la caustique touchent non la géodésique $OR$, mais son prolongement; s'il s'agit d'un maximum, les deux branches de la caustique sont dirigées de $R$ vers $O$.

Nous sommes ainsi amenés à distinguer 4 sortes de foyers.

1°. Les *foyers ordinaires*, correspondant aux points non singuliers de la caustique ($p = 0$).

2°. Les *foyers en pointe*, correspondant aux points de rebroussement ordinaires ($p = 1$), qui sont des minima de $u$.

3°. Les *foyers en talon*, correspondant aux points de rebroussement ordinaires, ($p = 1$) qui sont des maxima de $u$.

Ces deux expressions sont empruntées à l'art des chemins de fer; je me représente une trifurcation, où nous avons la voie principale représenté par la géodésique $OR$ et deux embranchements représentés par les deux branches de la caustique; un train qui suivrait la ligne principale en allant de $O$ vers $R$ rencontrerait les aiguilles en pointe dans le 1er cas et en talon dans le 2d.

4°. Les *foyers singuliers*, correspondant aux points singuliers d'ordre plus élevé ($p > 1$). On voit pour ce qui précède que ces points singuliers rentrent dans tous les cas dans une classe très particulier.

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Nous introduisons maintenant la notion de *ligne de partage*. Supposons que l'on joigne chaque point de la surface $S$ au point $O$ par le plus court chemin. Ce plus court chemin sera un arc de géodésique. Si $OAPM$ est le plus court chemin de $O$ à $M$; si le point $P$ est situé sur l'arc $OAPM$, l'arc $OAP$ sera manifestement le plus court chemin de $O$ à $P$. Il suit de là que le plus court chemin de $O$ à $M$ ne peut jamais croiser le plus court chemin de $O$ à $M'$.

Soit $OM$ une géodésique quelconque passant par $O$; on pourra trouver sur cette géodésique un point $P$, tel que le plus court chemin de $O$ à un point $Q$ situé sur la géodésique $OM$ entre $O$ et $P$, soit précisément l'arc $OQ$ de cette géodésique $OM$, mais que cela ne soit plus vrai si le point $Q$ est au delà de $P$. On dit alors que $P$ est *l'extrémité* du plus court chemin $OP$.

Nous pouvons alors conclure que par tout point de $S$ passe un plus court chemin et un seul. Il y a exception pour les extrémités des divers plus courts chemins; si $P$ est l'une de ces extrémités; du point $P$ partiront *au moins deux* plus courts chemins qui auront l'un et l'autre leur extrémités en $P$. Tous les plus courts chemins qui ont leur extrémité en $P$ ont même longueur.

Le lieu des points qui sont les extrémités de deux ou plusieurs plus courts chemins forment un ensemble de lignes que l'on peut appeler *lignes de partage*, Si un point $P$ partent seulement deux plus courts chemins, par le point $P$ passe une seule ligne de partage dont la tangent est la bissectrice de l'angle formé par les tangentes aux deux plus courts chemins.

Si du point $P$ partent plus de deux plus courts chemins, au point $P$ aboutissent plusieurs lignes de partage, dont les tangentes sont les bissectrices des angles formés par les tangentes à deux plus courts chemins consécutifs.

L'ensemble des lignes de partage ne divise pas la surface $S$ en deux régions, puisque l'on peut aller du point $O$ à un point quelconque $M$ de la surface sans traverser aucune ligne de partage; il suffit pour cela d'aller de $O$ en $M$ par le plus court chemin.

L'ensemble des lignes de partage, ou une partie de ces lignes, ne peut donc jamais constituer un polygone fermé; il formera une sorte de *système rameux*. où les bifurcations seront représentées par les points où aboutissent plus de deux plus courts chemins. Que représenteront alors les extrémités des rameaux? Supposons que nous suivions une ligne de partage $PQR$ et que le point $R$ soit l'extrémité de cette ligne. Du point $Q$ partiront deux plus courts chemins de même longueur; quand le point $Q$ ira de $P$ en $R$, ces deux plus courts chemins varieront d'une manière continue. Au point $R$ ils devront se confondre en un seul.

Il est aisé de voir que les points $R$ sont les seuls points des lignes de partage qui se trouvent sur la 1ère caustique; (je veux dire bien entendu qu'en aucun autre point d'une ligne de partage, l'un des plus courts chemin qui y'aboutissent

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ne touche la 1ère caustique). On voit d'ailleurs que ces points $R$ ne peuvent être des foyers ordinaires, mais seulement des foyers en pointe.

## § 3. *Géodésiques d'un sphéroïde.*

Cherchons les géodésiques d'une surface très peu différente d'une sphère; nous n'aurons pour cela qu'à appliquer la méthode de la variation des constantes de LAGRANGE. Sur la sphère, les lignes géodésiques sont les grands cercles et l'on peut adopter comme éléments définissant le mouvement d'un point sur ce grand cercle de la même manière que les éléments elliptiques définissent le mouvement Képlerien d'une planète, les quatre quantités suivantes: les coordonnées du pôle du grand cercle, la vitesse uniforme du mouvement ou une fonction de cette vitesse, la longitude du point mobile sur ce grand cercle, comptée à partir d'une certaine origine. Mais il convient d'abord de mettre les équations sous la forme canonique. Si nous mettons l'élément linéaire d'une surface quelconque sous la forme

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2,$$

on aura pour l'expression $T$ de la demi-force vive qui figure dans les équations de Lagrange ou de Hamilton

$$T = \tfrac{1}{2}(Eu'^2 + 2Fu'v' + Gv'^2),$$

d'où

$$U = \frac{dT}{du'} = Eu' + Fv',$$

$$V = \frac{dT}{dv'} = Fu' + Gv',$$

et

$$T = \tfrac{1}{2}(\mathfrak{E}U^2 + 2\mathfrak{F}UV + \mathfrak{G}V^2),$$

où $\mathfrak{E}$, $\mathfrak{F}$ et $\mathfrak{G}$ ont des valeurs faciles à calculer. Dans la cas de la sphère on a simplement

$$T = \tfrac{1}{2}(u'^2 + \sin^2 u \cdot v'^2),$$

d'où

$$T = \tfrac{1}{2}\left(U^2 + \frac{1}{\sin^2 u} V^2\right).$$

Dans le cas d'une surface très peu différente de la sphère, on aura

$$T = T_0 + \mu T_1,$$

où

$$T_0 = \tfrac{1}{2}\left(U^2 + \frac{1}{\sin^2 u} V^2\right),$$

où $\mu$ sera très petit et où

$$T_1 = \tfrac{1}{2}(eU^2 + 2fUV + gV^2)$$

$e$, $f$ et $g$ étant des fonctions quelconques de $u$ et de $v$.

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Formons l'équation de Jacobi.

Supposons d'abord $\mu = 0$ de façon que notre sphéroïde se réduise à une sphère.

Soit $\theta$ la longitude du noeud de l'orbite circulaire décrite par le point sur la sphère, sur le plan de l'équateur; soit $i$ l'inclinaison de cette orbite; soit $\lambda$ la longitude du point mobile sur la sphère comptée à partir du noeud; soit $\omega$ la vitesse de circulation; (le rayon de la sphère est pris pour unité ainsi que la masse du point mobile); alors la demi-force vive sera $\omega^2/2$; la constante des aires sera $\omega$ dans le plan de l'orbite et $\omega \cos i$ dans le plan de l'equateur.

Il est aisé d'exprimer $u$, $v$, $U$, $V$ en fonctions de $\theta$, $\lambda$, $i$, $\omega$, et on verrait comme en mécanique céleste que, en posant $G = \omega \cos i$,

$$\omega d\lambda + G d\theta - U du - V dv$$

est une differentielle exacte.

Si nous ne supposons plus $\mu = 0$, nous pourrons toujours considérer des variables nouvelles $\theta$, $\lambda$, $i$, $\omega$, liées à $u$, $v$, $U$ et $V$ par les mêmes relations que dans le cas de la sphère et que nous considérerons comme "les éléments de l'orbite osculatrice." Les équations conservent alors leur forme canonique et peuvent s'écrire

$$\frac{d\lambda}{dt} = \frac{dT}{d\omega}, \qquad \frac{d\omega}{dt} = -\frac{dT}{d\lambda}, \qquad \frac{dG}{dt} = -\frac{dT}{d\theta}, \qquad \frac{d\theta}{dt} = \frac{dT}{dG}.$$

Quelle est la forme de $T$? Observons d' abord que $T_0 = \omega^2/2$. Remarquons en outre que $u$, $v$, $u'/\omega$, $v'/\omega$, $U/\omega$, $V/\omega$ et par conséquent $T/\omega^2$ ne dépendent que de $i$, $\lambda$, $\theta$. Posons donc

$$\omega t = \tau, \quad T = \omega^2 S, \quad T_0 = \omega^2 S_0, \quad T_1 = \omega^2 S_1;$$

nos équations deviendront

$$\frac{d\omega}{\omega d\tau} = -\frac{dS}{d\lambda}, \qquad \frac{di}{d\tau} \sin i = \frac{dS}{d\theta} - \frac{dS}{d\lambda} \cos i,$$

$$\frac{d\lambda}{d\tau} = 2S + \frac{dS}{di} \operatorname{cotg} i, \qquad \frac{d\theta}{d\tau} = -\frac{1}{\sin i} \frac{dS}{di},$$

qui admettent comme il convient l'intégrale $\omega^2 S = \text{const.}$

Comme $S_0$ se réduit à $1/2$ nous pouvons écrire

$$\frac{di}{d\tau} = \frac{\mu}{\sin i} \frac{dS_1}{d\theta} - \mu \operatorname{cotg} i \frac{dS_1}{d\lambda}, \qquad \frac{d\theta}{d\tau} = -\frac{\mu}{\sin i} \frac{dS_1}{di},$$

$$\frac{d\lambda}{d\tau} = 1 + 2\mu S_1 + \mu \operatorname{cotg} i \frac{dS_1}{di}.$$

Comme $\mu$ est très petit, nous pouvons dans les seconds membres remplacer les inconnus par leurs valeurs approchées qui sont celles qui correspondent au cas

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de la sphère; c'est à dire des valeurs constantes pour $i$ et $\theta$, pour $\lambda$, la valeur approchée $\tau +$ const.

Au point de vue qui nous occupe, il sera d'ailleurs préférable de prendre $\lambda$ pour variable indépendante et d'écrire

$$\frac{di}{d\lambda} = \frac{\frac{\mu}{\sin i}\frac{dS_1}{d\theta} - \mu \operatorname{cotg} i \frac{dS_1}{d\lambda}}{1 + 2\mu S_1 + \mu \operatorname{cotg} i \frac{dS_1}{di}},$$

ou en négligeant $\mu^2$,

$$\frac{di}{d\lambda} = \frac{\mu}{\sin i}\frac{dS_1}{d\theta} - \mu \operatorname{cotg} i \frac{dS_1}{d\lambda}, \tag{1}$$

et de même

$$\frac{d\theta}{d\lambda} = -\frac{\mu}{\sin i}\frac{dS_1}{di}. \tag{2}$$

Dans les 2$^{ds}$ membres, on conservera alors $\lambda$ et on remplacera $i$ et $\theta$ par leurs valeurs approchées qui sont des constantes.

Rendons-nous compte de la signification géométrique de cette fonction $S_1$; comment d'abord ferons-nous correspondre les points du sphéroïde à ceux de la sphère? Le choix de cette correspondance est arbitraire dans une assez large mesure. Le plus simple est de faire correspondre les points où le plan tangent a même direction, c'est adopter ce qu'on appelle la représentation sphérique des surfaces. Dans ces conditions, si l'on veut me permettre le langage de la Géodésie, $u$ et $v$ représenteront, sur le sphéroïde la colatitude et la longitude *astronomiques*.

Considérons alors sur le sphéroïde deux points infiniment voisins $M$ et $M'$, et sur la sphère les deux points correspondants $M_1$ et $M'_1$. Soient $u$ et $v$ les coordonnées de $M$ (ou de $M_1$); soient $u + u'dt$, $v + v'dt$ celles de $M'$ (ou de $M'_1$). Soit $ds$ l'arc $MM'$ et $ds_1$ l'arc $M_1M'_1$, nous voyons que le rapport $ds/ds_1$ depend non-seulement de la position du point $M$, mais de l'orientation de l'arc $MM'$; lorsque cet arc appartient à une ligne de courbure, il n'est pas autre chose que le rayon de courbure principal correspondant.

Si $M$ et $M'$ représentent deux positions d'un point mobile sur le sphéroïde aux instants $t$ et $t + dt$, on aura pour l'énergie cinétique

$$T = \tfrac{1}{2}\left(\frac{ds}{dt}\right)^2.$$

Il est manifeste que les deux couples de points $MM'$ et $M_1M'_1$ correspondent aux mêmes valeurs de $u, v, u', v'$; mais ils ne correspondent pas aux mêmes valeurs de $u, v, U, V$, ni par conséquent de $i, \theta, \omega, \lambda$; car la relation qui lie $U$ et $V$ à $u, v, u', v'$, n'est pas la même pour $\mu = 0$ c'est à dire pour la sphère, et pour $\mu > 0$, c'est à dire pour le sphéroïde. Soit alors sur la sphère un couple

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de points infiniment voisins $M_1 M'_2$, choisis de façon à correspondre aux mêmes valeurs de $u$, $v$, $U$, $V$ (ou de $i$, $\theta$, $\omega$, $\lambda$) que le couple de points $MM'$ du sphéroïde.

Soient alors $u$ et $v$ les coordonnées du point $M_1$, $u + u'_0 dt$, $v + v'_0 dt$ celles du point $M'_2$; soit $ds_2$ l'arc $M_1 M'_2$. Alors $i$ et $\theta$ désigneront la colatitude et la longitude du pôle du grand cercle qui passe par $M_1$ et $M'_2$; $\lambda$ sera l'arc de ce grand cercle compris entre le point $M_1$ et l'équateur, et on aura enfin

$$M_1 M'_2 = ds_2 = \omega dt.$$

Soit

$$T_0 = \tfrac{1}{2}(u'^2 + v'^2 \sin^2 u),$$

$$T'_0 = \tfrac{1}{2}(u'^2_0 + v'^2_0 \sin^2 u),$$

on aura évidemment,

$$T_0 = \tfrac{1}{2}\left(\frac{ds_1}{dt}\right)^2, \qquad T'_0 = \tfrac{1}{2}\left(\frac{ds_2}{dt}\right)^2 = \frac{\omega^2}{2}.$$

Si nous désignons de même par $T'_1$ ce que devient $T_1$ quand on y remplace $u'$ et $v'$ par $u'_0$ et $v'_0$, la différence $T'_1 - T_1$ sera de l'ordre de $\mu$ de même que $dT'_1/du'_0 - dT_1/du'$. Comme $U$ et $V$ doivent avoir même valeur pour le couple $MM'$ sur la sphère et pour le couple $M_1 M'_1$ sur le sphéroïde, on aura

$$U = \frac{dT'_0}{du'_0} = \frac{dT}{du'},$$

$$V = \frac{dT'_0}{dv'_0} = \frac{dT}{dv'},$$

d'où en négligeant $\mu^2$

$$\frac{dT'_0}{du'_0} = \frac{dT_0}{du'} + \mu \frac{dT'_1}{du'_0},$$

$$\frac{dT'_0}{dv'_0} = \frac{dT_0}{dv'} + \mu \frac{dT'_1}{dv'_0}.$$

Je multiplie la 1ère par $u'_0$; la 2de par $v'_0$ et j'ajoute; j'obtiens alors par les propriétés des formes quadratiques,

$$\Sigma u'_0 \frac{dT'_0}{du'_0} = \Sigma u'_0 \frac{dT_0}{du'} + \mu \Sigma u'_0 \frac{dT'_1}{du'_0} = u' \frac{dT'_0}{du'_0} + 2\mu T'_1,$$

d'où

$$\Sigma (u'_0 - u') \frac{dT'_0}{du'_0} = 2\mu T'_1.$$

D'autre part, nous pouvons développer $T_0$ par la formule de Taylor suivant les puissances croissantes de $u' - u'_0$, $v' - v'_0$; nous avons alors en négligeant $\mu^2$, c'est à dire en négligeant les termes du 2d ordre,

$$T_0 = T'_0 + \Sigma \frac{dT'_0}{du'_0}(u' - u'_0) = \frac{\omega^2}{2} \cdot 2\mu T'_1,$$

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ou toujours en négligeant $\mu^2$,

$$T_0 = \frac{\omega^2}{2} - 2\mu T_1,$$

d'où

$$T = \frac{\omega^2}{2} - \mu T_1.$$

Des deux relations

$$T = T'_0 - \mu T_1, \qquad T = T_0 + \mu T_1$$

on tire d'abord

$$2T = T_0 + T'_0,$$

c'est à dire

$$2ds^2 = ds_1^2 + ds_2^2,$$

ou en négligeant $\mu^2$,

$$2ds = ds_1 + ds_2.$$

Ce que nous appelons $S_1$, c'est le rapport $(T - \omega^2)/\mu\omega^2$, c'est à dire

$$(ds^2 - ds_2^2)/\mu ds_2^2,$$

ou en négligeant les puissances supérieures de $\mu$

$$S_1 = \frac{ds_1^2 - ds^2}{\mu ds^2} = \frac{2}{\mu}\frac{ds_1 - ds}{ds} = \frac{2}{\mu}\frac{ds_1 - ds}{ds_1}.$$

Telle est la signification géométrique de $S_1$.

Reprenons maintenant les équations (1) et (2). Les 2$^{ds}$ membres sont des fonctions périodiques de $\lambda$. Pour que $i$ et $\theta$ soient également des fonctions périodiques de $\lambda$, c'est à dire pour que la géodésique soit fermée il faut et il suffit que la *valeur moyenne* de ce 2$^{d}$ membre, considéré comme fonction périodique de $\lambda$, soit nulle.

Quelle est cette valeur moyenne? Développons $S_1$ suivant les cosinus et les sinus des multiples de $\lambda$ et soit $R$ la valeur moyenne de $S_1$, c'est à dire l'ensemble des termes indépendants de $\lambda$. Il est clair que $dR/di$ sera la valeur moyenne de $ds_1/di$, et $dR/d\theta$ celle de $dS_1/d\theta$. Quant à la valeur moyenne de $dS_1/d\lambda$, elle sera nulle.

Pour que les valeurs moyennes de nos 2$^{ds}$ membres soient nulles, il faut donc et il suffit que

$$\frac{dQ}{di} = \frac{dR}{d\theta} = 0.$$

Les géodésiques fermées répondront donc aux maxima, aux minima, et aux minimax de la fonction $R$. A l'exemple des Anglais, j'appelle minimax les points où s'annulent les dérivées du 1$^{er}$ ordre d'une fonction de deux variables sans qu'on ait ni maximum, ni minimum.

Il faut d'abord rechercher la signification géométrique de la fonction $R$. Si nous donnons à $i$ et à $\theta$ des valeurs constantes et que nous fassions varier $\lambda$,

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comment variera le point $M$. Le point $M_1$ décrira sur la sphère un grand cercle, le plan tangent au point $M$ restera donc parallèle à une droite $D$ dont la direction est définie par la colatitude $i$ et la longitude $\theta$. Construisons un cylindre circonscrit au sphéroïde et dont les génératrices sont parallèles à $D$; il touche le sphéroïde suivant une courbe fermée $C$. *Le point $M$ restera sur cette courbe $C$.* Cette courbe $C$ sera ce qu'on pourrait appeler, dans le langage de la géodésie, un grand cercle astronomique.

On a alors

$$R = \frac{1}{2\pi}\int_0^{2\pi} S_1\, d\lambda.$$

Soient $M$, $M'$ deux points infiniment voisins de la courbe $C$, soit $M_1$, $M'_1$ les deux points correspondants de la sphère. Soit comme plus haut

$$MM' = ds, \qquad M_1M'_1 = ds_1.$$

On aura $ds_1 = d\lambda$ et

$$S_1 = \frac{2}{\mu}\frac{ds_1 - ds}{ds_1},$$

d'où

$$R = \frac{1}{2\pi}\int \frac{2}{\mu}\frac{ds_1 - ds}{d\lambda}\, d\lambda = \frac{1}{\pi\mu}(s_1 - s).$$

Ici $s_1 = 2\pi$ est la longueur totale du grand cercle de la sphère, et $s$ est la longueur totale de la courbe $C$.

Ainsi les maxima, minima et minimax de $R$ correspondent aux minima, maxima et minimax de la longueur totale des courbes $C$.

Considérons sur la sphère le grand cercle qui correspond à une courbe $C$, soient $P$ et $P'$ les deux pôles de ce grand cercle; la colatitude et la longitude du point $P$ sont $i$ et $\theta$; et $R$ est une fonction de $i$ et de $\theta$. Remarquons d'abord que cette fonction a même valeur aux deux points diamétralement opposés $P$ et $P'$. Construisons les courbes $R =$ const., c'est à dire les courbes qui joignent les différents points $P$ où la fonction $R$ a la même valeur. Nous aurons une série de courbes telles que par chaque point de la sphère passe une de ces courbes et une seule. Aux minima et aux maxima de $R$ correspondent des points isolés de ces courbes, aux minimax des points doubles à tangentes réelles. Un théorème d'Analysis Situs nous apprend que le nombre total des minima et des maxima, c'est à dire des points isolés, surpasse de 2 unités celui des minimax, c'est à dire des points doubles. Je me bornerai à renvoyer à ce sujet à mon mémoire sur les courbes définies par les équations différentielles (Journal de Liouville, sér. 3, tome 7).

Le nombre total des minima, maxima et minimax est donc un multiple de 4, plus 2. Mais à chaque courbe $C$ correspondent deux pôles $P$ et $P'$ diamétralement opposés, donc à chaque géodésique fermée correspondent *deux* minima,

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maxima ou minimax. *Le nombre total des géodésiques fermées est donc impair*, puisque c'est la moitié du nombre total des minima, maxima et minimax.

Il ne s'agit bien entendu ici que des géodésiques fermées qui subsistent *quelque petit que soit* $\mu$. Il peut y en avoir et il y en a une infinité d'autres. Une question reste à traiter. L'existence des géodésiques fermées dont nous venons de parler à été établie par un calcul approximatif puisque nous avons négligé $\mu^2$. Cette existence peut-elle être démontrée rigoureusement? Ou plutôt cette démonstration rigoureuse ne peut elle pas se tirer directement de notre calcul approximatif? Oui, cela peut se faire et par un procédé connu; on n'a qu'à appliquer les principes exposés dans le Chapitre 3 du Tome 1 de mon ouvrage, *Les Méthodes nouvelles de la Mécanique céleste.*

## § 4. *Le principe de continuité analytique.*

Soient $S$ et $S'$ deux surfaces quelconques satisfaisant aux conditions que nous nous sommes imposées dans les paragraphes précédents et que j' appellerai pour abréger les conditions $A$. *On peut toujours passer de l'une à l'autre d'une manière continue.* Je veux dire que l'on peut supposer une surface variable $\Sigma$ qui ayant pour équation générale

$$F(x, y, z, t) = 0,$$

où le 1er membre $F$ est une fonction *analytique* des coordonnées $x$, $y$, $z$ et d'un paramètre variable $t$ (qui restera analytique au moins pour tous les systèmes de valeurs de $x$, $y$, $z$, $t$, tels que 1°. $t$ soit compris entre 0 et 1; 2°. $x$, $y$, $z$ soient réels 3°. $F$ soit nul.) et de telle façon: 1°. Que cette surface $\Sigma$ se réduit à $S$ pour $t = 0$ et à $S'$ pour $t = 1$; 2°. Que quand $t$ varie d'une manière continue depuis 0 jusqu'à 1, cette surface $\Sigma$ ne cesse jamais de satisfaire aux conditions $A$.

Ainsi la surface $\Sigma$ quand $t$ variera d'une manière continue entre 0 et 1, restera convexe et analytique, ses rayons de courbure restent compris entre des limites déterminées. Considérons sur cette surface variable $\Sigma$ une ligne géodésique satisfaisant à une certaine condition, par exemple à celle d'être fermée. Soit $y = \phi(x)$, $z = \psi(x)$ l'équation de cette géodésique. Définissons cette géodésique, par les données initiales; appelons par exemple $y_0$ et $z_0$ les valeurs de $y$ et de $z$ pour $x = 0$, de sorte que

$$y_0 = \phi(0), \qquad z_0 = \psi(0).$$

Soient de même $y'_0$, $z'_0$ les valeurs initiales de $dy/dx$ et $dz/dx$, de telle façon que

$$y'_0 = \phi'(0), \qquad z'_0 = \psi'(0).$$

Ces données initiales $y_0, z_0, y'_0, z'_0$ suffisent pour distinguer notre géodésique de toutes les autres géodésiques tracées sur la surface $\Sigma$.

Cela posé, si nous écrivons que cette géodésique est fermée, nous obtiendrons

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entre $y_0$, $z_0$, $y'_0$, $z'_0$ certaines relations qui seront analytiques. Donc $y_0$, $z_0$, $y'_0$, $z'_0$ sont liés à $t$ par des relations analytiques; si nous considérons $y_0$, $z_0$, $y'_0$, $z'_0$, $t$ comme les coordonnées d'un point dans l'espace à 5 dimensions, ces relations définiront une certaine courbe analytique que j'appelle $C$. Nous envisagerons une branche de cette courbe analytique que j'appelle $B$. Il est aisé de définir cette branche et de la distinguer des autres par continuité. Dans le voisinage de $t = t_0$, les quantités $y_0$, $y'_0$, $z_0$, $z'_0$ sont susceptibles d'un ou de plusieurs développements et chacun de ces développements procède suivant les puissances de $(t - t_0)^{1/q}$; la branche $B$ se composera de tous les points réels de la courbe $C$ qui correspondent à l'un de ces développements; quand le développement ne sera plus valable, on la poursuivra par le procédé de la continuation analytique. Si $q$ est impair, cette branche comprendra un point et un seul, tant pour $t = t_0 - e$ que pour $t = t_0 + e$; si $q$ est pair, elle comprendra deux points pour $t = t_0 - e$ et zéro pour $t = t_0 + e$; ou inversement.

Nous dirons que les différentes géodésiques fermées qui appartiennent aux différents points d'une même branche $B$ font partie d'une même *série continue.*

Considérons les différents géodésiques fermées d'une même série continue qui appartiennent à une même surface $\Sigma$. Elles correspondent aux points d'intersection de la branche $B$ avec le plan $t = t_0$. Le nombre de ces points ne peut varier que quand $t_0$ prend une valeur telle que le nombre que nous appelions plus haut $q$ soit pair. Dans ce cas, ce nombre varie de deux unités. D'où cette conséquence:

*Le nombre des géodésiques d'une surface $\Sigma$ qui font partie de une, deux ou plusieurs séries continues déterminées est constamment pair ou constamment impair.*

Examinons maintenant les différentes sortes de courbes fermées que l'on peut tracer sur une surface convexe. Ces courbes au point de vue de l'Analysis Situs se répartiront en une infinité de *types* caractérisés par le nombre et la disposition des points doubles.

Considérons une série continue de géodésiques fermées; elles pourront appartenir à différents types; comment pourront elles passer d'une type à l'autre? D'abord *elles ne pourront jamais avoir de point de rebroussement;* car les géodésiques des surfaces satisfaisant aux conditions $A$ n'ont aucun point singulier. Ensuite, il ne pourra jamais arriver que deux branches de cette géodésique viennent à se toucher. Car deux géodésiques ne peuvent se toucher sans se confondre, attendu qu'un point et la tangente en ce point sur une surface convexe détermine complètement une géodésique.

Il résulte de là que *dans une même série continue, le nombre des points doubles d'une géodésique fermée demeure constant.* Ce nombre en effet, pourrait varier de deux manières: 1°. Si un point double à tangentes réelles devenait un point isolé à tangentes imaginaires, mais alors il faudrait passer par un

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point de rebroussement. 2°. Si deux points doubles réels devenaient imaginaires conjugués, mais alors il faudrait qu'au moment du passage, deux branches de courbe vinssent à se toucher.

Nous venons de voir que tout cela est impossible; un seul point reste à éclaircir. J'ai dit que deux géodésiques ne peuvent se toucher *à moins de se confondre*. Ne serait il pas possible que pour certaines valeurs de $t$, la géodésique fermée que nous envisageons *dégénère* parce que deux de ses parties se confondraient entre elles? Il est clair que cela est possible. Une courbe fermée d'un certain type, peut se réduire en dégénérant à une courbe fermée d'un autre type parcourue plusieurs fois. C'est ainsi qu'un limaçon de Pascal, peut, à la limite, se réduire à un cercle parcouru deux fois. Il n'y a pas de raison pour que cela n'arrive pas pour les géodésiques et en fait cela arrive.

Si une courbe fermée du type $T$, peut dégénérer en une courbe fermée du type $T'$ parcourue plusieurs fois, nous dirons que le type $T'$ est *subordonné* au type $T$.

Supposons donc que nous considérions une série continue de géodésiques fermées correspondant à la branche de courbe $B$ définie plus haut; que $M$, $M'$, $M''$ soient 3 points infiniment voisins de cette branche, que $M'$ soit entre $M$ et $M''$, qu'en $M$ la géodésique appartienne an type $T$; qu'en $M'$ elle se réduise à une courbe fermée du type $T'$ parcourue plusieurs fois; qu'en $M''$ elle appartienne an type $T''$, qui pourra être d'ailleurs identique à $T$. Le type $T'$ est ainsi subordonné tant à $T$ qu'à $T''$. Quand on passera de $M$ en $M''$ on peut se demander si le nombre des points doubles a varié.

Soit $t_0$ la valeur de $t$ qui correspond an point $M'$; nous poserons $\tau = (t - t_0)^{1/q}$ en choisissant l'entier $q$ de façon que $y_0$, $z_0$, $y'_0$, $z'_0$ soient développables suivant les puissances de $\tau$. Considérons l'un des points doubles de notre géodésique, quand $t$ tend vers $t_0$, ce point se rapprochera d'un certain point $D$ de la géodésique singulière du type $T'$. Prenons ce point pour origine et choisissons les axes de façon que le plan des $yz$ ne soit tangent à aucune des branches de géodésique qui passent par ce point. Dans ces conditions, nous pourrons mettre les équations d'une quelconque de nos branches de géodésique sous la forme suivante; $y$ et $z$ seront développables suivant les puissances de $x$ et de $\tau$. Considérons deux de ces branches et soient

$$y = \Sigma \tau^m \phi_m(x), \qquad z = \Sigma \tau^m \psi_m(x),$$
$$y = \Sigma \tau^m \phi'_m(x), \qquad z = \Sigma \tau^m \psi'_m(x),$$

leurs équations. Je suppose que ces deux branches se confondent pour $\tau = 0$, on a donc

$$\phi_0(x) = \phi'_0(x), \qquad \psi_0(x) = \psi'_0(x).$$

Cherchons les points doubles, c'est à dire les points d'intersection de nos deux branches. Ils nous seront donnés par l'équation

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$$\Sigma\tau^m[\phi_m(x)-\phi'_m(x)]=0$$

(laquelle entraîne d'ailleurs l'autre équation

$$\Sigma\tau^m[\psi_m(x)-\psi'_m(x)]=0$$

en tenant compte de l'équation de la surface $\Sigma$).

Soit $\phi_p(x)-\phi'_p(x)$ le premier des coefficients $\phi_m(x)-\phi'_m(x)$ qui ne s'annule pas identiquement; quand $\tau$ tendra vers zero, les points doubles tendront vers des limites, qui seront les solutions de l'équation

$$\phi_p(x)-\phi'_p(x)=0.$$

Par l'hypothèse le point $x=0$ est une de ces limites; on a donc

$$\phi_p(0)-\phi'_p(0)=0.$$

Nos points doubles sont alors donnés par l'équation

$$\Sigma\tau^{m-p}[\phi_m(x)-\phi'_m(x)]=0$$

dont le 1er membre, développable suivant les puissances de $x$ et de $\tau$, s'annule pour $x=\tau=0$; on peut en tirer $x$ développé suivant les puissances de $\tau$ ce qui prouve que $x$ reste réel aussi bien pour $\tau>0$ que pour $\tau<0$, et par conséquent qu'un point double réel, par exemple au point $M$, ne peut devenir imaginaire au point $M'$.

Il n'y aurait d'exception que si la dérivée

$$\frac{d}{dx}[\phi_p(x)-\phi'_p(x)]$$

s'annulait pour $x=0$. Or cela n'arrive pas; cela voudrait dire que deux branches se coupent en deux points infiniment voisins, c'est à dire qu'un point se confond avec son propre foyer, ou peut en devenir aussi voisin que l'on veut en comptant les distances sur la géodésique. Or nous avons vu plus haut que la distance d'un point à son foyer est toujours au moins égale à $r/2L_2$.

Donc même dans ce cas, le nombre des points doubles ne peut varier.

La disposition des points doubles peut elle varier? Oui, mais uniquement de la manière suivante: supposons trois branches de courbe qui se coupent en 3 points doubles de façon à former un petit triangle; à la limite ce petit triangle se réduit à un point et les 3 points doubles se confondent en un seul point triple; ensuite les 3 points doubles se séparent de nouveau, et les trois branches forment de nouveau un triangle, mais dont la disposition est différente. Si la branche 1 parcourue dans un certain sens rencontrait dans la 1ère disposition, la branche 2 d'abord, puis le branche 3; dans la 2de disposition, elle rencontrera la branche 3 d'abord puis la branche 2.

Il est aisé de voir que si les branches de courbe au lieu de déterminer un triangle



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déterminent un polygone de plus de 3 côtés; la disposition des points doubles ne pourra changer, car il faudrait que deux de ces branches devinssent tangentes. Lorsqu'on pourra passer de cette manière du type $T$ au type $T'$, nous dirons que ces deux *types sont équivalents.* Il y a des types qui ne sont équivalents à aucun autre, par exemple le type sans point double. Une courbe fermée à points doubles partagera la surface convexe en un certain nombre de polygones curvilignes dont quelques-uns pourront être des biangles ou des uniangles. Si aucun de ces polygones n'est un triangle, le procédé précédent ne pourra être appliqué et le type ne sera équivalent à aucun autre. Considérons maintenant deux séries continues de géodésiques; soient $G$ la géodésique de la 1ère serie, $G'$ celle de la 2de; le nombre des points d'intersection de $G$ et de $G'$ ne pourrait varier que si ces deux géodésiques devenaient tangentes l'une à l'autre, ce qui est impossible.

*Le nombre des points d'intersection de deux géodésiques appartenant à deux séries continues determinées est invariable.*

Supposons que pour $t$ voisin de $t_0$, on trouve que le nombre $q$, dont nous avons parlé plus haut et qui figure dans l'expression $(t-t_0)^{1/q}$ est pair pour la 1ère série continue, et égal à 1 pour la 2de série. Il arrivera alors que pour $t < t_0$, nous aurons 2 géodésiques $G$ et $G''$ de la 1ère serie, et que pour $t > t_0$ nous n'en aurons plus. Au contraire nous aurons toujours une géodésique $G'$ et une seule de la 2de série, quel que soit $t$. Nous pouvons poser

$$(t-t_0)^{1/q} = \tau$$

et faire varier $\tau$ d'une manière continue; alors dans la 1ère série nous passerons de la géodésique $G$ à la géodésique $G''$; et dans la 2de série nous partirons de la géodésique $G'$ et reviendrons finalement à cette même géodésique $G'$.

*Donc le nombre des points d'intersection de $G$ et de $G'$ est le même que celui des points d'intersection de $G$ et de $G''$.*

Quant à la disposition relative des points doubles de $G$ et de $G'$ et de leurs points d'intersection, elle ne pourra varier que de la façon que j'ai dite plus haut à propos de la disposition des points doubles d'une seule géodésique.

Comme première application de ce principe; étudions les géodésiques fermées sans point double; ces géodésiques formeront évidemment une ou plusieurs séries continues. D'après ce qui précède ces séries continues ne pourront contenir que des géodésiques fermées sans point double.

Donc le nombre total des géodésiques fermées sans point double est toujours pair ou toujours impair. Or nous avons vu au § précédent que pour un sphéroïde ce nombre est impair. De plus on peut passer d'un sphéroïde à une surface convexe quelconque d'une manière continue.

*Donc sur une surface convexe quelconque, il y a toujours au moins une géodésique fermée sans point double et il y en a toujours un nombre impair.*

Par exemple pour un ellipsoïde nous avons les trois sections principales par les plans de symétrie.

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## § 5. *Stabilité et instabilité.*

Etudions maintenant les géodésiques peu différentes d'une géodésique fermée (avec ou sans point double). Pour cela nous nous servirons du système de coordonnées suivant. Soit $G_0$ la géodésique fermée, $O$ un point fixe de cette géodésique; prenons sur $G_0$ à partir du point $O$, une longueur $OP = u$; au point $P$ menons une géodésique $G_1$ normale à $G_0$; sur $G_1$ à partir de $P$ prenons une longueur $PM = v$; les coordonnées du point $M$ seront $u$ et $v$. L'élément d'arc sera donné par

$$ds^2 = dv^2 + h^2 du^2.$$

On voit que $h = 1$ et $dh/dv = 0$ pour $v = 0$, et quel que soit $u$. Les équations d'une géodésique seront

$$\frac{d}{du}\frac{v'}{\sqrt{v'^2 + h^2}} = h\frac{dh}{dv},$$

$v'$ étant la dérivée de $v$ par rapport à $u$; ou en supposant $v$ très petit pour chercher les géodésiques très voisines de $G_0$,

$$v'' = h''v, \tag{1}$$

où $h''$ est la valeur de $d^2h/dv^2$ pour $v = 0$. Alors $-h''$, qui represente la courbure de la surface au point $(u, 0)$, est une fonction de $u$ qui est d'ailleurs périodique puisque $G_0$ est fermée.

L'équation (1) est une équation linéaire à coefficients périodiques; elle admet deux solutions remarquables de la forme suivante:

$$v = e^{\alpha u}\phi(u), \qquad v = e^{-\alpha u}\psi(u), \tag{2}$$

$\phi(u)$ et $\psi(u)$ étant périodiques en $u$.

Quatre cas peuvent se présenter:

1°. Ou bien $\alpha$ est purement imaginaire et les deux solutions (2) imaginaires conjugués. On dit alors que $G_0$ est *stable.*

2°. Ou bien $\alpha$ est réel et différent de zéro et les deux solutions (2) réelles. On dit alors que $G_0$ est *instable.*

3°. Ou bien on a $\alpha = \alpha' + i\pi/U$, $\alpha'$ étant réel, et $U$ étant la longueur totale de $G_0$; nos deux solutions peuvent alors se mettre sous la forme $v = e^{\alpha' u}\phi_1(u)$, $v = e^{-\alpha' u}\psi_1(u)$, les fonctions $\phi_1$ et $\psi_1$ étant réelles et telles que

$$\phi_1(u + U) = -\phi_1(u), \qquad \psi_1(u + U) = -\psi_1(u).$$

On dit encore dans ce cas que $G_0$ est *instable.*

4°. Ou bien $\alpha$ est multiple de $i\pi/U$; c'est un cas limite, qui ne se présentera pas en général et que nous laisserons de côté pour le moment; dans ce cas les expressions (2) peuvent prendre une forme dégénérescente où figure $\log u$; ainsi qu'il arrive dans le cas des racines multiples pour les équations linéaires à coefficients constants.

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Plaçons nous d'abord dans le 1er cas, et soient $\rho$ et $\theta$ le module et l'argument de la 1ère solution (2); alors l'equation (1) admettra comme solutions particulières,

$$v = \rho e^{i\theta}, \qquad v = \rho e^{-i\theta},$$

et comme solution générale

$$v = A\rho \sin(\theta - \beta), \tag{3}$$

où $\beta$ et $A$ sont des constantes d'intégration.

Soit en particulier

$$v_1 = \rho \sin\theta, \qquad v_2 = \rho\cos\theta,$$

on aura

$$\operatorname{tg}\theta = \frac{v_1}{v_2} = B\int\frac{du}{v_2^2}, \tag{4}$$

$B$ étant une constante. La formule (4) où tout est d'ailleurs réel nous montre que tg $\theta$ varie toujours dans le même sens, par exemple toujours en croissant; car si tg $\theta$ allait toujours en décroissant, il nous suffirait d'intervertir le rôle des deux solutions (2). Donc $\theta$ est une fonction croissante de $u$.

Supposons qu'on ait pris l'unité de longueur de telle façon que la longueur totale de $G_0$ soit égale à $2\pi$. Qu'arrivera-t-il quand $u$ augmentera de $2\pi$? Il arrivera que $\rho$ ne changera pas et que $\theta$ augmentera de $2\alpha\pi/i$ (ce qui nous montre d'ailleurs que $\alpha/i$ doit être positif si l'on veut que $\theta$ soit croissant).

Cela posé, nous appellerons *argument réduit* d'un point sur la géodésique fermée $G_0$ la quantité

$$\frac{i\theta}{\alpha}.$$

Si nous avons une unité de longueur quelconque et que $U$ soit la longueur totale de $G_0$, l'argument réduit sera

$$\frac{2i\pi\theta}{\alpha U}.$$

Il résulte de là que l'argument réduit augmente constamment de 0 à $2\pi$ quand on parcourt la géodésique tout entière. Cet argument réduit peut donc servir à définir la position d'un point sur $G_0$.

Revenons à la solution générale (3); elle représente une géodésique $G$ très peu différente de $G_0$; les points d'intersection successifs de $G$ et $G_0$ seront donnés par la formule

$$\theta = \beta + K\pi,$$

où $K$ est un entier quelconque.

Le $K^{\text{ème}}$ foyer du point $\theta = \beta$, sera donc le point $\theta = \beta + K\pi$; d'où cette conséquence: *Les arguments réduits d'un point et de son $K^{\text{ème}}$ foyer diffèrent d'une quantité constante et positive*

$$\frac{Ki\pi}{\alpha}\,\frac{2\pi}{U}.$$

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L'analyse qui précède ne diffère pas de celle du No. 347 du tome 3 de mon ouvrage sur les méthodes nouvelles de la mécanique céleste. Au No. 349 j'ai envisagé les solutions instables et j'ai montré qu'elles se répartissent en trois catégories.

1°. Ou bien $\alpha$ est réel et $\phi(u)$ ne s'annule jamais (solutions instables de la 1ère catégorie). On a alors

$$e^{-2\alpha u}\frac{\psi(u)}{\phi(u)} = B\int\frac{du}{e^{-2\alpha u}\phi^2(u)},$$

ce qui montre que le 1er membre qui est toujours fini et croissant ne peut s'annuler qu'une fois; comme $e^{-2\alpha u}$ ne s'annule pas, il en résulte que $\psi(u)$ ne peut s'annuler qu'une fois; et comme $\psi(u)$ est périodique, $\psi(u)$ ne s'annule jamais. D'ailleurs la solution générale de (1) sera encore de le forme

$$v = e^{\alpha u}\phi(u)B\int\frac{du}{e^{-2\alpha u}\phi^2(u)}, \tag{5}$$

ce qui montre que $v$ ne peut s'annuler qu'une fois: *Aucun point de $G_0$ n'a donc de foyer.*

2°. Ou bien $\alpha$ est réel et $\phi(u)$ s'annule (solutions instable, de la 2de catégorie).

La formule (5) nous montre alors que (l'intégrale du 2d membre étant toujours croissante et devenant infini quand $\phi$ s'annule) entre deux zéros consécutifs de $\phi(u)$, il y a un zéro de $v$ et un seul.

En particulier entre deux zéros de $\phi(u)$, il y a un zéro de $\psi(u)$ et un seul. Si donc $u_0$, $u_1$, $u_2$, etc. sont les zéros successifs de $\phi(u)$, les valeurs de $\psi(u_0)$, $\psi(u_1)$, $\psi(u_2)$, $\cdots$ sont alternativement positifs et négatifs; mais la fonction $\psi(u)$ étant périodique, doit revenir à la même valeur quand on fait tout le tour de $G_0$. Donc *le nombre des zéros de $\phi(u)$ est toujours pair.* Soient donc

$$u_0, u_1, u_2, \cdots, u_n,$$

les zéros successifs de $\phi(u)$ (pair-$n$). Tout point situé entre $u_0$ et $u_1$ aura son foyer entre $u_1$ et $u_2$ et quand l'argument d'un point croîtra de $u_0$ à $u_1$ l'argument de son foyer croîtra *constamment* de $u_1$ à $u_2$; car s'il cessait de croître, c'est qu'un même point aurait plusieurs foyers, ce qui est impossible.

Supposons que $\alpha$ soit positif; alors pour $u$ positif et très grand $v$ sera sensiblement proportionnel à $e^{\alpha u}\phi(u)$; pour $u$ négatif et très grand $v$ sera sensiblement proportionnel à $e^{-\alpha u}\psi(u)$.

Les foyers d'ordre positif et très grand seront donc très voisins de l'un des zéros de $\phi(u)$.

Les foyers d'ordre négatif et très grand seront très voisins de l'un des zéros de $\psi(u)$. Soient alors

$$u_0', u_1', \cdots, u_n'$$

les zéros successifs de $\psi(u)$; $u_0'$ étant compris entre $u_0$ et $u_1$.

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Soient $M_i$ et $M'_i$ des points de $G_0$ qui correspondent respectivement à $u_i$ et $u'_i$. Soit $F_0$ un point quelconque situé entre $M_0$ et $M'_0$, $F_k$ son $K^{\text{ème}}$ foyer. Alors pour $k < n$, $F_k$ sera compris entre $M_k$ et $M'_k$; on voit ensuite que $F_n$ est compris entre $M_0$ et $M'_0$; je dis que c'est entre $M_0$ et $F_0$; si en effet $F_n$ était compris entre $M'_0$ et $F_0$, $F_{2n}$ serait compris entre $M'_0$ et $F_n$ et par conséquent entre $M'_0$ et $F_0$; plus généralement $F_{kn}$ serait compris entre $M'_0$ et $F_0$. Donc $F_{kn}$ ne pourrait tendre vers l'un des zéros de $\phi(u)$ quand $k$ tend vers $+\infty$. En conséquence les points

$$F_n, F_{2n}, \cdots, F_{kn}, \cdots$$

sont entre $M_0$ et $F_0$ et quand $K$ croît constamment et indéfiniment, ils se rapprochent constamment et indéfiniment de $M_0$. Au contraire les points

$$F_{-n}, F_{-2n}, \cdots, F_{-kn}$$

sont entre $M'_0$ et $F_0$ et quand $K$ croît constamment et indéfiniment, ils se rapprochent constamment et indéfiniment de $M'_0$. Les zéros de $\phi(u)$ et ceux de $\psi(u)$ peuvent en conséquence recevoir le nom de *foyers limites*. Telle est la loi de la distribution des foyers pour les géodésiques fermées instables de la $2^{\text{de}}$ catégorie.

3°. Ou bien enfin $\alpha$ est de la forme $\alpha' + i\pi/U$, $\alpha$ étant réel (solutions instables de la $3^{\text{ème}}$ catégorie).

D'après ce que nous avons vu au No. 359 (loc. cit.), ces solutions jouissent des mêmes propriétés que celles de la $2^{\text{de}}$ catégorie. On a d'après ce qu'on a vu plus haut deux solutions de la forme:

$$v = e^{\alpha' u}\phi_1(u), \qquad v = e^{-\alpha' u}\psi_1(u),$$

avec

$$\phi_1(u+U) = -\phi_1(u), \qquad \psi_1(u+U) = -\psi_1(u),$$

et la solution générale s'écrit:

$$v = e^{\alpha' u}\phi_1(u) B \int \frac{du}{e^{-2\alpha' u}\phi_1^2(u)}. \tag{5 bis}$$

Nous voyons que $\phi_1(u)$ doit s'annuler, puisque cette fonction change de signe, qu'entre deux zéros de $\phi_1(u)$, il y a un zéro de $\psi_1(u)$ et un seul.

Ce que nous avons dit de la distribution des foyers successifs et de leurs relations avec les zéros de $\phi$ et de $\psi$ (dont le rôle est joué ici par les zéros de $\phi_1$ et de $\psi_1$) subsiste sans changement; la seule différence c'est que *le nombre des zéros de $\phi_1(u)$ au lieu d'être toujours pair, est toujours impair.*

En effet la fonction $\psi_1(u)$ doit, non pas revenir à la même valeur, mais au contraire changer de signe quand on fait tout le tour de $G_0$.

Dans le cas des surfaces convexes tout point a un foyer dont la distance à ce point est inférieure à une limite donnée; il n'y a donc pas de géodésiques fermées instables de la $1^{\text{ère}}$ catégorie; il y en a au contraire pour les surfaces à courbures opposées.

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Au No. 355 (loc. cit.) j'ai montré que la condition nécessaire et suffisante pour qu'une trajectoire fermée corresponde à un minimum de l'action, c'est qu'elle soit instable de la 1ère catégorie. Il résulte de là que sur une surface convexe, il n'y a pas de géodésique fermée qui soit plus courte que toutes les courbes fermées infiniment voisines. Cela est d'ailleurs aisé à établir directement.

Considérons un ellipsoïde et ses trois ellipses principales; la plus grande et la plus petite sont des géodésiques fermées stables; la moyenne est une géodésique fermée instable de la 2de catégorie. Sur un hyperboloïde à 1 nappe, l'ellipse de gorge est une géodésique fermée instable de la 1ère catégorie.

Il est aisé de voir (loc. cit., No. 378) que si deux géodésiques fermées viennent à se confondre, pour disparaitre ensuite en devenant imaginaires (lorsqu'on fait varier d'une manière continue le paramètre que nous appelions $t$ dans le § précédent), l'une d'elles est stable et l'autre instable. D'où il resulte que dans une même série continue, l'excès du nombre des géodésiques stables sur celui des géodésiques instables est constant. Or nous avons vu que les géodésiques fermées sans point double forment une ou plusieurs séries continues, et d'autre part que l'on peut toujours passer d'une manière continue d'une surface convexe quelconque à une autre.

Donc *si sur une surface convexe quelconque on envisage toutes les géodésiques fermées sans point double l'excès du nombre de celles qui sont stables sur le nombre de celles qui sont instables est constant*; il est donc le même que pour l'ellipsoïde, *il est donc égal à* 1.

*Sur une surface convexe, il y a donc toujours au moins une géodésique fermée stable sans point double.*

Remarquons que $\alpha$ que l'on appelle *l'exposant caractéristique* de $G_0$ est par ce qui précède entièrement déterminé. On pourrait croire le contraire, car si $U$ est la longueur totale de $G_0$, on a

$$e^{\alpha u}\phi(u) = e^{\alpha + \frac{2i\pi u}{U}}\left[\phi(u)e^{-\frac{2i\pi u}{U}}\right],$$

où $\phi(u)e^{-2i\pi u/U}$ est comme $\phi(u)$ une fonction périodique. Il semble donc que l'on puisse remplacer $\alpha$ par $\alpha + 2i\pi/U$ ou par $\alpha + 2ki\pi/U$ ($k$ entier positif ou négatif). Mais dans le cas des géodésiques instables, il n'y a qu'une des quantités $\alpha + 2ki\pi/U$ qui soit réelle et c'est celle qu'il convient d'adopter.

Dans le cas des géodésique stables, $\alpha$ est entièrement déterminé par la condition que l'argument $\theta$ augmente de $\alpha U/i$ quand on fait le tour de la géodésique $G_0$. Autre remarque: considérons une géodésique instable de la 2de ou de la 3ème catégories; elle fera partie d'une série continue au sens du § 4. Je dis que le nombre des zéros de $\phi(u)$, d'où dépend comme nous venons de le voir, la loi de distribution des foyers, demeurera constant; car il ne pourrait varier que si $\phi(u)$ et $\phi'(u)$ s'annulaient en même temps, ce qui ne peut avoir lieu. Une

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série continue peut se partager en *sections*, de telle façon que dans une section par exemple toutes les géodésiques soient stables, dans la suivante toutes instables, dans la suivante toutes stables, et ainsi de suite. Aux points où l'on passe d'une section à l'autre, l'exposant caractéristique est multiple de $i\pi/U$. Eh bien *dans une même section d'une même série continue, le nombre des foyers limites est invariable.*

Dans le cas des géodésiques instables, il y a plusieurs manières de définir l'exposant $\alpha$; nous pouvons, comme nous l'avons fait jusqu'ici, supposer $\alpha$ réel pour celles de la 1ère et de la 2de catégorie, tandis que pour celles de la 3ème catégorie, on supposera la partie imaginaire de $\alpha$ égale à $i\pi/U$. Mais on peut également adopter une autre solution.

On peut supposer que la partie imaginaire de $\alpha$ est égale à $ni\pi/U$, $n$ étant le nombre des zéros de $\phi(u)$ (ou de $\phi_1$); l'exemple qui va suivre nous fera comprendre les raisons qui pourraient justifier cette convention. Envisageons la surface

$$x^2 + y^2 + \frac{q^2z^2}{4} - \frac{q_1z^2x}{4} = 1,$$

$q$ et $q_1$ étant deux constantes.

Cette surface est convexe et fermée si $q_1$ est suffisamment petit; car elle diffère très peu de l'ellipsoïde de révolution

$$x^2 + y^2 + \frac{q^2z^2}{4} = 1.$$

Elle admet une géodésique fermée qui est le cercle

$$z = 0, \qquad x^2 + y^2 = 1.$$

Si pour ce cercle, nous exprimons $x$ et $y$ en fonctions de l'arc $u$, nous trouvons

$$x = \cos u, \qquad y = \sin u.$$

L'équation des géodésiques infiniment peu différentes de ce cercle sera

$$v'' + \tfrac{1}{4}(q^2 - q_1 \cos u)v = 0.$$

Nous reconnaissons une équation qui a fait l'objet de travaux nombreux (loc. cit., Tome 2, Chapitre 17), les notations sont les mêmes que dans ce chapitre 17 à la condition de changer $x$ en $v$ et $2t$ en $u$. Le rôle de l'exposant $\alpha$ est le même que celui que jouait dans l'ouvrage cité la quantité

$$\frac{ih}{2}.$$

Si donc $q_1$ est petit nous avons sensiblement

$$\alpha = \frac{iq}{2}.$$

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D'ailleurs $\alpha$ est imaginaire et la géodésique stable, à moins que $q$ ne soit voisin d'un entier.

Si $q$ est voisin d'un entier, $n$ de telle façon que sur la figure de la page 243 (loc. cit.) le point de coordonnées $q$, $q_1$ se trouve dans une des régions couvertes de hachures, la géodésique est instable. D'après le No. 352 (loc. cit., Tome 3) elle est toujours de la 2$^{de}$ ou de la 3$^{ème}$ catégorie).

Quel est le nombre des zéros de $\phi(u)$ ou de $\phi_1(u)$? Il suffit de remarquer que sur l'une des courbes qui limitent les régions couvertes de hachures (loc. cit., No. 352) la fonction $\phi$ ou $\phi_1$ est impaire et se réduit sensiblement (à cause de la petitesse de $q_1$) à $\sin qu/2$, ou à $\sin nu/2$ puisque $q$ est très voisin de $n$.

Le nombre des zéros de $\phi(u)$ ou de $\phi_1(u)$ est donc $n$ et celui des foyers limites est $2n$, dont $n$ pour les zéros de $\phi(u)$ et $n$ pour les zéros de $\psi(u)$. Cela nous montre:

1°. qu'il existe des surfaces qui admettent des géodésiques fermées instables de la 2$^{de}$ et de la 3$^{ème}$ catégories, le nombre des foyers limites pouvant être quelconque.

2°. que si l'on veut que $\alpha$ soit une fonction continue, il faut lui attribuer pour partie imaginaire $ni\pi / U$, $2n$ étant le nombre des foyers limites.

## § 6. *De quelques types de géodésiques fermées.*

Soit $G_0$ une géodésique fermée stable dont la longueur totale est $U$ et qui appartient à une série continue. Soit $\alpha$ son exposant caractéristique. Il est clair que $U$ et $\alpha$ sont des fonctions continues du paramètre que nous avons appelé $t$ au § 1 et qui correspond à cette série continue.

Supposons que pour une certaine valeur de $t$, pour $t = t_0$ par exemple, le produit $\alpha U$ soit commensurable avec $i\pi$ par, exemple égal à

$$2\frac{mi\pi}{n},$$

$m$ et $n$ étant deux entiers premiers entre eux. L'argument réduit d'un point et de son $K^{ème}$ foyer différeront de

$$\pi\frac{nK}{m},$$

ce qui montre que tout point de notre géodésique $G_0$ coïncidera avec son $2m^{ème}$ foyer. Si l'on considère un point $M$ de $G_0$, et la géodésique issue du point $M$ et infiniment voisine de $G_0$ on voit qu'elle viendra repasser en $M$, aux infiniment petits près d'ordre supérieur après avoir recoupé $G_0$ $2m$ fois (le point $M$ compris). Elle aura ainsi fait $n$ fois le tour de la géodésique $G_0$. Les principes exposés dans le Chapitre 28 (loc. cit.) nous permettent d'interpréter ce résultat. Il existe outre la géodésique $G_0$, deux autres géodésiques fermées $G_1$ et $G_1'$ jouissant des propriétés suivantes:

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1°. Pour $t < t_0$ (par exemple) ces deux géodésiques n'existent pas ou plutôt sont imaginaires; 2°. Pour $t > t_0$ elles sont toutes deux réelles; 3°. Pour $t = t_0$ elles se confondent entre elles et se réduisent l'une et l'autre à la géodésique $G_0$ parcourue $n$ fois; 4°. Elles appartiennent à une même série continue; 5°. l'une d'elles est stable et l'autre instable.

La relation de $G_1$ et $G'_1$ avec $G_0$, est la même que celle qu'on rencontre en mécanique céleste, entre les solutions périodiques de la 2de et de la 3ème sorte et celles de la 1ère sorte, ou entre les solutions du 2d genre et celles du 1er genre. La série continue $S_1$ à la quelle appartiennent $G_1$ et $G'_1$ peut donc être regardée comme *engendrée* par la série $S_0$ à la quelle appartient $G_0$; comme le produit $\alpha U$, en variant d'une manière continue, passera une infinité de fois par des valeurs commensurables avec $i\pi$, la série $S_0$ engendrera une infinité d'autres séries continues. Et comme l'une des deux géodésiques $G_1$ et $G'_1$ est stable, le série $S_1$ va à son tour engendrer de la même manière une infinité d'autres séries.

Nous avons vu qu'il y a toujours une géodésique fermée stable, sans point double. Appelons-la $G_0$. Supposons qu'elle engendre comme nous venons de l'expliquer deux autres géodésiques fermées $G_1$ et $G'_1$; soit $mi\pi/n$ la valeur correspondante de $\alpha U$. Pour des valeurs de $t$ très voisines de $t_0$, ces deux géodésiques différeront très-peu de $G_0$ et nous obtiendrons leur équation approximative en reprenant l'analyse du § précédent; nous trouverons alors (en reprenant les équations (3) du § 2 précédent)

$$v = A\rho \sin(\theta - \beta) \quad \text{pour } G_1,$$

$$v = A'\rho \sin(\theta - \beta') \quad \text{pour } G'_1,$$

$A$, $\beta$, $A'$, $\beta'$ étant des constantes.

Les points d'intersection de $G_0$ et de $G_1$ par exemple nous sont donnés par la formule

$$\theta = \beta + K\pi \quad (K \text{ entier}),$$

et les valeurs correspondantes de l'argument réduit sont

$$\frac{2i\pi\beta}{\alpha U} + \frac{K\pi \cdot 2i\pi}{\alpha U} = \frac{n\beta}{m} + \frac{n}{m} K\pi.$$

Combien l'expression

$$\frac{nK}{m},$$

peut-elle prendre de valeurs distinctes, deux valeurs n'étant pas regardées comme distinctes lorsqu'elles diffèrent d'un entier pair? Evidemment $2m$; donc *le nombre des points d'intersection de $G_0$ avec $G_1$, et par conséquent avec une géodésique quelconque de le série $S_1$ est égal à $2m$.*

Cherchons maintenant le nombre des points doubles. Supposons la géodésique fermée $G_1$ partagée en $2m$ arcs par ses $2m$ points d'intersection avec $G_0$; ces

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arcs seront alternativement d'un côté et de l'autre de $G_0$; d'où il suit qu'un arc de rang pair ne peut pas couper un arc de rang impair, mais que deux arcs de rang pair, ou deux arcs de rang impair peuvent se couper.

J'observe ensuite que $v =$ partie réelle $e^{au}\phi(u)$ est une fonction uniforme de $u$ de sorte que les points doubles correspondront aux valeurs de $u$ pour lesquelles on aura

$$(h) \qquad v(u) = v(u + 2K\pi),$$

$K$ étant entier, et qu'il n'y en aura pas d'autre. Nons avons d'ailleurs

$$\frac{v(u + 2K\pi)}{v(u)} = B\int \frac{du}{v^2(u)},$$

car $v(u)$ et $v(u + 2K\pi)$ sont deux intégrales de l'équation linéaire (1) du § précédent ce qui montre que le rapport de ces deux intégrales va constamment en croissent ou constamment en décroissant. Entre deux zéros consécutifs de $v(u)$, qui correspondent aux valeurs $+\infty$ et $-\infty$ de ce rapport, il prendra donc une fois et une seule la valeur zéro, une fois et une seule la valeur 1. Entre deux points d'intersection consécutifs de $G_0$ et de la branche $v(u)$ de la géodésique $G_1$ (points donnés par l'équation $v(u) = 0$), il y aura un point d'intersection de $G_0$ et de la branche $v(u + 2K\pi)$ de la géodésique $G_1$ (point donné par l'équation $v(u + 2K\pi) = 0$) et il n'y en aura qu'un seul; de plus il y aura un point double donné par l'équation (4) et il n'y en aura qu'un seul.

Cela nous permet d'énumérer les points doubles. Donnons à l'entier $K$ une valeur déterminée et faisons varier $u$ de 0 à $2n\pi$, nous aurons $2m$ points d'intersection de $G_0$ avec la branche $v(u)$ de $G_1$; cela nous donnera donc $2m$ points doubles. Maintenant nous pouvons donner à $K$ les valeurs $1, 2, \cdots n-1$, ce qui fait en tout $2m(n-1)$ points doubles. Mais chaque point double est ainsi compté deux fois, puisqu'on retrouve le même en changeant $u$ et $K$ en $u + 2K\pi$ et $n - K$.

*Le nombre total des points doubles pour la géodésique $G_1$ qui diffère infiniment peu de $G_0$, et par conséquent aussi pour toutes les géodésiques fermées de la série $S_1$, est donc $m(n-1)$.*

Pour $n = 1$, on retrouve des géodésiques fermées sans point double; pour $n = 2$ on constate la circonstance suivante: Les divers arcs de $G_1$ partagent la surface en $m + 2$ regions, dont $m$ sont des polygones curvilignes de deux côtés et les deux autres des polygones curvilignes de $m$ côtés. Si $m$ n'est pas égal à 3 aucun de ces polygones n'est un triangle, de sorte que d'après ce que nous avons dit au § 4, la disposition des points doubles ne peut varier quand on parcourt la série $S_1$ d'une manière continue; *toutes les géodésiques de cette série appartiennent donc au même type.*

Il semble qu'il y ait exception pour $m = 3$ et que deux types soient possibles,

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représentés par les figures 1 et 2; mais la disposition de la figure 2 est impossible. En effet la courbe 2 limite quatre régions $A$, $B$, $C$, $D$; la région $D$ est un triangle dont les trois angles sont $\alpha$, $\beta$, $\gamma$; la courbure totale de $D$ est $\alpha+\beta+\gamma-\pi$; de même celles de $A$, $B$, $C$ sont respectivement $\alpha+\pi$, $\beta+\pi$, $\gamma+\pi$; de sorte que

$$A+B+C>D+4\pi>4\pi,$$

chacune des grandes lettres représentant la courbure totale de la région correspondante. Ce qui voudrait dire que la courbure totale des trois régions

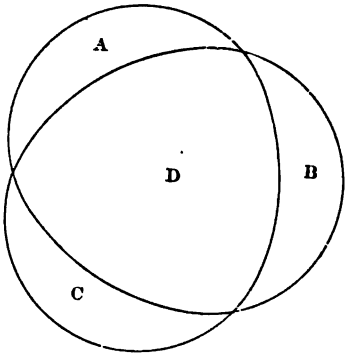


FIG. 1.

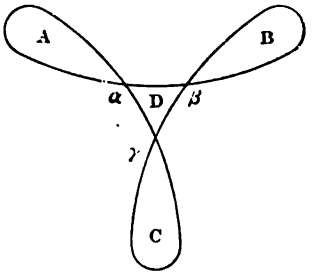


FIG. 2.

$A$, $B$, $C$ réunies serait plus grande que celle de la surface entière, résultat évidemment absurde.

Supposons maintenant une géodésique $G_1$ de la série $S_1$ qui ne soit plus très-peu différente de $G_0$. Je considère le produit $\alpha U$ correspondant et je suppose qu'il passe par la valeur

$$\frac{2m_1 i\pi}{n_1},$$

commensurable avec $i\pi$. D'après ce que nous venons de voir, la série $S_1$ engendrera une autre série $S_2$; soit $G_2$ une des géodésiques de cette série et par exemple celle qui diffère très peu de la géodésique $G_1$ parcourue $n_1$ fois. On voit d'abord qu'elle rencontrera $G_0$ en $2mn_1$ points. En combien de points rencontre-t-elle $G_1$? Les points d'intersection seront de deux sortes. D'abord à chaque point double de $G_1$ correspondront $2n_1$ points d'intersection. En effet en chaque point double viennent passer 2 branches de $G_1$, soient $B$ et $B'$, et alors $B$ rencontre $n_1$ branches de $G_2$ très peu différentes de $B'$; et $B'$ rencontre $n_1$ branches de $G_2$ très peu différentes de $B$. On obtient ainsi $2mn_1(n-1)$ points d'intersection. Il faut y ajouter les $2m_1$ points d'intersection de $G_2$ avec la branche correspondante de $G_1$, soit en tout

$$2mn_1(n-1)+2m_1.$$

Combien $G_2$ aura-t-elle de points doubles? Chaque point double de $G_1$ nous

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donnera d'abord $n_1^2$ points doubles de $G_2$, car nous avons les $n_1$ branches peu différentes de $B$, qui coupent en $n_1^2$ points, les $n_1$ branches peu différentes de $B'$; nous avons en outre $m_1(n_1 - 1)$ points doubles obtenus par des équations analogues à (4), soit en tout

$$mn_1^2(n-1) + m_1(n_1 - 1).$$

*Ainsi $G_2$ et par conséquent toutes les géodésiques de la série $S_2$ couperont $G_0$ en $2mn_1$ points, $G_1$ en $2mn_1(n-1) + 2m_1$ points et auront $mn_1^2(n-1) + m_1(n-1)$ points doubles.*

Il y a une autre quantité qui demeure invariable dans toutes les géodésiques d'une même série, c'est la courbure totale de la région limitée par la géodésique; dans le cas d'une courbe fermée sans point double, c'est $2\pi$; dans le cas de la figure 1, c'est $4\pi$ en comptant $A$, $B$ et $C$ une fois, et $D$ deux fois, conformément aux conventions habituelles.

## § 7. *Existence d'une géodésique fermée.*

Nous avons vu plus haut qu'il y a toujours au moins une géodésique fermée sans point double. Bien que la démonstration ne laisse rien à désirer au point de vue de la simplicité, je crois cependant devoir en donner une seconde, quoique beaucoup moins simple.

Soit une géodésique fermée sans point double, elle partagera la surface en deux régions et la courbure totale de chacune de ces deux régions sera $2\pi$. Considérons maintenant toutes les courbes fermées sans point double qui partagent la surface en deux régions dont la courbure totale est $2\pi$. La longueur de l'une de ces courbes fermées ne peut pas devenir plus petite que toute quantité donnée; car si cette courbe se réduisait à un contour infiniment petit, la courbure totale de l'une des régions limitées par ce contour serait elle-même infiniment petite.

Parmi ces courbes, il y en a donc une qui est plus courte que toutes les autres, et je dis que c'est une géodésique.

Contentons-nous d'abord d'un premier aperçu, afin de faire compendre le principe de la démonstration.

Soit $C$ une courbe fermée quelconque et $R$ l'une des régions limitées par $C$, soit

$$U = \int ds, \qquad \Omega = \int \frac{d\omega}{\rho},$$

la longueur totale de $C$ et la courbure totale de $R$; dans ces formules $ds$ représente l'élément d'arc de $C$, $d\omega$ l'élément de surface de $R$, et $\rho$ le produit des deux rayons de courbure principaux.

Considérons une courbe $C'$ très peu différente de $C$ et limitant une région

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$R'$, soit $\zeta$ la distance de ces deux courbes comptée sur la normale à $C$. Soit $U + \delta U$ la longueur de $C'$, et $\Omega + \delta\Omega$ la courbure totale de $R'$; on aura

$$\delta U = \int \gamma\zeta ds, \qquad \delta\Omega = \int \frac{\zeta ds}{\rho},$$

$\gamma$ représentant la courbure géodésique de $C$. Nous nous sommes imposé la condition $\Omega = 2\pi$, et nous voulons que $U$ soit minimum; nous devons donc avoir

$$\delta U = 0, \qquad \delta\Omega = 0.$$

Ces deux équations doivent donc être équivalentes, c'est à dire qu'on doit avoir

$$\gamma = \frac{K}{\rho},$$

$K$ étant une constante.

Mais pour une courbe fermée quelconque $C$ limitant une région $R$ on a

$$\int \gamma ds = \Omega - 2\pi.$$

Dans notre cas, on a $\Omega = 2\pi$; d'où

$$K\int \frac{ds}{\rho} = 0.$$

Comme la surface est convexe, $\rho$ est essentiellement positif de sorte que l'intégrale $\int ds/\rho$ ne peut s'annuler; on aura donc

$$K = 0, \qquad \gamma = 0,$$

ce qui veut dire que *la courbe $C$ est une géodésique.*

Examinons maintenant les objections qu'on pourrait faire à ce raisonnement incomplet. Considérons l'ensemble des courbes analytiques $C$ fermées et sans point double, limitant une région $R$ de courbure totale $2\pi$; il est clair que la longueur de ces courbes aura une limite inférieure; mais on peut se demander si cette limite sera effectivement atteinte; et si elle le sera par une courbe faisant partie de l'ensemble.

Cette courbe pour laquelle le minimum serait atteint sera-t-elle analytique, de telle façon que les règles du calcul des variations puissent lui être appliquées?

Et si elle est analytique, sera-t-elle dépourvue de points doubles, ou bien ne pourrait-il se faire que tout en étant infiniment voisine de courbes sans point double elle possédât elle-même des points singuliers où deux de ses branches viendraient à se toucher?

Cela fait en réalité deux objections distinctes.

La 1ère est d'ordre analytique; c'est celle que l'on rencontre dans tous les problèmes analogues; malgré son importance, je n'en parlerai pas ici, je me bornerai à renvoyer aux récents travaux de M. HILBERT.

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La 2^de^ est d'ordre physique, pour ainsi dire; elle est spéciale au problème qui nous occupe et nous allons en faire un examen approfondi.

Pour la bien faire comprendre, nous allons donner à notre problème une signification physique et concrète. Supposons d'abord un gaz enfermé dans une enceinte en partie déformable, il exercera sur les parois de cette enceinte une pression qui tendra à en augmenter le volume, le travail effectué par la pression de ce gaz dépendra des variations de ce volume. Au lieu d'un gaz, nous pouvons supposer un fluide compressible comme un gaz, mais tel que la pression soit liée au volume non par la loi de Mariotte, mais par une relation quelconque. Alors le travail virtuel de la pression pour une déformation virtuelle quelconque de l'enceinte sera $p\delta V$, $p$ designant la pression du gaz, laquelle doit être uniforme, et $\delta V$ l'accroissement virtuel du volume.

Si l'enceinte n'est susceptible que de certaines déformations, l'equilibre sera atteint quand le volume sera maximum, en supposant que toutes les déformations possibles se fassent sans résistance. Si au contraire certaines forces opposent à la déformation de l'enceinte et que le travail virtuel de ces forces soit $\delta W$ la condition d'équilibre sera

$$p\delta V + \delta W = 0.$$

Supposons maintenant que notre enceinte soit limitée: 1°. Par notre surface convexe $S$ elle-même, 2°. Par une autre surface convexe $S'$ infiniment peu différente de $S$ et telle que la distance des deux surfaces, comptée suivant la normale à $S$, soit égale à $\epsilon/\rho$, $\epsilon$ étant une constante infiniment petite et $\rho$ le produit des rayons de courbure, 3°. Par un ruban infiniment étroit dont les bords s'appuieront sur $S$ et sur $S'$, (c'est ce ruban qui va épouser la forme de notre courbe fermée $C$, de telle façon que notre enceinte sera l'espace infiniment mince compris entre la région de $S$ que nous avons appelée $R$ et la région correspondante de $S'$). Le volume de notre enceinte sera alors représenté par l'intégrale

$$\int \frac{\epsilon}{\rho} d\omega = \epsilon\Omega.$$

Nous supposerons que notre ruban résiste plus on moins à l'extension, mais qu'il est d'ailleurs flexible sans résistance; sa largeur pourra être supposée une peu plus grande que la plus grande valeur de $\epsilon/\rho$, de telle sorte que ses bords s'appliquent sur les surfaces $S$ et $S'$ et soient *collés* contre ces surfaces par la pression du fluide. La pression du fluide tendra à allonger le ruban et mettra en jeu la résistance du ruban à l'extension, c'est à dire ce qu'on appelle la *tension* du ruban; cette tension sera constante tout le long du ruban, puisque la pression du fluide est normale à ce ruban; nous l'appellerons $T$; on aura alors

$$\delta W = -T\delta U,$$

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$U$ étant la longueur totale du ruban, c'est à dire la courbe $C$. L'équation d'équilibre s'écrit alors

$$p\epsilon\delta\Omega = T\delta U.$$

Si le fil est inextensible, c'est à dire si un accroissement très petit de $U$ amène un accroissement très grand de $S$: la longueur $U$ est constante et doit être regardée comme une des données de la question, et l'équilibre est atteint quand $\Omega$ est maximum.

Si au contraire le fluide est incompressible, c'est à dire si une diminution très petite du volume amène un accroissement tres grand de $p$, c'est $\Omega$ qui est constant et qui est une donnée de la question (comme dans le cas qui nous occupe où $\Omega = 2\pi$) et l'équilibre est atteint quand $U$ est minimum.

Considérons un segment très petit $ds$ du ruban, limité en $AB$ et $A'B'$; tout se passera comme si le ruban était coupé en $AB$ et $A'B'$ et soumis à deux forces appliquées l'une au milieu de $AB$, l'autre au milieu de $A'B'$ et représentant l'une l'action de la portion du ruban située an dela de $AB$, l'autre l'action de la portion du ruban située au dela de $A'B'$; ces deux forces ne sont autre chose que la tension du ruban; elles sont donc égales en grandeur et égales à $T$; elles doivent faire équilibre à la pression exercée par le fluide sur le segment $ABA'B'$. Projetons tout sur la normale à la courbe $C$, dans le plan tangent à la surface $S$. La projection de la pression du fluide sera

$$\frac{p\epsilon}{\rho} ds,$$

celle de la tension sera $Td\alpha$, $d\alpha$ étant l'angle de contingence géodésique de $C$. On aura donc

$$\frac{p\epsilon}{\rho} ds = Td\alpha.$$

Comme $T$, $p$ et $\epsilon$ sont des constantes, nous voyons que le rapport de la courbure géodésique $d\alpha/ds$ à $1/\rho$ doit être une constante; nous retrouvons ainsi, dans un autre langage, le résultat obtenu plus haut.

L'équilibre pourra certainement être atteint d'une façon quelconque, ce qui ne peut se faire que quand $C$ sera une géodésique fermée *à moins que*, et c'est ici que nous retrouvons la difficulté signalée plus haut, deux portions du ruban ne viennent se coller l'une contre l'autre. Dans ce cas les deux portions du ruban ainsi collées l'une sur l'autre, subiraient la pression du fluide des deux côtés, de telle façon que l'effet de cette pression se trouverait annulée.

Dans ce cas l'équation d'équilibre s'écrirait, pour cette portion du ruban,

$$Td\alpha = 0,$$

de sorte que l'angle de contingence géodésique étant nul cette partie du ruban devrait affecter la forme d'une géodésique.

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Ainsi nous devrons distinguer les *parties libres* du ruban qui devraient satisfaire à la condition $\rho d\alpha/ds = \text{const.}$, et les *parties collées* qui se réduiraient à des arcs de géodésiques. Que se passerait-il maintenant aux points de raccordement? Je dis que les diverses parties du ruban devraient se raccorder par contact, c'est à dire sous un angle nul et de façon que l'ensemble du ruban ne présente pas de point anguleux.

Soient en effet $BAC$, $EDF$ deux brins du ruban, $AB$ étant appliqué sur $ED$; nous savons que la tension est constante tout le long de $AB$, de même que tout le long de $AC$, de $ED$ ou de $DF$; mais nous ne savons pas encore si elle est la même par exemple sur $AC$ et sur $AB$.

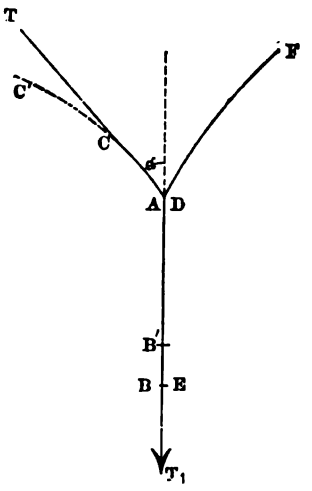

FIG. 3.

Mais il est aisé d'établir ce dernier point; le brin $BAC$ est en équilibre sous l'action des deux tensions $CT$ et $ET'$ appliquées à ses deux extrémités, de la pression du fluide et de l'action exercée par le brin $DE$ sur le brin $AB$ sur lequel il s'applique. Cette action (égale d'ailleurs au sens près à la réaction du brin $AB$ sur le brin $DE$) est normale au brin $AB$, puisque les deux brins, quoique appliqués l'un sur l'autre, peuvent librement glisser l'un sur l'autre.

Supposons alors que le brin $BAC$ se déplace pour venir en $B'AC'$ (c'est à dire glisse en passant par le point fixe $A$); la somme des travaux virtuels doit être nulle. Or le travail de la pression et celui de l'action de $DE$ sur $AB$ sera nul, puisque ces forces sont normales aux brins sur lesquels elles s'exercent. Les travaux des deux tensions seront donc égaux et de signe contraire, ce qui exige que les deux tensions $CT$ et $BT$, soient égales. C. Q. F. D.

Supposons maintenant les arcs $AC$ et $AB$ très petits; soit $AM$ le prolongement de la tangente en $A$ à $AB$; soit $\alpha$ l'angle de $CT$ avec $AM$. Projetons sur une perpendiculaire à $AM$ toutes les forces qui aggissent sur $ABC$. Les pressions seront négligeables puisque les arcs sont très petits de sorte que le brin $ABC$ devra être en équilibre sous l'action des deux tensions et de l'action de $DE$ sur $AB$; il faut donc (puisque les tensions sont égales) que cette action soit dirigée suivant la bissectrice des deux tensions; c'est à dire qu'elle fasse avec la perpendiculaire à $AB$ un angle $\alpha/2$; or elle est normale à $AB$. Donc l'angle de raccordement $\alpha$ est nul. C. Q. F. D.

Ainsi le ruban forme une courbe sans point anguleux le long de laquelle la tension $T$ est constante.

Alors la courbure totale de $R$ sera encore

$$\frac{p\epsilon}{T}\int\frac{ds}{\rho} + 2\pi,$$



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l'intégrale étant étendue à toute la partie libre du ruban; on aura donc

$$\frac{p\epsilon}{T}\int\frac{ds}{\rho}=0,$$

ce qui veut dire encore que $p=0$, ou que la courbure géodésique est partout nulle, ou que le ruban entier affecte partout la forme d'une géodésique fermée.

Si alors $AB$ et $A_1B_1$ sont deux parties du ruban collées l'une sur l'autre, si $AC$ et $A_1C_1$ sont les parties libres du ruban au delà de $A$ et de $A_1$; si $BD$ et $B_1D_1$ sont les parties libres du ruban au delà de $B$ et de $B_1$, toutes ces parties devront appartenir à une même géodésique; donc $AC$ est la continuation analytique de $AB$, et $A_1C_1$ celle de $A_1B_1$. Ce qui est absurde, puisque par hypothèse $AB$ coïncide avec $A_1B_1$ et que $AC$ ne coïncide pas avec $A_1C_1$. L'hypothèse où deux parties du ruban viennent se coller l'une sur l'autre doit donc être écartée.

Nous devons de même écarter celle où deux parties du ruban viendraient se toucher en un point, car nous avons vu que deux géodésiques ne peuvent être tangentes l'une à l'autre.

L'équilibre ne pourra donc être atteint que quand le ruban prendra la forme d'une géodésique fermée sans point double.

## § 8. *Discussion du minimum.*

La courbe $C$ dont la longueur est un minimum, doit d'après ce qui précède se réduire à une géodésique; mais cette condition nécessaire n'est pas suffisante; c'est celle qui se rapporte à la 1ère variation et nous avons à examiner celle qui se rapporte à la 2de variation.

Soit $C$ la courbe considérée, qui est représentée sur la figure en $AMBNEPA$. Supposons qu'il existe une courbe $AM_1BN_1E$ infiniment peu différente de la 1ère et satisfaisant comme elle à la condition:

$$\text{Courbure géodésique}=\frac{K}{\rho},$$

$K$ étant une constante qui pourra ne pas avoir la même valeur que pour la courbe $C$ (c'est à dire la valeur zéro).

Je suppose de plus que la courbure totale de l'aire infiniment petite $AM_1BMA$, soit égale à la courbure totale de l'aire $BNEN_1B$, de telle façon que les deux courbes fermées $AMBNEPA$ et $AM_1BN_1EPA$ enveloppent des aires de même courbure totale.

Je considère les angles sous lesquels les deux courbes $AMBNE$, $AM_1BN_1E$ se coupent en $A_1$ en $B$ et en $E$ comme des infiniment petits du 1er ordre, et je remarque d'abord que la différence des longueurs des deux courbes est infiniment petite du 3ème ordre.

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Prenons sur $AM_1B$ et $AP$ deux points $A''$ et $A'$ voisins de $A$ sans en être infiniment voisins (je les suppose simplement assez près pour que les géodésiques qui les joignent en restant très voisins de $C$ restent le plus court chemin d'un point à un autre); et de même sur $EN_1B$ et $EP$ deux points $E''$ et $E'$ voisins de $E$. Joignons par des arcs de géodésique marqués en traits pointillés $A'A''$ d'une part, $E'E''$ d'autre part. Il est clair:

1°. Que la longueur $A'A''$ sera plus petite que $A'A + AA''$, et de même la longueur $E'E''$ plus petite que $E'E + EE''$, et que les différences seront des infiniment petits du 2d ordre (et non pas d'ordre supérieur au 2d); en effet

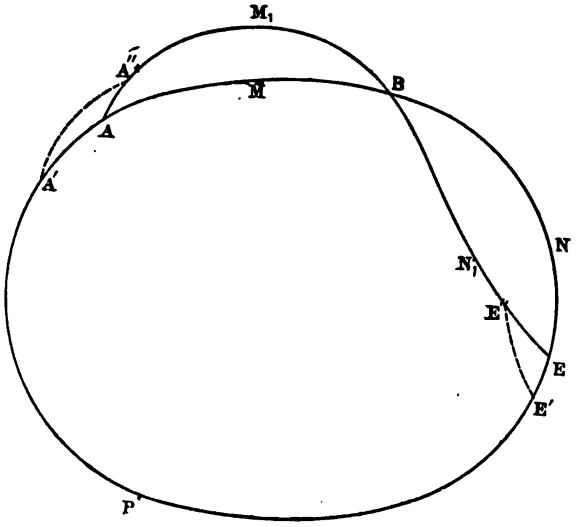


FIG. 4.

$A'A''$ étant une géodésique $AA' + AA'' - A'A''$ par exemple, est au moins égal à $\overline{AA'} + \overline{AA''} - A'A''$, en designant par $\overline{AA'}$ et $\overline{AA''}$ les géodésiques qui joignent $A$ à $A'$ ou à $A''$; (comme $AA'$ est une géodésique, elle coïncide avec $\overline{AA'}$). Les deux géodésiques se coupent sous de angles infiniment petits du 1er ordre, (et non d'ordre supérieur, puisque la distance de $A''$ à $C$ est du 1er ordre) de sorte que $\overline{AA'} + \overline{AA''} - A'A''$ est positif et plus grand qu'un infini petit du 2d ordre.

Il résulte de tout cela que la courbe fermée $A'A''M_1BN_1E''E'PA'$ est plus courte que la courbe fermée $AMBNEPA$, ou C.

2°. Il est clair ensuite que l'on peut placer les points $A'$, $A''$, $E'$, $E''$ de telle façon que les deux triangles $AA'A''$ et $EE'E''$ aient même courbure totale. Dans ces conditions, les deux courbes $AMBNEPA$ et $A'A''M_1BN_1E''E'PN$ enveloppent des aires ayant même courbure totale.

Si donc on peut construire une courbe $AM_1BN_1E$ placée comme sur la figure et jouissant des propriétés énoncées, $C$ n'est certainement pas la plus courte de

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toutes les courbes fermées qui enveloppent des aires de courbure totale $2\pi$. *La courbe $C$ ne correspond pas à un vrai minimum.* Supposons maintenant que $C$ soit une géodésique fermée stable et que le produit $\alpha U/i$ soit plus grand que $2\pi$. Je dis que $C$ ne pourra pas correspondre à un vrai minimum.

Soit en effet $A$ un point quelconque de $C$, $AM_1B$ une géodésique très voisine de $C$ et passant par $A$; elle viendra recouper $C$ en un point $B$ qui sera le 1er foyer de $A$ puis en un point $E$ qui sera le 2d foyer de $A$. Si $\alpha U/i$ est $> 2\pi$, les deux arcs $AMB$, $BNE$ ne couvriront pas le périmètre de $C$ tout entier et les deux courbes seront disposées comme sur la figure.

Si de plus l'angle sous lequel les deux géodésiques se coupent en $A$ est égal à celui sous lequel les deux géodésiques se coupent en $E$, les deux aires $AM_1BA$, $BN_1EMB$ auront même courbure totale; nous serons dans les conditions du théorème précédent et $C$ ne sera pas un vrai minimum.

La question à résoudre est donc la suivante: peut-on choisir le point $A$ de telle façon que les deux angles en question soient égaux?

Reprenons les rotations du § 5, la géodésique $AM_1B$ aura pour équation

$$v = \rho \sin(\theta - \beta),$$

et les points $A$, $B$, $E$ correspondront à

$$\theta = \beta, \qquad \theta = \beta + \pi, \qquad \theta = \beta + 2\pi,$$

l'angle sous lequel les deux géodésiques se couperont en chacun de ces trois points sera représenté par la valeur correspondante de $\rho d\theta/du$. La condition à remplir est donc

$$\left(\rho \frac{d\theta}{du}\right)_{\theta=\beta} = \left(\rho \frac{d\theta}{du}\right)_{\theta=\beta+2\pi}.$$

Mais $\rho$ et $d\theta/du$ sont des fonctions périodiques de $u$ se reproduisant quand $u$ augmente de $U$. Ce sont donc aussi des fonctions périodiques de l'argument réduit

$$\frac{2i\pi\theta}{\alpha U}.$$

Nous aurons donc

$$\rho \frac{d\theta}{du} = F\left(\frac{2i\pi\theta}{\alpha U}\right),$$

$F$ étant développable en série de Fourier suivant les cosinus et les sinus des multiples de $2i\pi\theta/\alpha U$.

Il s'agit donc de savoir si l'on peut déterminer $\beta$ de telle façon que

$$F\left(\frac{2i\pi\beta}{\alpha U} + \frac{4i\pi^2}{\alpha U}\right) - F\left(\frac{2i\pi\beta}{\alpha U}\right) = 0.$$

Or le premier membre est une fonction périodique de $2i\pi\beta/\alpha U$, développable en

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série trigonométrique suivant les sinus et les cosinus des multiples de $2i\pi\beta/\alpha U$, *et dont la valeur moyenne est nulle.* Ce premier membre ne peut donc être toujours de même signe et il faut bien qu'il s'annule. *Donc $C$ n'est pas un vrai minimum.* C. Q. F. D.

Supposons maintenant que $C$ soit instable, soit $n$ le nombre des zéros des $\phi(u)$ (ou de $\phi_1$) et par conséquent $n$ le nombre des zéros de $\psi(u)$ et $2n$ celui des foyers limites. Soient

$$u_0, u_1, u_2, \cdots, u_{n-1}, u_n = u_0 + U, u_{n+1} = u_1 + U,$$

les zéros successifs de $\phi(u)$ (ou de $\phi_1$); soient

$$u'_0, u'_1, u'_2, \cdots, u'_{n-1}, u'_n = u'_0 + U, u'_{n+1} = u'_1 + U,$$

les zéros successifs de $\psi(u)$.

Les zéros de $\phi(u)$ partagent le périmètre de $C$ en $n$ segments, de telle façon que si un point est sur le $K^{\text{ème}}$ segment, son foyer est sur le $(K+1)^{\text{ème}}$, son second foyer sur le $(K+2)^{\text{ème}}$, etc.

Reprenons la figure 3, soit $A$ un point quelconque de $C$, $AM_1BN_1E$ une géodésique infiniment voisine de $C$, de sorte que $B$ et $E$ soient le 1er et le 2d foyers de $A$. Si $n > 1$, cette courbe est disposée comme sur la figure, et la seule condition à remplir pour que notre théorème soit applicable, c'est que l'angle sous lequel cette courbe coupe $C$ en $A$ soit égal à l'angle sous lequel elle coupe $C$ en $E$.

Soit $u$ la valeur de $u$ au point $A$ et soit $F(u)$ le rapport de l'angle sous lequel les deux courbes se coupent en $E$ à l'angle sous lequel elles se coupent en $A$. Le rapport $F(u)$ est toujours positif, et la question est de savoir si l'on peut disposer de $u$ de telle façon que

$$F(u) = 1.$$

Soit $u = u_0$; l'équation de la géodésique $AM_1BN_1E$ se réduit alors à

$$v = e^{\alpha u}\phi(u)$$

(ou à $v = e^{\alpha' u}\phi_1(u)$ dans le cas des géodésiques instables de la 3ème catégorie). Elle vient couper $C$ successivement en $A_0, A_1, \cdots$, points correspondant aux arguments $u_0, u_1, \cdots$; de sorte que $A_{k+1}$ est le foyer de $A_k$, et l'intersection a lieu sous les angles

$$\theta(u_0), \theta(u_1) \cdots,$$

en posant

$$\theta(u) = e^{\alpha u}[\alpha\phi(u) + \phi'(u)].$$

On aura donc

$$F(u_0) = \frac{\theta(u_2)}{\theta(u_0)}, \quad F(u_2) = \frac{\theta(u_1)}{\theta(u_2)}, \quad \cdots, \quad F(u_{2n-2}) = \frac{\theta(u_{2n})}{\theta(u_{2n-2})};$$

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d'où

$$F(u_0)F(u_2)\cdots F(u_{2n-2})=\frac{\theta(u_{2n})}{\theta(u_0)}=\frac{\theta(u_0+2U)}{\theta(u_0)}=e^{2\alpha U}>1,$$

ce qui prouve que l'un au moins des facteurs

$$F(u_0),\ F(u_2),\ \cdots,\ F(u_{2n-2})$$

est plus grand que 1, et par conséquent que $F(u)$ peut devenir $>1$. Soit maintenant $u=u'_0$, la géodésique se réduit à

$$v=e^{-\alpha u}\psi(u)$$

et l'on a

$$F(u'_0)=\frac{\omega(u'_2)}{\omega(u'_2)},\quad F(u'_2)=\frac{\omega(u'_4)}{\omega(u'_2)},\quad \cdots,\quad F(u'_{2n-2})=\frac{\omega(u'_{2n})}{\omega(u'_{2n-2})},$$

en posant

$$\omega(u)=e^{-\alpha u}[\psi'(u)-\alpha\psi(u)];$$

d'où,

$$F(u'_0)F(u'_2)\cdots F(u'_{2n-2})=\frac{\omega(u_{2n})}{\omega(u_0)}=\frac{\omega(u_0+2U)}{\omega(u_0)}=e^{-2\alpha U}<1,$$

ce qui prouve que l'un au moins des facteurs

$$F(u'_0),\ F(u'_2),\ \cdots,\ F(u'_{2n-2})$$

est $<1$, et par conséquent que $F(u)$ peut devenir $<1$.

Le rapport $F(u)$ étant une fonction continue et périodique de $u$ et pouvant devenir $<1$ et $>1$, pourra devenir égal à 1.

Ce qui montre encore que $C$ n'est pas un vrai minimum.

Ainsi pour que la géodésique fermée $C$, soit la plus courte de toutes les courbes fermées sans point double qui enveloppent une aire de courbure totale $2\pi$, il faut, si elle est stable que $\alpha U/i$ soit plus petit que $2\pi$ et si elle est instable que le nombre $2n$ des foyers limites soit égal à 2.

Ces conditions nécessaires sont-elles suffisantes? Je n'ai pas besoin de traiter ici la question que les procédés ordinaires du calcul des variations permettraient de résoudre.

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# THE CLASSIFICATION OF QUADRICS*

BY

T. J. I'A. BROMWICH

In a former number of the Transactions† Mr. J. L. COOLIDGE has given a classified list of the possible species of quadrics in hyperbolic space; his method is based on that used by CLEBSCH for discussing the mutual relations of two quadrics. It seems desirable, however, that the classification should be associated with the algebraic problem of reducing two quaternary quadratic forms to canonical types; the solution of this problem is, of course, based on WEIERSTRASS's theory of *invariant factors*.‡ One obvious advantage of using invariant factors as a basis of classification is that they can be determined *without any preliminary reduction* of the equation to the quadric; indeed, it is generally necessary to find the factors before we can begin the reduction. Another advantage may be found in the fact that the canonical types given by WEIERSTRASS are the most compact possible.§

The first classification of two quadrics by means of invariant factors is due to SYLVESTER.‖ From the geometrical point of view, an elaborate discussion was carried out, independently of WEIERSTRASS's methods, by PAINVIN;¶ PAINVIN touches on the question of *reality*, but only in the simplest case (when the invariant factors are *linear*). According to MUTH, the application to quadrics

* Presented to the Society at the St. Louis meeting, September 17, 1904. Received for publication March 4, 1904.

† Transactions of the American Mathematical Society, vol. 4 (1903), p. 161.

‡ *Elementartheiler;* see his paper on bilinear and quadratic forms, Berliner Monatsberichte, 1868, p. 310 = *Gesammelte Werke*, vol. 2, p. 19. For a bibliography, consult a review of MUTH's *Elementartheiler* (Leipzig, 1899), in the Bulletin of the American Mathematical Society, vol. 7 (1901), p. 308. An elementary account of the application of this theory to the reduction of two quadratic forms will be found in a paper by the writer in the Quarterly Journal of Mathematics, vol. 33 (1901), p. 85.

§ As we shall see later, some of COOLIDGE's types contain superfluous coefficients, which may be removed by suitable linear substitutions.

‖ Philosophical Magazine, ser. 4, vol. 1 (1851), pp. 119, 295; this paper is, of course, earlier than WEIERSTRASS's. But it is not complete, as no method for determining the reducing substitutions is given, and thus it is not certain that the classification is exhaustive. In fact the two "singular cases" are omitted; i. e., when the quadrics are cones having (i) a common tangent-plane, which they touch along a common generator (ii) a common vertex.

¶ Nouvelles Annales de Mathématiques, ser. 2, vol. 7 (1868), pp. 481, 529; vol. 8 (1869), pp. 49, 145, 193. The last section of his paper contains a list of the chief results obtained.



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of WEIERSTRASS's results (and of KRONECKER's,* on the "singular case") has been fully carried out by KILLING.†

WEIERSTRASS's method of determining the reducing substitutions does not show how to modify them in case we restrict ourselves to real quadratic forms and real linear substitutions. This difficulty was first surmounted by KLEIN in his celebrated memoir on the classification of line-complexes of the second degree‡; KLEIN shows there that if one of the quadratic forms has a known type, and if we are restricted to real forms (and real substitutions), then the possible invariant-factors are restricted by a certain relation.‖ KLEIN gives full details of all possible cases, if the number of variables is six, and the signature of the one known quadratic form is zero (see §1 below, for definition of *signature*).

BÔCHER § has carried out a similar classification for the case of five variables, with a quadratic form of signature three.

In what follows we shall consider the case of four variables, the given signature being two (§2), four (§3), or three (§4).

## §1. *Preliminary results on* REAL *transformations of quadratic forms.*

Any real quadratic form can be expressed in the shape

$$a_1x_1^2 + a_2x_2^2 + \cdots + a_nx_n^2$$

(where $x_1, x_2, \cdots, x_n$ are real and linearly independent) in an infinity of ways; but, however the reduction is effected, the number of positive coefficients and the number of negative coefficients (in $a_1, a_2, \cdots, a_n$) are both *fixed*. This is SYLVESTER's *law of inertia* (*Trägheitsgesetz*) of quadratic forms.¶

* KRONECKER published a number of papers on this case; a list would be too long for enumeration here. The earlier papers will be found in the second half of volume 1 of his works; the last paper (in which the question was finally settled) appeared in the Berliner Sitzungsberichte of 1890, 1891. Alternative methods have been given by the present writer, Proceedings of the London Mathematical Society, vol. 32 (1900), pp. 87, 117, 326.

† *Der Flächenbüschel zweiter Ordnung, Inauguraldissertation*, Berlin, 1872; unfortunately this memoir has not been accessible. It does not appear from MUTH's references, whether or no KILLING concerns himself with the question of reality.

‡ *Inauguraldissertation*, Bonn, 1868; reprinted in vol. 23 of the Mathematische Annalen, p. 539.

‖ Mathematische Annalen, vol. 23, pp. 561, 562; the theorem has been extended to "HERMITE's forms" by A. LOEWY, Crelle's Journal für die Mathematik, vol. 122 (1900), p 69; see also a paper by the writer, Proceedings of the London Mathematical Society, vol. 32 (1901), p. 349.

§ *Reihenentwickelungen der Potentialtheorie*, Leipzig, 1894; the classification (of cyclides) will be found in Kapitel 3. It is perhaps of interest to note that the results obtained in §2 below can also be used to classify the cyclic curves (i. e. bicircular quartics), mentioned by BÔCHER.

¶ Philosophical Magazine, ser. 4, vol. 2 (1852), p. 138; Philosophical Transactions of London, 1853, p. 407. BORCHARDT (Crelle's Journal für die Mathematik,

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For our present purpose, the difference between these two numbers is the most useful constant; this difference we call the *signature* of the quadratic form, after FROBENIUS.*

We are now in a position to state KLEIN's theorem (given in the dissertation already cited): —

*If $A$ is a real quadratic form, whose signature is known to be $m$, then, whatever quadratic form $B$ may be (provided that it is real), the characteristic determinant $|\lambda A - B|$ has at least $m$* REAL *invariant factors of* ODD *degree.*†

In the present application, $A$ is the absolute quadric (of hyperbolic space) and takes the form

$$x_1^2 + x_2^2 + x_3^2 - x_4^2$$

so that its signature is 2 (i. e. $3 - 1$). Accordingly, whatever quadric $B$ may be, the determinant $|\lambda A - B|$ has at least two real invariant-factors of odd degree, Now the degree of this determinant is 4, so that the two real invariant-factors must have degrees 1, 1 or 3, 1. Hence the only possible combinations of indices for the invariant-factors are given by the sets

$$1111;\ 211;\ 31.$$

It is now clear that complex roots can occur only in the first set; further, in this set there may be one pair of such roots (not more). Thus the four fundamental types are

| I | II | III | IV |
|---|---|---|---|
| 1111 | $11\bar{1}\bar{1}$ | 211 | 31 |

vol. 53 (1857), p. 275) points out that JACOBI was acquainted with the result as early as 1847; but apparently JACOBI had no occasion to publish it.

The restrictions that $x_1, x_2, \cdots, x_n$ must be real and linearly independent are essential; for example, we have

$$(x+iy)^2 + (x-iy)^2 = 2x^2 - 2y^2, \text{ and } x^2 + y^2 + (x+y)^2 + (x-y)^2 = 3x^2 + 3y^2.$$

*Crelle's Journal für die Mathematik, vol. 114 (1894), p. 187: the paper was also published in the Berliner Sitzungsberichte of that year.

†This theorem requires modification in case $|A| = 0$. For instance, if there are three variables $x, y, z$ and if $A = x^2 + y^2$, $B = 2yz$, the signature of $A$ is 2, but there is only *one* invariant-factor of $|\lambda A + \mu B|$ which has an odd degree; the invariant-factors are actually $\lambda$, $\mu^2$. Of course in KLEIN's work the case $|A| = 0$ is excluded owing to the nature of the problem.

The modified theorem may be stated as follows: *Let $|\lambda A + \mu B|$ have $n_1$ invariant-factors of the type $(\lambda + \mu c)^\gamma$, $c$ being real (including zero) and $\gamma$ being odd; and let $n_2$ be the number of invariant-factors of the type $\mu^\beta$ where $\beta$ is even; then*

$$n_1 + n_2 \geqq m.$$

In the illustration above we have $n_1 = 1$, $n_2 = 1$, and the theorem is verified. For a discussion of the last theorem, see A. LOEWY, Crelle's Journal für die Mathematik, vol. 122 (1900), p. 61; BROMWICH, Proceedings of the London Mathematical Society, vol. 32 (1900), p. 162.

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where, in case II, we have one pair of conjugate complex roots, indicated by the bar over the third and fourth indices.

We can now write down the standard types of terms which occur in the reduced* equations to the quadrics $A$, $B$; lists of these types will be found in the papers already cited and in MUTH's *Elementartheiler*.

Let $c$ be the typical real root of the determinant $|\lambda A - B|$, then the standard types are given by the following table: —

| Invariant-factor | $\lambda - c$ | $(\lambda - c)^2$ | $(\lambda - c)^3$ |
|---|---|---|---|
| Terms in $A$ | $\pm x^2$ | $2xy$ | $2xy + z^2$ |
| Terms in $B$ | $\pm cx^2$ | $2cxy \pm x^2$ | $c(2xy + z^2) + 2zx$ |

where $x$, $y$, $z$ are all real and do not appear in any other terms.

Some modifications must, however, be introduced, in order to retain the regular type for the absolute $A$: and so we put (in the second and third cases) $x = \xi - \eta$, $2y = \xi + \eta$, obtaining the table: —

| Invariant-factor | $(\lambda - c)^2$ | $(\lambda - c)^3$ |
|---|---|---|
| Terms in $A$ | $\xi^2 - \eta^2$ | $\xi^2 - \eta^2 + z^2$ |
| Terms in $B$ | $c(\xi^2 - \eta^2) \pm (\xi - \eta)^2$ | $c(\xi^2 - \eta^2 + z^2) + 2z(\xi - \eta)$ |

Another change is necessary, when we have a pair of *complex* roots; if these are $\alpha + i\beta$, $\alpha - i\beta$, the corresponding reduced terms will be given by: —

| Invariant-factor | $(\lambda - \alpha)^2 + \beta^2$ |
|---|---|
| Terms in $A$ | $\frac{1}{2}(x + iy)^2 + \frac{1}{2}(x - iy)^2 = x^2 - y^2$ |
| Terms in $B$ | $\frac{1}{2}[(\alpha - i\beta)(x + iy)^2 + (\alpha + i\beta)(x - iy)^2] = \alpha(x^2 - y^2) + 2\beta xy$ |

where $x$ and $y$ are real.

We have now sufficient material to write down all the types of quadrics by inspection.†

## § 2. *Canonical forms for the possible quadrics in hyperbolic space.*

*Type I.* Indices 1111.

$$A = y_1^2 + y_2^2 + y_3^2 - y_4^2,$$
$$B = c_1 y_1^2 + c_2 y_2^2 + c_3 y_3^2 - c_4 y_4^2, \qquad |B| = -c_1 c_2 c_3 c_4$$

where $c_1$, $c_2$, $c_3$, $c_4$ are roots of the determinantal equation in $\lambda$, $|\lambda A - B| = 0$.

* It is to be borne in mind that the invariant-factors can be calculated from the equation to $B$, given in any form; indeed they must, in general, be found before the reduction can be carried out.

† Since $|A| = -1$, there is here no reason to put down the types corresponding to the *identical* vanishing of $|\lambda A - B|$; nor to an infinite root of $|\lambda A - B| = 0$.

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I($a$). $c_1$, $c_2$, $c_3$, $c_4$ are all different, represented by the symbol [1111]. This case gives the *ellipsoid* and four *hyperboloids* found by COOLIDGE* (pp. 162, 163, l. c.).

I($b$). $c_1 = c_2$ *or* $c_3 = c_4$, represented by [(11)11]. The case $c_1 = c_2$ gives the *spheroids* and *hyperboloids of revolution* (C., p. 168); $c_3 = c_4$ gives the *surfaces of translation* (C., § 4, p. 169).

I($c$). $c_1 = c_2 = c_3$ *or* $c_1 = c_2 = c_4$; represented by [(111)1]. These cases appear as a *sphere* and an *equidistant surface*, respectively (C., p. 170).

I($d$). $c_1 = c_2$ *and* $c_3 = c_4$; represented by [(11)(11)]. This gives the *analogue of Clifford's surface* in elliptic space (C., end of § 3, p. 169).

I($e$). $c_1 = c_2 = c_3 = c_4$; represented by [(1111)]. This case is trivial, geometrically, as the quadric coincides with the absolute; it is, of course, algebraically distinct.

*Type II.* Indices $11\,\bar{1}\bar{1}$.

$$A = y_1^2 + y_2^2 + y_3^2 - y_4^2,$$
$$B = c_1 y_1^2 + c_2 y_2^2 + \alpha(y_3^2 - y_4^2) + 2\beta y_3 y_4, \qquad |B| = -c_1 c_2(\alpha^2 + \beta^2),$$

where the roots of $|\lambda A - B| = 0$ are $c_1$, $c_2$, $\alpha + i\beta$, $\alpha - i\beta$.

II($a$). $c_1$, $c_2$ are unequal; represented by [$11\,\overline{11}$]. This is the *surface with imaginary centers* given by COOLIDGE (C., p. 165). But it may be well to point out that the apparently more general expression given by him is not actually more general than this (it contains five coefficients, one of which can be removed by linear transformations)†; and further, his equation has the disadvantage that its coefficients cannot be put down directly from the determinant $|\lambda A - B|$, before reduction.

II($b$). $c_1 = c_2$; represented by [$(11)\overline{11}$]. This is a *corresponding surface of revolution* (C. § 3, sub-case 4); here COOLIDGE's equation has the right number of arbitrary coefficients, but again they are not those naturally suggested by the determinant $|\lambda A - B|$; namely, $\alpha = \frac{1}{2}(a_3 - 1)$, $\beta = \frac{1}{2}[b^2 - (a_3 - 1)^2]^{\frac{1}{2}}$.

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* I do not stay to repeat COOLIDGE's sub-classification, which depends on the signs and relative magnitudes of the $c$'s; and, in future, I quote his paper by the letter C.

† More exactly, two of them can be made equal, thus: Let $p$ be that root of the quadratic $p + p^{-1} = 4b(a_3 + a_4)^{-1}$, which is numerically less than 1. Write

$$\alpha = \tfrac{1}{2}(a_3 - a_4), \qquad \beta = \tfrac{1}{4}(a_3 + a_4)(1 - p^2)p^{-1},$$
$$y_3 = (x_3 - px_4)(1 - p^2)^{-\frac{1}{2}}, \qquad y_4 = (px_3 - x_4)(1 - p^2)^{-\frac{1}{2}};$$

then we find

$$x_3^2 - x_4^2 = y_3^2 - y_4^2, \qquad a_3 x_3^2 - 2bx_3 x_4 + a_4 x_4^2 = \alpha(y_3^2 - y_4^2) + 2\beta y_3 y_4;$$

where

$$\alpha = \tfrac{1}{2}(a_3 - a_4) \text{ and } \beta = [b^2 - \tfrac{1}{4}(a_3 + a_4)^2]^{\frac{1}{2}}.$$

Of course $p$ is real, because of the condition $4b^2 > (a_3 + a_4)^2$.

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It might be thought that we could have cases to correspond to I($c$), I($d$); but this is not so, on the ground of our restriction to reality.

*Type III.* Indices 211.

$$A = y_1^2 + y_2^2 + y_3^2 - y_4^2,$$
$$B = c_1 y_1^2 + c_2 y_2^2 + c_3(y_3^2 - y_4^2) \pm (y_3 - y_4)^2, \qquad |B| = -c_1 c_2 c_3^2,$$

the roots of $|\lambda A - B| = 0$ being $\lambda = c_1, c_2, c_3, c_3$.

III($a$). $c_1, c_2, c_3$ all different; represented by [211]. This gives the *elliptic and hyperbolic paraboloids* * (C., p. 166).

III($b$). $c_2 = c_3$; represented by [(21)1]. This is the *horocyclic paraboloid.* (C., p. 167).

III($c$). $c_1 = c_2 = c_3$; represented by [(211)]. This is the *horocyclic surface of zero curvature* (C., p. 170); in this case the arbitrary sign must be positive, to avoid an ideal surface.

III($d$). $c_1 = c_2$; represented by [2(11)]. This gives the *paraboloids of revolution* (C., p. 169).

*Type IV.* Indices 31.

$$A = y_1^2 + y_2^2 + y_3^2 - y_4^2,$$
$$B = c_1 y_1^2 + c_2(y_2^2 + y_3^2 - y_4^2) + 2y_2(y_3 - y_4), \qquad |B| = -c_1 c_2^3,$$

the roots of $|\lambda A - B| = 0$, being $c_1, c_2, c_2, c_2$.

IV($a$). $c_1 \gtrless c_2$; represented by [31]. This is a surface to which COOLIDGE has not assigned a name; it is the first on p. 167 (C.). The equation given there contains two superfluous constants $a_2, a_3$; $a_2$ may be replaced by 1, and $a_3$ by 0, without loss of generality.†

* Again I do not stop to give the sub-classifications. But it may be well to point out that the coefficients $a_3$, $a_4$ in COOLIDGE's equation are not invariants; the essentials are $c_3 = \frac{1}{2}(a_3 + a_4)$ and the *sign* of $(a_3 - a_4)$. For if $a_3 - a_4 = 2eq^2$, where $e = \pm 1$, and $q$ is real, we can write $y_3 - y_4 = q(x_3 - x_4)$, $y_3 + y_4 = (x_3 + x_4)/q$ and then $y_3^2 - y_4^2 = x_3^2 - x_4^2$, while

$$a_3 x_3^2 + (a_4 - a_3) x_3 x_4 - a_4 x_4^2 = c_3(y_3^2 - y_4^2) + e(y_3 - y_4)^2.$$

Similarly in COOLIDGE's type for III($b$), the constant $a$ is superfluous, except as to sign; in the type for III($c$), $a$ must be positive and may be put equal to $+1$.

† To prove this statement, call COOLIDGE's equation $B$; $A$ being the equation to the absolute. Then

$$B - A = (a_1 - 1)x_1^2 + (x_3 - x_4)[2a_2 x_2 + a_3(x_3 - x_4)] = (a_1 - 1)x_1^2 + 2y_2(y_3 - y_4),$$

provided that $2a_2 y_2 = 2a_2 x_2 + a_3(x_3 - x_4)$, $y_3 - y_4 = a_2(x_3 - x_4)$. Now substitute in $A$ for $x_2$ in terms of $y_2, y_3, y_4$; thus we find

$$A = x_1^2 + y_2^2 + (y_3 - y_4)\left[-\frac{a_3 y_2}{a_2^2} + \frac{a_3^2}{4a_2^4}(y_3 - y_4) + \frac{1}{a_2}(x_3 + x_4)\right].$$

Call the last expression in square brackets $(y_3 + y_4)$, then

$$A = x_1^2 + y_2^2 + y_3^2 - y_4^2, \qquad B - A = (a_1 - 1)x_1^2 + 2y_2(y_3 - y_4).$$

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IV($b$). $c_1 = c_2$; represented by [(31)]. This is the third surface considered on p. 167 (C.); in the equation there given we may take $r = 2$ without loss of generality.

We have now exhausted all possible quadric surfaces, in hyperbolic space. There are thus *twelve* fundamental classes, represented by the symbols given in the table

| | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| I | 1111 | (11)11 | (111)1 | (11)(11) |
| II | $11\bar{1}\bar{1}$ | (11)11 | | |
| III | 211 | (21)1 | (211) | 2(11) |
| IV | 31 | (31) | | |

This classification can be completely carried out by an examination of $|\lambda A - B|$, without any preliminary reduction. But, for the further subdivisions, a certain amount of reduction is necessary; as it is not possible to foretell in I which of the four roots $c_1, c_2, c_3, c_4$ is to be associated with the negative term in $A$. More can be done in III; for $c_3$ is necessarily associated with the negative square; but here the sign of $(y_3 - y_4)^2$ cannot be foretold. In cases II and IV we can obtain complete information from the determinant; as may be seen by considering the reduced forms.

## § 3. *Quadrics in elliptic space.*

The absolute is here a positive quadratic form and its signature is 4; hence the only indices possible are 1111, and the roots $c_1, c_2, c_3, c_4$ are all *real.** Thus we are restricted to the possibilities given by the row numbered I in the table at the end of § 2.

Hence every quadric in elliptic space can be reduced to the form

$$c_1 y_1^2 + c_2 y_2^2 + c_3 y_3^2 + c_4 y_4^2 = 0$$

the absolute being $y_1^2 + y_2^2 + y_3^2 + y_4^2 = 0$; the coefficients $c_1, c_2, c_3, c_4$ are all real, being the roots of $|\lambda A - B| = 0$.

The details of the cases are:

I($a$). [1111]; *surfaces not of revolution*, ruled or not according as the product $c_1 c_2 c_3 c_4$ is positive or negative.

I($b$). [(11)11]; *surfaces of revolution* ($c_1 = c_2$), ruled or not according as $c_3 c_4$ is positive or negative.

I($c$). [(111)1]; *spheres* (not ruled), $c_1 = c_2 = c_3$.

* This fact follows also from two theorems due to WEIERSTRASS (Berliner Monatsberichte, 1858, p. 207; *Gesammelte Werke*, vol. 1, p. 233; the theorems are given in §§ 3, 4).

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I($d$). [(11)(11)]; CLIFFORD'S *surfaces*, (ruled) $c_1 = c_2$, $c_3 = c_4$.
It may be remarked that there are no hyperboloids in any of these cases.

## § 4. *Quadrics in parabolic (Euclidean) space.*

The classification of quadrics in Euclidean space is carried out in most of the ordinary text-books on solid geometry, and details need hardly be given here; but a few remarks may be added, to indicate the contrast to the cases considered in §§ 2, 3.

In parabolic space the absolute is degenerate; its equation can be expressed only in terms of plane-coördinates $u_1, u_2, u_3, u_4$. If then $A = 0$ is the absolute, it will be found that the determinant $|A| = 0$, and that the signature of $A$ is 3. Thus, if $B = 0$ is any quadric (in plane-coördinates) the determinant* $|\lambda A + \mu B|$ has

*either* (i) four linear invariant-factors $\lambda + \mu c_1, \lambda + \mu c_2, \lambda + \mu c_3, \mu$;

*or* (ii) two linear invariant-factors $\lambda + \mu c_1, \lambda + \mu c_2$ and one $\mu^2$;

where, in each case, $c_1, c_2, c_3$ (or $c_1, c_2$) are *real*.

From these facts we see that, by applying WEIERSTRASS'S methods (in conjunction with a slight modification† so as to include the factors $\mu$, or $\mu^2$), the reduced types are

$$\text{(i)} \qquad \begin{cases} A = v_1^2 + v_2^2 + v_3^2, \\ B = c_1 v_1^2 + c_2 v_2^2 + c_3 v_3^2 + c_4 v_4^2, \end{cases}$$

$$\text{(ii)} \qquad \begin{cases} A = v_1^2 + v_2^2 + v_3^2, \\ B = c_1 v_1^2 + c_2 v_2^2 + 2f v_3 v_4. \end{cases}$$

Here the coefficients $c_4$, $f$ cannot be found from the determinant $|\lambda A + \mu B|$; it is not usually necessary to find them merely for the purpose of classification.‡ But they must be found if we require the precise size of the surface $B$: one method has been given by the present writer elsewhere.§

Another possibility may occur if $|B| = 0$; it may happen that

$$|\lambda A + \mu B| = 0,$$

* Either by the modified form of KLEIN'S theorem given above; or by a theorem due to GUNDELFINGER which is given in his edition of HESSE'S lectures on geometry and also in his lectures on conics (p. 67). See MUTH'S *Elementartheiler*, p. 180.

† Proceedings of the London Mathematical Society, vol. 32 (1900), p. 158.

‡ To distinguish between ruled and non-ruled surfaces, the sign of $|B|$ is sufficient

§ Proceedings of the Cambridge Philosophical Society, vol. 10 (1901), p. 358 (see § 3); see also BRÜCKEL'S collection of GUNDELFINGER'S formulae in Crelle's Journal für die Mathematik, vol. 119 (1898), pp. 210, 313.

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identically. If so, the reduced forms are

$$\text{(iii)} \qquad \begin{cases} A = v_1^2 + v_2^2 + v_3^2, \\ B = c_1 v_1^2 + c_2 v_2^2 + c_3 v_3^2, \end{cases}$$

$B$ being a conic in the plane at infinity.

But if a quadric given in point-coördinates is degenerate the method just explained breaks down entirely; for the transformation of such an equation to plane-coördinates is no longer possible.* To meet this difficulty an alternative method of reduction is explained in §§1, 2 of the paper just quoted; and a simple general method of classification is obtained. But the process by which the result is obtained is not very easy; and it may therefore be worth while to state the result in order to make it accessible to those who do not care to go through the details which are unavoidable in finding the reducing substitution. Let $R = 0$ be the given quadric, expressed in point-coördinates $x_1, x_2, x_3, x_4$; $S = 0$ any sphere, the coefficients of $S$ having been so chosen that the expression $S/(p_1x_1 + p_2x_2 + p_3x_3 + p_4x_4)^2$ represents the square of the tangent from $(x_1, x_2, x_3, x_4)$ to the sphere, where $p_1x_1 + p_2x_2 + p_3x_3 + p_4x_4 = 0$ is the plane at infinity. Write $\Delta_0$ for the determinant $|\lambda R - S|$; and $\Delta_1$ for the symmetrical determinant formed by adding a fifth row and column $(p_1, p_2, p_3, p_4, 0)$ to $\Delta_0$. Then the elements of classification are†:

(i) the factors of $\Delta_1$; say $1 - \lambda c_1$, $1 - \lambda c_2$, $1 - \lambda c_3$; where $c_1$, $c_2$, $c_3$ are all real and some may be zero.

(ii) the first term (say $k\lambda^\alpha$) in the expansion of $(-\Delta_0/\Delta_1)$ in *descending* powers of $\lambda$.

When the quadric is transformed to orthogonal Cartesian coördinates (say $y_1, y_2, y_3$), the terms of the second degree can be reduced to

$$c_1y_1^2 + c_2y_2^2 + c_3y_3^2$$

and, as to the remaining terms, three cases are possible, according to the value of the integer $\alpha$. To put the work in as short a form as possible, we assume (from the results of the elementary investigations) that $R$, $S$ must take the forms

$$R = c_1y_1^2 + c_2y_2^2 + c_3y_3^2 + 2fy_3 + c_4,$$

$$S = y_1^2 + y_2^2 + y_3^2 + 2(l_1y_1 + l_2y_2 + l_3y_3) + m,$$

* More precisely, the transformed equation either vanishes identically or fails to give sufficient information. It should be observed that the corresponding difficulty does not arise in hyperbolic or elliptic space; for the absolute may be expressed either in point or in plane-coördinates.

† It is the second of these which constitutes the chief simplification introduced by this method of classification.

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where, in $R$, *either* $f$ is zero, *or* both $c_3$, $c_4$ are zero. It is then easy to see that

$$\Delta_1 = -M^2(\lambda c_1 - 1)(\lambda c_2 - 1)(\lambda c_3 - 1),$$

$$\Delta_0 = M^2[(\lambda c_1 - 1)(\lambda c_2 - 1)(\lambda c_3 - 1)(\lambda c_4 - m) - l_1^2(\lambda c_2 - 1)(\lambda c_3 - 1) - l_2^2(\lambda c_3 - 1)(\lambda c_1 - 1) - (f\lambda - l_3)^2(\lambda c_1 - 1)(\lambda c_2 - 1)],$$

$M$ being the determinant of a certain linear substitution. From this we find that $c_1$, $c_2$, $c_3$ are the letters obtained in (i) from the factors of $\Delta_1$; also that

$$-\frac{\Delta_0}{\Delta_1} = \lambda c_4 - m - \frac{l_1^2}{\lambda c_1 - 1} - \frac{l_2^2}{\lambda c_2 - 1} - \frac{(f\lambda - l_3)^2}{\lambda c_3 - 1}.$$

Now, according to (ii), $k\lambda^\alpha$ is the first term of this expression when expanded in powers of $1/\lambda$; thus, we have three cases:

(1) $f = 0$; then $k\lambda^\alpha = \lambda c_4$; so that $k = c_4$, $\alpha = 1$,

(2) $f = 0$, $c_4 = 0$; here $\alpha < 1$,

(3) $c_3 = 0$, $c_4 = 0$; then $k\lambda^\alpha = (f\lambda)^2$; so that $k = f^2$, $\alpha = 2$.

Thus, classifying by means of the index $\alpha$, we may write the results

$$\alpha < 1, \qquad R = c_1y_1^2 + c_2y_2^2 + c_3y_3^2,$$

$$\alpha = 1, \qquad R = c_1y_1^2 + c_2y_2^2 + c_3y_3^2 + k,$$

$$\alpha = 2, \qquad R = c_1y_1^2 + c_2y_2^2 + 2k^{\frac{1}{2}}y_3,$$

giving a complete classification of all quadrics* in point-coördinates.

QUEEN'S COLLEGE, GALWAY,
*February* 22, 1904.

ADDITION. Mr. COOLIDGE has kindly recalled to my memory the fact that a classification of quadrics, in hyperbolic and elliptic space, had been previously given by P. BARBARIN.† In this classification of hyperbolic space, the classes I, II, III, are the same as the classes which bear these numbers in § 2 above; but class IV is left out. The omission is due to an oversight in connection with the transformations given at the top of p. 113 (l. c.), § 28; in fact if the absolute is $x^2 + y^2 + z^2 - u^2 = 0$, the quadric $z^2 + 2x(y - u) = 0$ has only one real principal plane ($z = 0$); and this quadric cannot be reduced to BARBARIN's typical equation (40), $S_1X^2 + S_2y^2 + S_2Z^2 - 2PUX = 0$. There is a similar difficulty with any quadric belonging to class IV of § 2.

* It may be worth while to point out that if the quadric is degenerate one or more of $c_1$, $c_2$, $c_3$ may be zero; but this involves no change in the calculation of $c_4$ and $f$.

† *Études de Géometrie Non-Euclidienne*, Bruxelles, 1900, pp. 101–126; this is an extract (published separately) from vol. 60 of the Mémoires couronnés et autres Mémoires publiés par l'Académie royale de Belgique.

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It is to be noticed that BARBARIN's nomenclature differs in some respects from COOLIDGE's, which has been used above; the former gives the generic name *paraboloid* to class II, and *horiquadric* to class III. Although there is an advantage in giving a name to each class, yet it does not appear to me that the name paraboloid is specially suited to class II, the quadrics of which have imaginary centers.

In conclusion, the classification of conics in the hyperbolic plane, carried out on the lines of §§ 1, 2, is:

I($a$): $[111]$; $c_1y_1^2 + c_2y_2^2 - c_3y_3^2 = 0$.

($b$): $[(11)1]$; $c_1(y_1^2 + y_2^2) - c_3y_3^2 = 0$, and $c_1y_1^2 + c_2(y_2^2 - y_3^2) = 0$.

($c$): $[(111)]$; $y_1^2 + y_2^2 - y_3^2 = 0$ (the *absolute*).

II: $[\overline{11}1]$; $c_1y_1^2 + \alpha(y_2^2 - y_3^2) + 2\beta y_2y_3 = 0$.

III($a$): $[21]$; $c_1y_1^2 + c_2(y_2^2 - y_3^2) \pm (y_2 - y_3)^2 = 0$.

($b$): $[(21)]$; $c_1(y_1^2 + y_2^2 - y_3^2) \pm (y_2 - y_3)^2 = 0$.

IV: $[3]$; $c_1(y_1^2 + y_2^2 - y_3^2) + 2y_1(y_2 - y_3) = 0$.

In each case the absolute is $y_1^2 + y_2^2 - y_3^2 = 0$; and, according to KLEIN's theorem, there is at least one real invariant factor of odd degree.

The numbering of classes I, II, III corresponds to that of BARBARIN (l. c. pp. 39–57), who omits class IV; in fact the method given on p. 33 for determining the rotation $\theta$ fails if applied to the conics IV; and these conics cannot be reduced to the form contained in BARBARIN's equation ($20_1$) on p. 40, which is the basis of his subsequent work.

It is of course possible to give a classification of conics in the elliptic and in the parabolic planes, by a suitable modification of the work above; but the necessary changes are so obvious that it is hardly worth while to enumerate them.

*March* 15, 1905



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# ON DIFFERENTIAL INVARIANTS*

BY

JOSEPH EDMUND WRIGHT

## *Introduction.*

In the consideration of differential equations there arise expressions, such as for instance, the JACOBI-POISSON alternant of two first order partial equations, which are in their nature invariantive to all contact transformations. An important problem immediately presents itself, namely, the obtaining of all invariants of this type, that is of all invariants, with respect to contact transformations, of differential expressions.

In this paper are obtained all such invariants of a restricted type.

The restrictions are the following:—

(1) The only expressions considered are: (*a*) expressions of the first order with $m$ dependent and $n$ independent variables; (*b*) expressions of the second order with one dependent variable.

(2) The invariants are only of the first order, that is to say, they involve only first derivatives of the differential expressions.

The variables assumed to occur in case (*a*) are

$$x_1, \cdots, x_n \text{ (the independent variables)},$$

$$z_1, \cdots, z_m \text{ (the dependent variables)},$$

$$p_k^i \equiv \frac{\partial z_i}{\partial x_k} \qquad (i=1, \cdots, m;\ k=1, \cdots, n),$$

$$p_{kl}^i \equiv \frac{\partial^2 z_i}{\partial x_k \partial x_l} \qquad (i=1, \cdots, m;\ k, l=1, \cdots, n),$$

and

$$\frac{\partial f_\lambda}{\partial x_k}, \quad \frac{\partial f_\lambda}{\partial z_i}, \quad \frac{\partial f_\lambda}{\partial p_k^i} \qquad (\lambda=1, \cdots, r;\ i=1, \cdots, m;\ k=1, \cdots, n),$$

where $f_\lambda(x, z, p_k^i)$, $(\lambda = 1, \cdots, r)$, are the differential expressions considered.

In addition the variables $dx_k$, $dz_i$, $dp_k^i$ will be assumed to enter, subject to the conditions

$$dz_i = \sum_{k=1}^{n} p_k^i dx_k \qquad (i = 1 \cdots m),$$

$$dp_k^i = \sum_{l=1}^{n} p_{kl}^i dx_l \qquad (i=1, \cdots, m;\ k=1, \cdots, n).$$

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In invariants corresponding to case (*b*) the variables are the same, with the exception that $m$ is 1; and that there are additional variables

$$\frac{\partial f_\lambda}{\partial p_{kl}} \qquad (\lambda = 1, \cdots, r; k, l = 1, \cdots, n),$$

where $f_\lambda(x, z, p_k, p_{kl})$ are the differential expressions considered.

The method used is a modification of that given by LIE in his paper "Ueber Differentialinvarianten,"* in which invariants for certain simpler types of infinitesimal transformations are obtained. In this paper LIE shows (p. 566) that by the method there outlined, a series of invariants may be obtained satisfying a system of linear differential equations which form a complete system.

In the paper mentioned LIE suggests a problem connected with invariants of surfaces, the class desired being that which does not change owing to "deformation" of the surface.

ŻORAWSKI † attempted the solution of this question and found a class of such invariants.

FORSYTH, ‡ in 1903 attacked the more general question of invariants due to a purely arbitrary point transformation performed on the surface, and also obtained in this manner the differential invariants of space. In his paper certain modifications are made on the LIE-ŻORAWSKI method, one modification being that he sought for relative, as well as absolute, invariants. The method as modified by FORSYTH will be used here.

The invariant sought will, therefore, be such that if $F$ denote its expression in the original, $F_1$ in the transformed variables, we shall have

$$F_1 = \Omega F,$$

where $\Omega$ is a function depending only on the transformation.

Now if the transformation were a general one in the variables considered, it is well known that $\Omega$ would be some power of the Jacobian of the transformation. But the transformation is not perfectly general.

In fact, in our most general case, the Jacobian of $X_k$, $Z_i$, $P^i_k$, $P^i_{kl}$ with reference to $x_k$, $z_i$, $p^i_k$, $p^i_{kl}$, where capitals denote transformed variables, and where $\{k, l = 1, 2, \cdots n; i = 1, 2 \cdots m\}$ breaks up into two factors, the first of which is

$$\bar{J}_1 \equiv J\left(\frac{X_1 X_2 \cdots X_n Z_1 \cdots Z_m P'_1 P'_2 \cdots P^m_n}{x_1 x_2 \cdots p^m_n}\right),$$

and the second

$$J_2 \equiv J \cdot \left(\frac{\cdots P^i_{kl} \cdots}{\cdots p^i_{kl} \cdots}\right).$$

* LIE, Mathematische Annalen, vol. 24 (1884), pp. 537–578.

† ŻORAWSKI, Acta Mathematica, vol. 16 (1892–3), pp. 1–64.

‡ FORSYTH, Philosophical Transactions, ser. A., vol. 201 (1903), pp. 329–402, and ser. A., vol. 202 (1904), pp. 277–333.

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Further, if the transformation is a point transformation, $\bar{J}_1$ breaks up into two factors,

$$J_0 \equiv J\left(\frac{X_1 X_2 \cdots X_n Z_1 \cdots Z_m}{x_1 x_2 \cdots x_n z_1 \cdots z_m}\right),$$

$$J_1 \equiv J\left(\frac{P_1' P_2' \cdots P_n^m}{p_1' p_2' \cdots p_n^m}\right).$$

Now if the number of dependent variables is greater than one, it may easily be shown that the most general contact transformation possible is an extended point transformation.

The discussion will be limited to those cases in which the factor $\Omega$ is of the form

$$J_0^{\mu_0} J_1^{\mu_1} J_2^{\mu_2} \cdots$$

when the number of dependent variables is greater than one, and

$$\bar{J}_1^{\mu_1} J_2^{\mu_2} \cdots$$

when there is only one dependent variable.

Now let $F$ be an invariant of the type considered, and let an infinitesimal contact transformation be performed on $F$.

The condition for invariance is that, $t$ being the parameter of the transformation,

$$F_1 = \Omega F, \qquad \text{or} \qquad \frac{dF}{dt} = F\frac{d\Omega}{dt}$$

when $d\phi/dt$ denotes the increment of $\phi$ due to the infinitesimal transformation and $F_1$ is $F$ in the tranformed variables. Expressing the fact that this equation holds for all such transformations as considered, we obtain a complete system of linear differential equations, the solutions of which are the invariants desired.

In the course of the work the values of certain increments are required, and they will be given now, before we consider the various cases in detail.

The following notation is used throughout:—

$$\frac{dx_k}{dt} \equiv \xi_k, \qquad \frac{dz_i}{dt} \equiv \zeta_i, \qquad \frac{dp_k^i}{dt} \equiv \pi_k^i, \qquad \frac{dp_{kl}^i}{dt} \equiv \pi_{kl}^i, \qquad \theta_i \equiv \zeta_i - \sum_{k=1}^{n} p_k^i \xi_k,$$

$$\frac{\partial f}{\partial x_k} \equiv X_k, \qquad \frac{\partial f}{\partial z_i} \equiv Z_i,$$

$$\frac{\partial f}{\partial p_k^i} \equiv P_k^i, \qquad \frac{\partial f}{\partial p_{kl}^i} \equiv P_{kl}^i,$$

where $f$ is used to denote one of the forms considered. If it is desired to specify any one of the $f$'s particularly, the notation

$$f_\lambda, \qquad X_{\lambda,k}, \qquad P_{\lambda,k}^i, \qquad P_{\lambda,kl}^i,$$

etc., is used.

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$\mathbf{S}$ denotes a summation taken over all the expressions $f$.

In addition,

$$dx_k \equiv a_k, \qquad dz_i \equiv b_i, \qquad dp_k^i \equiv c_k^i, \qquad dp_{kl}^i \equiv c_{kl}^i,$$

$$\frac{d}{dx_k} \equiv \frac{\partial}{\partial x_k} + \sum_{\lambda=1}^{m} p_k^\lambda \frac{\partial}{\partial z_\lambda} + \sum_{\lambda,\, l} p_{kl}^\lambda \frac{\partial}{\partial p_l^\lambda}.$$

Using this notation we have the following increments:

$$\frac{dJ_0}{dt} \equiv \sum_{i=1}^{m} \frac{\partial \theta_i}{\partial z_i} + \sum_{k=1}^{n} \frac{d\xi_k}{dx_k},$$

$$\frac{dJ_1}{dt} \equiv n \sum_{i=1}^{m} \frac{\partial \theta_i}{\partial z_i} - m \sum_{k=1}^{n} \frac{d\xi_k}{dx_k},$$

$$\frac{dJ_2}{dt} \equiv \tfrac{1}{2} n(n+1) \sum_{i=1}^{m} \frac{\partial \theta_i}{\partial z_i} - m(n+1) \sum_{k=1}^{n} \frac{d\xi_k}{dx_k}$$

when $m$ is greater than unity, and

$$\frac{d\bar{J}_1}{dt} \equiv (n+1) \frac{\partial \theta}{\partial z},$$

$$\frac{dJ_2}{dt} \equiv (n+1) \sum_{\lambda=1}^{n} \frac{d}{dx_\lambda} \left( \frac{\partial \theta}{\partial p_\lambda} \right) + \tfrac{1}{2} n(n+1) \frac{\partial \theta}{\partial z}$$

when $m$ is unity.

The quantities $\pi_k^i$, $\pi_{hk}^i$, etc., are readily obtained in terms of the $\xi$'s and $\zeta$'s and their derivatives by the method given in LIE-ENGEL, *Theorie der Transformationsgruppen*, vol. 1, p. 544, *et seq.*

In the case, however, where $m$ is unity the increment of the variables in an extended infinitesimal contact transformation may be expressed in a particularly simple manner.

The theorem is as follows:

*Let there be an extended infinitesimal contact transformation in the $n$ independent variables $x_1, x_2, \cdots, x_n$ and the dependent variable $z$, and let $p_{hkl\cdots}$ denote $\partial^r z / \partial x_h \partial x_k \partial x_l \cdots$, where there are $r$ letters $h, k, l, \cdots$. Then the increment of $p_{hkl\cdots}$ due to the transformation is $(d^r\theta / dx_h dx_k dx_l \cdots)\delta t$, where $\theta$, with the usual notation, is equal to*

$$\zeta - \sum_{l=1}^{n} p_l \xi_l,$$

*and $(d^r\theta / dx_h dx_k dx_l \cdots)$ denotes a total differentiation of $\theta$ in which the terms containing the highest derivatives of $z$ are omitted.*

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For example, in the case when there are two independent variables,

$$\pi_1 = \frac{\partial \theta}{\partial x_1} + p_1 \frac{\partial \theta}{\partial z}, \qquad \pi_2 = \frac{\partial \theta}{\partial x_2} + p_2 \frac{\partial \theta}{\partial z},$$

$$\pi_{11} = \frac{\partial^2 \theta}{\partial x_1^2} + 2p_1 \frac{\partial^2 \theta}{\partial z\, \partial x_1} + p_1^2 \frac{\partial^2 \theta}{\partial z^2}$$

$$+ 2p_{11}\left(\frac{\partial^2 \theta}{\partial x_1 \partial p_1} + p_1 \frac{\partial^2 \theta}{\partial z \partial p_1}\right) + 2p_{12}\left(\frac{\partial^2 \theta}{\partial x_1 \partial p_2} + p_1 \frac{\partial^2 \theta}{\partial z \partial p_2}\right)$$

$$+ p_{11}^2 \frac{\partial^2 \theta}{\partial p_1^2} + 2p_{11}p_{12} \frac{\partial^2 \theta}{\partial p_1 \partial p_2} + p_{12}^2 \frac{\partial^2 \theta}{\partial p_2^2} + p_{11} \frac{\partial \theta}{\partial z}, \text{ etc.}$$

This theorem is known to be true * for the increments of the first derivatives of $z$, and it may be easily proved for higher derivatives by induction.

The increments of the quantities $X_{\lambda,k}$, $Z_{\lambda,i}$, $P^i_{\lambda,k}$, $P^i_{\lambda,hk}$, owing to the infinitesimal contact transformation are determined by the method given by FORSYTH in his paper already quoted.†

Using this method, we have the results

$$-\frac{dX_\lambda}{dt} = \sum_{k=1}^{n} \frac{\partial \xi_k}{\partial x_\lambda} X_k + \sum_{i=1}^{m} \frac{\partial \zeta_i}{\partial x_\lambda} Z_i + \sum_{ik} \frac{\partial \pi^i_k}{\partial x_\lambda} P^i_k + \sum_{ihk} \frac{\partial \pi^i_{hk}}{\partial x_\lambda} P^i_{hk},$$

$$-\frac{dZ_\lambda}{dt} = \sum_{k=1}^{n} \frac{\partial \xi_k}{\partial z_\lambda} X_k + \sum_{i=1}^{m} \frac{\partial \zeta_i}{\partial z_\lambda} Z_i + \sum_{ik} \frac{\partial \pi^i_k}{\partial z_\lambda} P^i_k + \sum_{ihk} \frac{\partial \pi^i_{hk}}{\partial z_\lambda} P^i_{hk},$$

$$-\frac{dP^\lambda_\mu}{dt} = \sum_{k=1}^{n} \frac{\partial \xi_k}{\partial p^\lambda_\mu} X_k + \sum_{i=1}^{m} \frac{\partial \zeta_i}{\partial p^\lambda_\mu} Z_i + \sum_{ik} \frac{\partial \pi^i_k}{\partial p^\lambda_\mu} P^i_k + \sum_{ihk} \frac{\partial \pi^i_{hk}}{\partial p^\lambda_\mu} P^i_{hk},$$

$$-\frac{dP^\lambda_{\mu\nu}}{dt} = \sum_{k=1}^{n} \frac{\partial \xi_k}{\partial p^\lambda_{\mu\nu}} X_k + \sum_{i=1}^{m} \frac{\partial \zeta_i}{\partial p^\lambda_{\mu\nu}} Z_i + \sum_{ik} \frac{\partial \pi^i_k}{\partial p^\lambda_{\mu\nu}} P^i_k + \sum_{ihk} \frac{\partial \pi^i_{hk}}{\partial p^\lambda_{\mu\nu}} P^i_{hk}.$$

The increments of the quantities $a_k$, $b_i$, $c^i_k$, etc., are readily calculated, for the transformation changes $x_k$ into $x_k + \xi_k \delta t$, and therefore $dx$ becomes $dx_k + d\xi_k \delta t$.

Hence

$$\frac{da_k}{dt} = \sum_{l=1}^{n} \frac{\partial \xi_k}{\partial x_l} a_l + \sum_{\lambda=1}^{m} \frac{\partial \xi_k}{\partial z_\lambda} b_\lambda + \sum_{l,\lambda} \frac{\partial \xi_k}{\partial p^\lambda_l} c^\lambda_l.$$

Similarly

$$\frac{db_i}{dt} = \sum_{l=1}^{n} \frac{\partial \zeta_i}{\partial x_l} a_l + \sum_{\lambda=1}^{m} \frac{\partial \zeta_i}{\partial z_\lambda} b_\lambda + \sum_{l,\lambda} \frac{\partial \zeta_i}{\partial p^\lambda_l} c^\lambda_l,$$

with similar expressions for the other increments of this type.

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* See LIE-ENGEL, *Theorie der Transformationsgruppen*, vol. 2, p. 82, 252.

† FORSYTH, Philosophical Transactions, vol. 201, p. 337, 338.

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§ 1.

We shall now proceed to the determination of invariants of expressions of the first order. The case in which there is only one dependent variable differs from the others in that the most general infinitesimal contact transformation is not an extended point transformation. This case will therefore be considered independently of the other.

Assuming $F$ to be an invariant, and performing on it the infinitesimal transformation corresponding to a function

$$\theta \equiv \zeta - \sum_{k=1}^{n} p_k \xi_k,$$

we have the equation

$$\frac{dF}{dt} + \mu(n+1)\theta_z F = 0.$$

If the variables occurring in $F$ are

$$x_1, \cdots, x_n, z, p_1, \cdots, p_n, X_{\lambda,1}, \cdots, X_{\lambda,n}, Z_\lambda, P_{\lambda,1}, \cdots, P_{\lambda,n} \quad (\lambda = 1 \cdots r),$$

this equation becomes on expansion,

$$\begin{aligned}
&\mu(n+1)\theta_z F + \left(\theta - \sum_{i=1}^{n} p_i \theta_{p_i}\right)\frac{\partial F}{\partial z} + \sum_{i=1}^{n}(\theta_{x_i} + p_i\theta_z)\frac{\partial F}{\partial p_i} - \sum_{i=1}^{n}\theta_{p_i}\frac{\partial F}{\partial x_i} \\
&- \mathbf{S}\sum_{i=1}^{n}\frac{\partial F}{\partial X_i}\left\{\sum_{\rho=1}^{n} X_\rho(-\theta_{p_\rho x_i}) + \left(\theta_{x_i} - \sum_{\rho=1}^{n} p_\rho \theta_{p_\rho x_i}\right)Z + \sum_{\rho=1}^{n}(\theta_{x_\rho x_i} + p_\rho \theta_{x_i z})P_\rho\right\} \\
&- \mathbf{S}\frac{\partial F}{\partial Z}\left\{\sum_{\rho=1}^{n} X_\rho(-\theta_{p_\rho z}) + \left(\theta_z - \sum_{\rho=1}^{n} p_\rho \theta_{p_\rho z}\right)Z + \sum_{\rho=1}^{n}(\theta_{x_\rho z} + p_\rho \theta_{zz})P_\rho\right\} \\
&- \mathbf{S}\sum_{i=1}^{n}\frac{\partial F}{\partial P_i}\left\{\sum_{\rho=1}^{n} X_\rho(-\theta_{p_\rho p_i}) + \sum_{\rho=1}^{n}(-p_\rho \theta_{p_\rho p_i})Z\right. \\
&\left. + \sum_{\rho=1}^{n}(\theta_{p_i x_\rho} + p_\rho \theta_{z p_i})P_\rho + \theta_z P_i\right\} = 0.
\end{aligned}$$

Now $\theta$ is a perfectly arbitrary function of the variables $x_1 \ldots x_n, z, p_1 \ldots p_n$ and the above equation must be satisfied for all values of $\theta$.

Hence we may equate to zero the coefficients of the derivatives of $\theta$, and thus obtain the system of linear partial equations which $F$ must satisfy.

The system is the following:

From $\theta$

(1) $$F_z = 0.$$

From $\theta_{x_i}$,

(2) $$F_{p_i} - \mathbf{S}ZF_{X_i} = 0 \qquad (i = 1 \cdots n).$$

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From $\theta_{p_i}$,

$$F_{z_i} = 0 \qquad (i=1, \cdots, n). \tag{3}$$

From $\theta_z$,

$$\mu(n+1)F + \sum_{i=1}^{n} p_i F_{p_i} - \mathbf{S}ZF_Z - \mathbf{S}\sum_{i=1}^{n} P_i F_{P_i} = 0. \tag{4}$$

From $\theta_{x_i x_k}$,

$$\mathbf{S}P_k F_{X_i} + \mathbf{S}P_i F_{X_k} = 0 \qquad (i=1, \cdots, n;\ k=1, \cdots, n). \tag{5}$$

From $\theta_{x_i z}$,

$$\mathbf{S}\sum_{\rho=1}^{n} p_\rho P_\rho F_{X_i} + \mathbf{S}P_i F_Z = 0 \qquad (i=1, \cdots, n). \tag{6}$$

From $\theta_{x_i p_k}$,

$$\mathbf{S}(X_k + p_k Z) F_{X_i} - \mathbf{S}P_i F_{P_k} = 0 \qquad (i=1, \cdots, n;\ k=1, \cdots, n). \tag{7}$$

From $\theta_{zz}$,

$$\mathbf{S}\left(\sum_{\rho=1}^{n} p_\rho P_\rho\right) F_Z = 0. \tag{8}$$

From $\theta_{zp_i}$,

$$\mathbf{S}(X_i + p_i Z) F_Z - \mathbf{S}\left(\sum_{\rho=1}^{n} p_\rho P_\rho\right) F_{P_i} = 0 \qquad (i=1, \cdots, n). \tag{9}$$

From $\theta_{p_i p_k}$,

$$\mathbf{S}(X_k + p_k Z) F_{P_i} + \mathbf{S}(X_i + p_i Z) F_{P_k} = 0. \qquad (i, k=1, \cdots, n). \tag{10}$$

It follows from these equations that $F$ must be a function of the variables

$$A_{1,1}, \cdots, A_{1,n}, A_{2,1}, \cdots, A_{2,n}, \cdots, Z_1, Z_2, \cdots, Z_r, P_{1,1}, \cdots, P_{r,n},$$

where

$$A_{i,k} \equiv X_{i,k} + p_k Z_i \qquad (i=1, \cdots, r;\ k=1, \cdots, n).$$

If we modify the system of equations by assuming $F$ to be a function of these variables only, it becomes

$$\mu(n+1)F = \mathbf{S}ZF_Z + \mathbf{S}\sum_{i=1}^{n} P_i F_{P_i}, \tag{11}$$

$$\mathbf{S}P_i F_{A_k} + \mathbf{S}P_k F_{A_i} = 0 \qquad (i, k=1, \cdots, n), \tag{12}$$

$$\mathbf{S}P_i F_Z = 0 \qquad (i=1, \cdots, n), \tag{13}$$

$$\mathbf{S}A_k F_{A_i} - \mathbf{S}P_i F_{P_k} = 0 \qquad (i, k=1, \cdots, n), \tag{14}$$

$$\mathbf{S}A_i F_Z = 0 \qquad (i=1, \cdots, n), \tag{15}$$

$$\mathbf{S}A_i F_{P_k} + \mathbf{S}A_k F_{P_i} = 0 \qquad (i, k=1, \cdots, n). \tag{16}$$

If we put aside for the present the first of these equations, the remaining equations form a complete system. The number of functionally independent solu-

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tions is, however, not immediately deducible, as some of the equations may depend algebraically on the others. In particular, equations (13) and (15) show that if the number $r$ of expressions $f$ considered is not greater than $2n$, and if all the determinants of order $r$ of the matrix

$$\begin{Vmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n} & P_{1,1} & \cdots & P_{1,n} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,n} & P_{2,1} & \cdots & P_{2,n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ A_{r,1} & A_{r,2} & \cdots & A_{r,n} & P_{r,1} & \cdots & P_{r,n} \end{Vmatrix}$$

do not vanish, $F_z$ must be zero.

Further, there is no necessity to consider more than $2n+1$ expressions $f$, for if there were more than this number they could all be expressed by means of any $(2n+1)$ of them.

Suppose that there is only one expression $f$, then $F_{A_i}$, $F_z$, $F_{P_i}$ $(i=1, \cdots, n)$ are all zero, and therefore it follows that there is no invariant of the type sought of a single expression $f$.

Next let there be two expressions $f$. Then $F_{z_1}$, $F_{z_2}$ are zero unless

$$\frac{A_{1,1}}{A_{2,1}} = \frac{A_{1,2}}{A_{2,2}} = \cdots = \frac{P_{1,i}}{P_{2,i}} = \cdots.$$

But these relations are equivalent to the conditions that any equation $\phi(x_i z p_i) = 0$ which lies in involution with $f_1 = a$, shall lie in involution with $f_2 = b$, where $a$ and $b$ are arbitrary constants.

Now corresponding to a particular value of $a$ there are $\infty^{2n-1}$ characteristic strips which go to build up the integrals of the equation $f_1 = a$.

Hence taking account of all values of $a$, all the surface elements $(z x_i p_i)$ in space of $(n+1)$ dimensions are arranged in $\infty^{2n}$ characteristic strips.

The condition given above is easily seen to be the same as the condition that $f_2 = b$ determines the same system of $\infty^{2n}$ characteristic strips.

Exactly similarly in the case of $r$ expressions $f$, there are obtained $\infty^{2n}$ $(2n-r+1)$-fold manifolds which are common to the first $(r-1)$ equations $f_1 = a_1, \cdots, f_{r-1} = a_{r-1}$, where $a_1, \cdots, a_{r-1}$ are arbitrary constants, and the conditions in virtue of which $F_{z_1}, F_{z_2}, \cdots, F_{z_r}$ are not zero are the conditions that these manifolds should also satisfy $f_r = a_r$ for all values of the constant $a_r$.

We accordingly neglect this particular case, and then all the differential coefficients $F_z$ are zero, provided $r < 2n+1$.

The system of equations which $F$ satisfies is readily integrated, and it is seen that $F$ must be a function of the Poisson alternants

$$[f_\lambda f_\mu] \qquad (\lambda, \mu = 1, 2 \cdots r),$$

and in addition must satisfy equation (11). This equation merely expresses the

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fact that $F$ must be homogeneous of degree $\mu(n+1)$ in the quantities $P$, and $F$ is therefore a homogeneous function of the quantities

$$[f_\lambda f_\mu] \qquad (\lambda, \mu = 1, 2 \cdots r).$$

There remains one case still to be considered, namely that in which we have $(2n+1)$ expressions $f$. In this case there exist invariants of the form $[f_\lambda f_\mu](\lambda, \mu = 1, 2, \cdots, 2n-1)$ and among these the only relations are those of the type $[f_\lambda f_\mu] + [f_\mu f_\lambda] = 0$.

We therefore have $n(2n+1)$ functionally independent solutions of our system of equations.

But returning to this system, we find that it consists of $2n^2 + 3n$ equations in addition to an equation which expresses a condition of homogeneity. There are $(2n+1)^2$ variables involved in the equations, and the equations are now algebraically independent. They therefore possess $(2n+1)^2 - (2n^2+3n)$ functionally independent solutions. Of this number, $(2n^2+n+1)$, $n(2n+1)$ are accounted for, and therefore one solution still remains to be discovered.

It is readily seen that this solution is

$$\begin{vmatrix} Z_1 & A_{1,1} & A_{1,2}, \cdots & P_{1,1} & P_{1,2}, \cdots & P_{1,n} \\ Z_2 & A_{2,1} & \cdots & & & P_{2,n} \\ \vdots & & & & & \\ Z_{2n+1} & A_{2n+1,1} & & & & P_{2n+1,n} \end{vmatrix} \equiv J.$$

If we substitute in (11) we see that, if $F$ is $J$, $\mu$ is equal to unity.

Collecting results we see that *the only functionally independent relative invariants of our type, of $r$ expressions $f$, are the alternants $[f_\lambda f_\mu]$ if $r$ is less than $2n+1$, and if $r$ is equal to $(2n+1)$ there is one additional invariant, the Jacobian of the forms with respect to the variables involved in them.*

It is well known that the alternants $[f_\lambda f_\mu]$ are all invariants of the forms $f$.

The theorem that these are the only invariants of the type sought, has been given by LIE,* who, however, merely suggests the method of proof. Further, LIE has apparently overlooked the additional invariant which arises in connection with $(2n+1)$ forms though he must have been perfectly familiar with the fact that the Jacobian is an invariant.

## § 2.

Let us now consider expressions involving one dependent variable, $n$ independent variables, and the derivatives of the dependent variable with respect to the independent ones of the first and second orders.

The variables involved are now

$$z, x_1, x_2, \cdots, x_n, p_1, p_2, \cdots, p_n, \cdots, p_{ik} \qquad (i, k = 1, 2 \cdots n).$$

* LIE, Mathematische Annalen, vol. 24 (1884), p. 578, and Göttingen Nachrichten (1872), pp. 478–479.

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The invariant sought will be a function of these, and of the first derivatives of the expression with respect to the variables involved in them.

Assume that the infinitesimal contact transformation is determined as before by a function $\theta$, and that $F$ is the invariant. Then the equation satisfied by $F$ is

$$\frac{dF}{dt} + \left\{ \mu_0(n+1)\frac{\partial\theta}{\partial z} + \mu_1\left[\frac{n(n+1)}{2}\frac{\partial\theta}{\partial z} + (n+1)\sum_{h=1}^{m}\frac{d}{dx_h}\left(\frac{\partial\theta}{\partial p_h}\right)\right]\right\} F = 0.$$

Expand this, and it becomes

$$F_z\frac{dz}{dt} + \sum_{r=1}^{n} F_{x_r}\frac{dx_r}{dt} + \sum_{r=1}^{n} F_{p_r}\frac{dp_r}{dt} + \sum_{\alpha\beta} F_{p_{\alpha\beta}}\frac{dp_{\alpha\beta}}{dt} + \mathbf{S}F_Z\frac{dZ}{dt} + \mathbf{S}\sum_{r=1}^{n} F_{X_r}\frac{dX_r}{dt}$$

$$+ \mathbf{S}\sum_{r=1}^{n} F_{P_r}\frac{dP_r}{dt} + \mathbf{S}\sum_{\alpha\beta} F_{P_{\alpha\beta}}\frac{dP_{\alpha\beta}}{dt} + \left[\lambda\theta_z + \mu\sum_{r=1}^{n}\frac{d}{dx_r}(\theta_{p_r})\right] F = 0,$$

where

$$\lambda = (n+1)\left(\mu_0 + \frac{n}{2}\mu_1\right),$$

$$\mu = (n+1)\mu_1.$$

As before, $F$ is an invariant to all contact transformations, and therefore, if we substitute in the above equation the values of

$$\frac{dz}{dt}, \frac{dx_r}{dt}, \cdots, \frac{dZ}{dt}, \cdots$$

in terms of $\theta$ and its derivatives, and if we then equate to zero the coefficients of the various derivatives of $\theta$, we obtain a system of linear differential equations which $F$ must satisfy.

If we equate to zero the coefficients of

$$\theta,\ \theta_{x_1},\ \cdots,\ \theta_{x_n},\ \theta_{x_1x_1},\ \cdots,\ \theta_{x_nx_n},\ \theta_{p_1},\ \cdots,\ \theta_{p_n},$$

we obtain the following system of equations:

$$F_z = 0, \qquad F_{x_i} = 0 \qquad (i = 1, 2, \cdots, n),$$

$$F_{p_i} - \mathbf{S}ZF_{X_i} = 0 \qquad (i = 1, 2, \cdots, n),$$

$$F_{p_{ii}} - \mathbf{S}P_iF_{X_i} = 0 \qquad (i = 1, 2, \cdots, n),$$

$$F_{p_{ik}} - \mathbf{S}P_iF_{X_k} - \mathbf{S}P_kF_{X_i} = 0 \qquad (i, k = 1, 2, \cdots, n, i \neq k).$$

We therefore introduce the variables

$$A_i = X_i + p_iZ + \sum_{k=1}^{n} p_{ik}P_k \qquad (i = 1, 2, \cdots, n),$$

and then these equations show that $F$ is a function of

$$A_1,\ \cdots,\ A_n,\ Z,\ P_1,\ \cdots,\ P_n,\ P_{11},\ P_{12},\ P_{nn},$$

only.

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The increments of the $A$'s due to the transformation are readily found, and we have

$$-\frac{dA_i}{dt} = -\sum_{k=1}^{n} A_k \left(\frac{d\theta_{p_k}}{dx_i}\right) + \sum_{jk} \left\{\frac{d}{dx_i}\left(\frac{d^2\theta}{dx_j dx_k}\right)\right\} P_{jk},$$

where $(d^2\theta/dx_\alpha dx_\beta)$ denotes as before the differential coefficient of $\theta$ with respect to $x_\alpha$ and $x_\beta$, in which $z, p_r, p_{\alpha\beta}$ are taken to be functions of the $x$'s, the terms containing third derivatives of $z$ being omitted, and $d/dx_i$ is written for

$$\frac{\partial}{\partial x_i} + p_i \frac{\partial}{\partial z} + \sum_{k=1}^{n} p_{ik} \frac{\partial}{\partial p_k}.$$

It is easy to show that

$$\frac{d}{dx_i}\left(\frac{d^2\theta}{dx_\alpha dx_\beta}\right) = \frac{d}{dx_\alpha}\left(\frac{d^2\theta}{dx_i dx_\beta}\right) = \frac{d}{dx_\beta}\left(\frac{d^2\theta}{dx_i dx_\alpha}\right) = \left(\left(\frac{d^3\theta}{dx_i dx_\alpha dx_\beta}\right)\right),$$

where the last expression denotes $d^3\theta/dx_i dx_\alpha dx_\beta$ with the terms containing third and fourth derivatives of $z$ omitted.

The increments expressed in the variables $A$, $Z$, $P$ are therefore as follows:

$$-\frac{dA_i}{dt} = -\sum_{k=1}^{n} A_k \frac{d\theta_{p_k}}{dx_i} + \sum_{jk} \left(\left(\frac{d^3\theta}{dx_i dx_j dx_k}\right)\right) P_{jk},$$

$$-\frac{dZ}{dt} = -\sum_{k=1}^{n} A_k \frac{\partial^2\theta}{\partial p_k \partial z} + Z\theta_z + \sum_{k=1}^{n} P_k \frac{d\theta_z}{dx_k} + \sum_{jk} \frac{\partial}{\partial z}\left(\frac{d^2\theta}{dx_j dx_k}\right) P_{jk},$$

$$-\frac{dP_i}{dt} = -\sum_{k=1}^{n} A_k \frac{\partial^2\theta}{\partial p_i \partial p_k} + \sum_{k=1}^{n} P_k \frac{d\theta_{p_i}}{dx_k} + P_i\theta_z + \sum_{jk} \frac{\partial}{\partial p_i}\left(\frac{d^2\theta}{dx_j dx_k}\right) P_{jk},$$

$$-\frac{dP_{hk}}{dt} = \sum_{\alpha\beta} \frac{\partial}{\partial p_{hk}}\left(\frac{d^2\theta}{dx_\alpha dx_\beta}\right) P_{\alpha\beta}.$$

If $F$ is taken as a function of $A_1, \cdots, A_n, Z, P_1, \cdots, P_n, P_{11}, P_{12}, \cdots, P_{nn}$, the equation

$$\frac{dF}{dt} + \left[\lambda\theta_z + \mu \sum_k \frac{d\theta_{p_k}}{dx_k}\right] F = 0$$

becomes

$$\begin{aligned}
&\mathbf{S} \sum_{i=1}^{n} F_{A_i} \left\{\sum_{k=1}^{n} A_k \frac{d\theta_{p_k}}{dx_i} - \sum_{jk} P_{jk} \left(\left(\frac{d^3\theta}{dx_i dx_j dx_k}\right)\right)\right\} \\
&+ \mathbf{S} F_Z \left\{\sum_{k=1}^{n} A_k \theta_{p_k z} - Z\theta_z - \sum_{k=1}^{n} \frac{d\theta_z}{dx_k} P_k - \sum_{jk} P_{jk} \frac{\partial}{\partial z}\left(\frac{d^2\theta}{dx_j dx_k}\right)\right\} \\
&+ \mathbf{S} \sum_{i=1}^{m} F_{P_i} \left\{\sum_{r=1}^{n} A_r \theta_{p_i p_r} - \sum_{r=1}^{n} P_r \frac{d\theta_{p_i}}{dx_r} - P_i \theta_z + \sum_{jk} P_{jk} \frac{\partial}{\partial p_i}\left(\frac{d^2\theta}{dx_j dx_k}\right)\right\} \\
&- \mathbf{S} \sum_{hk} F_{P_{hk}} \left\{\sum_{\alpha\beta} P_{\alpha\beta} \frac{\partial}{\partial p_{hk}}\left(\frac{d^2\theta}{dx_\alpha dx_\beta}\right)\right\} + \left[\lambda\theta_z + \mu \sum_{k=1}^{n} \frac{d\theta_{p_k}}{dx_k}\right] F = 0.
\end{aligned}$$

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Equating coefficients of the derivatives of $\theta$ to zero, we obtain the following system of equations:

$$\mathbf{S}\,[P_{jk}F_{A_i} + P_{ki}F_{A_j} + P_{ij}F_{A_k}] = 0 \qquad (i \neq j,\ j \neq k,\ k \neq i), \tag{17}$$

$$\mathbf{S}\,[P_{ij}F_{A_i} + P_{ii}F_{A_j}] = 0, \tag{18}$$

$$\mathbf{S}\,[P_{ij}F_Z] = 0, \tag{19}$$

$$\mathbf{S}\,[P_{ij}F_{P_k}] = 0, \tag{20}$$

$$\mathbf{S}\left[A_iF_{A_i} - P_iF_{P_i} - P_{ii}F_{P_{ii}} - \sum_{j=1}^{n} P_{ij}F_{P_{ij}}\right] + \mu F = 0, \tag{21}$$

$$\mathbf{S}\left[A_iF_{A_j} - P_jF_{P_i} - \sum_{k=1}^{n} P_{kj}F_{P_{ki}} - P_{jj}F_{P_{ji}}\right] = 0 \qquad (i \neq j), \tag{22}$$

$$\mathbf{S}\,[P_iF_Z] = 0, \tag{23}$$

$$\mathbf{S}\,[A_iF_Z] = 0, \tag{24}$$

$$\mathbf{S}\,[A_jF_{P_i} + A_iF_{P_j}] = 0, \tag{25}$$

$$\mathbf{S}\left[ZF_Z + \sum_{i=1}^{n} P_iF_{P_i} + \sum_{\alpha\beta} P_{\alpha\beta}F_{P_{\alpha\beta}}\right] = \lambda F \qquad (i, j, k = 1, 2, \cdots, n). \tag{26}$$

There now arise two cases to be considered. In the first case $P_{ij} = 0$ $(i, j = 1, 2, \cdots, n)$, and our expressions are therefore of the first order. In this case the equations become

$$\mathbf{S}\,[A_iF_{A_i} - P_iF_{P_i}] + \mu F = 0,$$

$$\mathbf{S}\,[A_iF_{A_j} - P_jF_{P_i}] = 0,$$

$$\mathbf{S}\,[P_iF_Z] = 0,\ \mathbf{S}\,[A_iF_Z] = 0,$$

$$\mathbf{S}\,[A_jF_{P_i} + A_iF_{P_j}] = 0,$$

$$\mathbf{S}\left[ZF_Z + \sum_{i=1}^{n} P_iF_{P_i}\right] = \lambda F \qquad (i, j = 1, 2, \ldots n).$$

These equations are almost identical with the set (11)...(16); they must of course possess the same integrals as that set, together with others arising from the facts that equations of the type (12) have not now to be satisfied, and that $\mu$, which is an arbitrary constant, has the particular value zero in the first set of equations.

It is easy to see that the integrals still to be found are functions of the variables $A$ alone, and therefore they satisfy the equations

$$\mathbf{S}A_iF_{A_i} + \mu F = 0,$$

$$\mathbf{S}A_iF_{A_j} = 0 \qquad (i, j = 1, 2, \cdots, n).$$

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Assuming that there are $r$ expressions of the first order whose invariants we are seeking, the solutions of the above equations are easily seen to be the $n$-row determinants of the matrix,

$$\left\| \begin{matrix} A_{1,1} & A_{2,1} & \cdots & A_{r,1} \\ A_{1,2} & & \cdots & \vdots \\ \vdots & & & \\ A_{1,n} & & \cdots & A_{r,n} \end{matrix} \right\|,$$

provided that $r$ is greater than $n$.

There are no additional solutions if $r$ is less than $n$.

These solutions are the Jacobians of sets of $n$ of the forms.

We shall now consider the case in which the quantities $P_{ij}$ are not all zero.

Before discussing the general case, we shall consider the case in which there are only two independent variables.

We shall take in order the cases in which there are one, two, three expressions of the second order whose invariants we are seeking.

In the case of one such expression,

$$\begin{aligned} F_{A_1} = F_{A_2} = 0 = F_Z = F_{P_1} = F_{P_2}, \\ 2P_{11}F_{P_{11}} + P_{12}F_{P_{12}} = \mu F, \\ P_{12}F_{P_{12}} + 2P_{22}F_{P_{22}} = \mu F, \\ 2P_{11}F_{P_{12}} + P_{21}F_{P_{22}} = 0, \\ 2P_{22}F_{P_{12}} + P_{21}F_{P_{11}} = 0. \end{aligned}$$

These equations show that $F$ must be a homogeneous function of the algebraic invariant of the binary form

$$(P_{11},\ \tfrac{1}{2}P_{12},\ P_{22}\ \ast\ )^2.$$

Hence $F \equiv$ const. $\times\ (P_{12}^2 - 4P_{11}P_{22})^{\mu/2}$.

To interpret this invariant, suppose that

$$f(x_1\, x_2\, z\, p_1\, p_2\, p_{11}\, p_{12}\, p_{22}) = 0$$

is a differential equation of the second order.

Let $z = \phi(x_1 x_2)$ be some non-singular solution of this equation. We define two directions on this integral surface by means of the equation

$$P_{22}dx_1^2 - P_{12}dx_1\,dx_2 + P_{11}dx_2^2 = 0.$$

Along one of the curves thus determined on the particular integral surface, $x_1$, $x_2$, $z$, $p_1$, $p_2$, $p_{11}$, $p_{12}$, $p_{22}$ are functions of a single parameter while $dz$, $dp_1$, $dp_2$ are determined from the equations

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$$dz = p_1 dx_1 + p_2 dx_2,$$

$$dp_1 = p_{11} dx_1 + p_{12} dx_2,$$

$$dp_2 = p_{12} dx_1 + p_{22} dx_2.$$

Also

$$\frac{df}{dx_1} + P_{11}\frac{dp_{11}}{dx_1} + P_{22}\frac{dp_{12}}{dx_2} = 0,$$

$$\frac{df}{dx_2} + P_{11}\frac{dp_{12}}{dx_1} + P_{22}\frac{dp_{22}}{dx_2} = 0.$$

The system of seven equations thus obtained are equivalent to six distinct relations, and they determine the "characteristics" of the given equation $f = 0$.

The fundamental property in connection with the curves obtained is that if two integral surfaces have contact of the first order along a characteristic, and if they have contact of the second order at any one point of this curve, they have contact of the second order all along the curve.*

We notice that the directions of the curves are given at every point by means of $z = \phi(x_1 x_2)$ and $P_{11} dx_2^2 - P_{12} dx_1 dx_2 + P_{22} dx_1^2 = 0$.

Now the transformation considered changes an integral surface into an integral surface, and also a characteristic upon an integral surface into a characteristic upon the transformed surface. We therefore expect $P_{11} dx_2^2 - \cdots = 0$ to be an invariant of the expression considered, and we also expect any function geometrically connected with it to be an invariant. $P_{12}^2 - 4P_{11}P_{22}$ was therefore *a priori* to be expected as an invariant.

We next consider the case of two expressions $f_1$ and $f_2$.

It is easy to see that the only invariants are the algebraic invariants of the two binary forms

$$(P_{1,11}\tfrac{1}{2}P_{1,12}P_{1,22}(\ *\ )^2,$$

$$(P_{2,11}\tfrac{1}{2}P_{2,12}P_{2,22}(\ *\ )^2,$$

unless the condition

$$I \equiv \begin{vmatrix} P_{1,12} & P_{2,12} \\ P_{1,11} & P_{2,11} \end{vmatrix} \times \begin{vmatrix} P_{1,22} & P_{2,22} \\ P_{1,12} & P_{2,12} \end{vmatrix} - \begin{vmatrix} P_{1,11} & P_{2,11} \\ P_{1,22} & P_{2,22} \end{vmatrix}^2 = 0$$

holds.

These invariants have an immediate interpretation from the theory of characteristics. $I$ is itself one of the invariants above mentioned, and $I = 0$ is the condition that the two quadratic forms above mentioned, when equated to zero, have a common root. Therefore, unless the characteristics of $f_1 = 0$ have one direction common with those of $f_2 = 0$ at every point on a common integral surface, the two expressions have only three functionally independent invariants of

* GOURSAT, *Equations aux dérivées partielles du second ordre*, vol. 1, pp. 170 seq.

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our type, namely those of two quadratic forms. Suppose now that $I$ is zero. Let $m$ be used to denote the common root mentioned above. Then

$$m = -\frac{P_{1,11}P_{2,22} - P_{2,11}P_{1,22}}{P_{1,12}P_{2,11} - P_{2,12}P_{1,11}} = -\frac{P_{1,22}P_{2,12} - P_{2,22}P_{1,12}}{P_{1,11}P_{2,22} - P_{2,11}P_{1,22}}.$$

Let $H$ denote $A_{1,1}P_{2,11} - A_{2,1}P_{1,11}$ and let $K$ denote $A_{1,2}P_{2,22} - A_{2,2}P_{1,22}$. Then it easily follows that $K + mH$ satisfies our system of equations provided it is zero, and further, this is the only additional solution the system can have.

We may verify that

$$\frac{d}{dt}(K + mH) = \left\{ 3\theta_s - 3\frac{d\theta_{p_1}}{dx_1} + m\frac{d\theta_{p_1}}{dx_2} + \frac{1}{m}\frac{d\theta_{p_2}}{dx_1} - \frac{d\theta_{p_2}}{dx_1} \right\} \times (K + mH).$$

Hence $K + mH$ is not an invariant of our type, although $K + mH = 0$ is an invariant relation.

These two equations

$$I = 0,$$

$$K + mH = 0$$

have an important signification in the theory of differential equations. They are* the conditions that the two equations $f_1 = 0, f_2 = 0$, form a system in involution, in other words, they are the conditions that the two equations have a system of common integrals depending on an infinite number of arbitrary constants.

We shall now consider invariants of three expressions $f_1, f_2, f_3$.

From the system of equations it follows that $F_Z = 0$ unless all the 3-row determinants of the matrix

$$\begin{Vmatrix} A_{1,1} & A_{1,2} & P_{1,1} & P_{1,2} & P_{1,11} & P_{1,12} & P_{1,22} \\ A_{2,1} & A_{2,2} & P_{2,1} & P_{2,2} & P_{2,11} & P_{2,12} & P_{2,22} \\ A_{3,1} & A_{3,2} & P_{3,1} & P_{3,2} & P_{3,11} & P_{3,12} & P_{3,22} \end{Vmatrix}$$

are zero.

Also $F_{P_\lambda}$ is zero unless all the 3-row determinants of the matrix

$$\begin{Vmatrix} A_{1,1} & A_{1,2} & P_{1,11} & P_{1,12} & P_{1,22} \\ A_{2,1} & A_{2,2} & P_{2,11} & P_{2,12} & P_{2,22} \\ A_{3,1} & A_{3,2} & P_{3,11} & P_{3,12} & P_{3,22} \end{Vmatrix}$$

are zero.

Assuming that the conditions mentioned are not satisfied we see that

$$F_Z = 0, \qquad F_{P_1} = 0, \qquad F_{P_2} = 0.$$

* GOURSAT, *Equations aux dérivées partielles du second ordre*, vol. 2, p. 40 and p. 76.

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The functionally independent solutions of our equations are then seen to be the algebraic invariants of the binary forms $(K, H \between *)$:

$$(P_{1,11},\ \tfrac{1}{2}P_{1,12},\ P_{1,22} \between *)^2,$$
$$(P_{2,11},\ \tfrac{1}{2}P_{2,12},\ P_{2,22} \between *)^2,$$
$$(P_{3,11},\ \tfrac{1}{2}P_{3,12},\ P_{3,22} \between *)^2,$$

where

$$H \equiv \begin{vmatrix} P_{1,11} & P_{2,11} & P_{3,11} \\ A_{1,1} & A_{2,1} & A_{3,1} \\ P_{1,22} & P_{2,22} & P_{3,22} \end{vmatrix} - \begin{vmatrix} A_{1,2} & \cdots \\ P_{1,12} & \cdots \\ P_{1,22} & \cdots \end{vmatrix},$$

$$K \equiv \begin{vmatrix} P_{1,11} & \cdots \\ A_{1,2} & \cdots \\ P_{1,22} & \cdots \end{vmatrix} - \begin{vmatrix} P_{1,11} & \cdots \\ P_{1,12} & \cdots \\ A_{1,1} & \cdots \end{vmatrix}.$$

It is important to notice the meaning of the equations $H = 0$, $K = 0$. They are in fact the conditions that the three equations $f_1 = 0$, $f_2 = 0$, $f_3 = 0$ have a common integral surface.*

The three quadratic binary forms and their invariants have as before, immediate interpretation from the theory of characteristics, but the linear form $Kdx_2 - Hdx_1$ has no such immediate interpretation.

As another example of invariants of this type we shall now consider the case in which there are two expressions, one of the second order and the other of the first.

The equations are readily solved, and the solutions are the algebraic invariants of the two forms

$$(P_{1,11},\ \tfrac{1}{2}P_{1,12},\ P_{1,22} \between *)^2,$$
$$(A_{2,2},\ -A_{2,1} \between *),$$

where $P_{2,11}$, $P_{2,12}$, $P_{2,22}$ are all zero. We know that

$$P_{1,11}dx_2^2 - P_{1,12}dx_1dx_2 + P_{1,22}dx_1^2 = 0$$

is the equation for the directions of the characteristics of $f_1 = 0$. It seems, therefore, important to consider the meaning of

$$A_{2,2}dx_2 - (-A_{2,1})dx_1,$$

or

$$A_{2,1}dx_1 + A_{2,2}dx_2.$$

But this is equal to $df_2$, provided

$$dz - p_1dx_1 - p_2dx_2 = 0,\quad dp_1 - p_{11}dx_1 - p_{12}dx_2 = 0,\quad dp_2 - p_{12}dx_1 - p_{22}dx_2 = 0.$$

* VALYI, Crelle's Journal, vol. 95 (1883) p. 100; GOURSAT, loc. cit., vol. 2, p. 199.



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We thus have an interpretation of both invariants.

We now return to the general case, when there are $n$ independent variables. The equations (17)—(26) possess all the invariants of our type as solutions.

Suppose that there are $r$ expressions $f$. Then from (19), (23), (24), $F_Z = 0$ unless $r$ is greater than $\frac{1}{2}n(n+1)+2n$, or unless all the $r$-row determinants of the matrix

$$\left\| \begin{matrix} P_{1,11} & P_{1,12} & \cdots & P_{1,nn} & P_{1,1} & \cdots & P_{1,n} & A_{1,1} & \cdots & A_{1,n} \\ \vdots & \vdots & & & & & & & & \\ P_{r,11} & P_{r,12} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & A_{r,n} \end{matrix} \right\|$$

vanish.

Also the equations (17) (18) show that unless the determinants of another matrix vanish, $F_{A_i} = 0$ for all values of $i$.

Suppose that $F_Z, F_{A_1}, \ldots, F_{A_n}$ are all zero. There remains the system of equations for $F$

$$\mathbf{S}(P_{ij} F_{P_i}) = 0,$$

$$\mathbf{S}\left[P_i F_{P_i} + P_{ii} F_{P_{ii}} + \sum_{i=1}^{n} P_{ij} F_{P_{ij}}\right] - \mu F = 0,$$

$$\mathbf{S}\left[P_j F_{P_i} + \sum_{k=1}^{n} P_{kj} F_{P_{ki}} + P_{jj} F_{P_{ji}}\right] = 0,$$

$$\mathbf{S}\left[A_j F_{P_i} + A_i F_{P_j}\right] = 0,$$

$$\mathbf{S}\left[\sum_{i=1}^{n} P_i F_{P_i} + \sum_{\alpha\beta} P_{\alpha\beta} F_{P_{\alpha\beta}}\right] = \lambda F \qquad (i, j, k = 1, \cdots, n),$$

and the fourth of these equations shows that $F_{P_i}$ is zero for all values of $i$. Hence the system becomes

$$\mathbf{S}\left[P_{ii} F_{P_{ii}} + \sum_{j=1}^{n} P_{ij} F_{P_{ij}}\right] - \mu F = 0,$$

$$\mathbf{S}\left[\sum_{k=1}^{n} P_{kj} F_{P_{ki}} + P_{jj} F_{P_{ji}}\right] = 0,$$

$$\mathbf{S}\left[\sum_{\alpha\beta} P_{\alpha\beta} F_{P_{\alpha\beta}}\right] = \lambda F \qquad (i, j, k = 1, \cdots, n).$$

Hence $\lambda = n\mu/2$, and the invariants required are the algebraic invariants of the system of quadratic forms

$$\sum_{\alpha\beta} P_{\alpha\beta} x_\alpha x_\beta,$$

where none of the magnitudes $P_{\alpha\beta}$ is repeated and $P_{\alpha\beta} = P_{\beta\alpha}$.

These quadratic forms are easily seen to be those which FORSYTH * calls

* FORSYTH, Philosophical Transactions, ser. A, vol. 191 (1898), p. 2.

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"characteristic invariants" when there are only three independent variables. This name might with advantage be extended to the general case in which there are $n$ independent variables.

We now return to the case in which $F_{A_1} F_{A_2}$, etc., are not all zero.

Suppose that there are three independent variables, and two expressions $f$.

Then if $F_{A_1}$, etc., are not zero, all the 6-row determinants of the matrix

$$\begin{Vmatrix} P_{1,11} & 0 & 0 & P_{1,12} & P_{1,22} & 0 & 0 & P_{1,31} & P_{1,33} & P_{1,23} \\ P_{2,11} & 0 & 0 & P_{2,12} & P_{2,22} & 0 & 0 & P_{2,31} & P_{2,33} & P_{2,23} \\ 0 & P_{1,22} & 0 & P_{1,11} & P_{1,12} & P_{1,23} & P_{1,33} & 0 & 0 & P_{1,31} \\ 0 & P_{2,22} & 0 & P_{2,11} & P_{2,12} & P_{2,23} & P_{2,33} & 0 & 0 & P_{2,31} \\ 0 & 0 & P_{1,33} & 0 & 0 & P_{1,22} & P_{1,23} & P_{1,11} & P_{1,31} & P_{1,12} \\ 0 & 0 & P_{2,33} & 0 & 0 & P_{2,22} & P_{2,23} & P_{2,11} & P_{2,31} & P_{2,12} \end{Vmatrix}$$

must vanish.

Let $S_i$ denote the characteristic invariant of $f_i$, then it is easily seen that if we construct the cubic forms

$$S_1x_1,\ S_1x_2,\ S_1x_3,\ S_2x_1,\ S_2x_2,\ S_2x_3,$$

the above matrix is the matrix of the coefficients.

Hence, if our conditions hold, the above six cubics must belong to a five fold linear system. Expressing this condition we see that $S_1L_1 + S_2L_2 \equiv 0$, where $L_1$ and $L_2$ are certain linear forms.

Hence either $S_1$ and $S_2$ both break up into linear factors, or $S_2$ is equivalent to $S_1$.

In the case in which there are three expressions $f$, the conditions give

$$S_1L_1 + S_2L_2 + S_3L_3 \equiv 0,$$

where $L_1$, $L_2$, $L_3$ are linear. Hence, in general $S_1$, $S_2$, $S_3$, regarded as conics, have two common points.

The generalization is immediate, and the condition in order that invariants involving the magnitudes $A$, of $r$ expressions $f$ in $n$ independent variables exist, are equivalent to the conditions that $r$ linear forms $L_1, \cdots, L_r$ should exist such that

$$\sum_{i=1}^{r} S_i L_i \equiv 0$$

identically, when $S_i$ is the characteristic invariant of $f_i$. It is readily seen that these conditions may be expressed by the vanishing of certain algebraic invariants of the $r$ quadratic forms $S$.

An upper limit to the number of these conditions may readily be obtained.

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This upper limit is $n(n+1)(n+2)/3!-nr+1$. The number may fall below this in certain cases, for example, if $n=3$ and $r=2$ it is 4, whilst

$$\frac{n(n+1)(n+2)}{3!}-nr+1=5.$$

If all the linear expressions $L$ are equivalent, the conditions require that constants $\lambda$ can be found such that $\sum \lambda S=0$.

The number of conditions for this is readily seen to be

$$\frac{n(n+1)}{2}-r+1.$$

This number is less than the previous one if $[n(n+1)/6-r](n-1)>0$, and the second conditions are all independent.

Hence, if $r$ is $<\frac{1}{6}n(n+1)$ the imposition of $n(n+1)/2-r+1$ conditions is sufficient to cause the remainder to be satisfied.

Further consideration of this question will be omitted from the present paper.

Suppose the conditions in question to be satisfied. We then obtain a solution of the set of equations in $F_A$ which is a determinant linear in the magnitudes $A$. Call this determinant $\Delta$, then in a manner strictly analogous to the case when $n=2$, it may be shown that $\Delta=0$ is an invariant relation, provided the previous set of conditions holds.

If there are solutions of the system of equations considered which involve the magnitudes $A$, the derivatives $F_{P_i}$ are not necessarily zero. From the equations of type (20), we see that, if $F_{P_i}\neq 0$, the matrix

$$\left\|\begin{matrix} P_{1,11} & \cdots & P_{1,ij} & \cdots & P_{1,nn} \\ \vdots & & \vdots & & \\ P_{r,11} & \cdots & P_{r,ij} & \cdots & P_{r,nn} \end{matrix}\right\|$$

must have all its $r$-row determinants zero.

In addition, all the equations of type (25),

$$\mathbf{S}(A_j F_{P_i}+A_i F_{P_j})=0,$$

must be satisfied.

Now take any one of the determinants of the above matrix, replace one of its columns by the magnitudes $P_{1,i}, P_{2,i}, \cdots, P_{r,i}$. Call the determinant then formed $\Delta_i$. Call the similar determinant with $A_i$ instead of $P_i$, $M_i$. Then $P_i$ only enters through $\Delta_i$, and the equations (25) become

$$M_j F_{\Delta_i}+M_i F_{\Delta_j}=0 \qquad (i,j=1,\cdots,r).$$

Hence if an invariant contains $P_i$, we see from the case when $i=j$, that $M_i$ must be zero. If $M_i=0$ and $F_{\Delta_i}\neq 0$, we see that $M_j=0$ $(j=1,\cdots,r)$; and if $F_{\Delta_i}=0$ and $M_i\neq 0$, we see that $F_{\Delta_j}=0$ $(j=1,2,\cdots,r)$.

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Hence either none of the magnitudes $P_i (i = 1, \cdots, r)$ occur in any invariant, or all the magnitudes $M_i (i = 1, \cdots, r)$ are zero.

It is easy to see that if all the given conditions are satisfied, then the magnitudes $\Delta_i$ satisfy the remaining equations, and therefore these magnitudes $\Delta_i$ are invariants.

There is no invariant involving $Z$ unless all the conditions given in connection with the magnitude $P_i$ are satisfied and, in addition,

$$\Delta_1 = \Delta_2 = \cdots = \Delta_r = 0.$$

If all these conditions are satisfied, there is an invariant involving $Z$ given by replacing any column in any $r$-row determinant of the matrix

$$\left\| \begin{matrix} P_{1,11} & \cdots & P_{1,ij} & \cdots & P_{1,nn} \\ \vdots & & \vdots & & \vdots \\ P_{r,11} & \cdots & P_{r,ij} & \cdots & P_{r,nn} \end{matrix} \right\|$$

by $Z_1, Z_2, \cdots, Z_r$.

§ 3.

We have not as yet considered invariants which involve the magnitudes $dx_i$, $dr$, $dp_i$, $dp_{ij} (i, j = 1, \cdots, n)$.

It is clear that invariants of this type do exist. For example, it is easy to verify that

$$\sum_{\alpha, \beta} P_{\alpha\beta}\, dx_\alpha\, dx_\beta$$

is such an invariant.

The work is somewhat simplified if we take as variables

$$\begin{aligned} dx_i &\equiv a_i, \\ dz - \sum_i p_i dx_i &\equiv u, \\ dp_i - \sum_j p_{ij} dx_j &\equiv v_i, \\ dp_{ij} &\equiv c_{ij}. \end{aligned} \qquad (i, j = 1, \cdots, n).$$

The increments of these magnitudes are readily obtained. We have

$$-\frac{da_i}{dt} = \sum_k \frac{d\theta_{p_i}}{dx_k} a_k + \frac{\partial \theta_{p_i}}{\partial z} u + \sum_k \frac{\partial^2 \theta}{\partial p_i \partial p_k} v_k, \qquad \frac{du}{dt} = \theta_z u,$$

$$\frac{dv_i}{dt} = \frac{d\theta_z}{dx_i} u + \sum_k \frac{d\theta_{p_k}}{dx_i} v_k + \theta_z v_i,$$

$$\frac{dc_{ij}}{dt} = \sum_k \left( \left( \frac{d^3\theta}{dx_i dx_j dx_k} \right) \right) a_k + \left( \frac{d^2\theta_z}{dx_i dx_j} \right) u$$

$$+ \sum_k \left( \frac{d^2\theta_{p_k}}{dx_i dx_j} \right) v_k + \frac{d\theta_z}{dx_j} v_i + \frac{d\theta_z}{dx_i} v_j + \sum_{\alpha\beta} \frac{\partial}{\partial p_{\alpha\beta}} \left( \frac{d^2\theta}{dx_i dx_j} \right) c_{\alpha\beta}.$$

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If $F$ is an invariant, the equation which it satisfies is similar to the one given in the previous section, but it contains the additional terms

$$-\sum_i F_{a_i}\left\{\sum_k \frac{d\theta_{p_i}}{dx_k}a_k + \frac{\partial\theta_{p_i}}{\partial z}u + \sum_k \frac{\partial^2\theta}{\partial p_i \partial p_k}v_k\right\}$$

$$+ F_u\theta_z u + F_{v_i}\left\{\frac{d\theta_z}{dx_i}u + \theta_z v_i + \sum_k \frac{d\theta_{p_k}}{dx_i}v_k\right\}$$

$$+\sum_{ij}\left\{\sum_k\left(\left(\frac{d^3\theta}{dx_i dx_j dx_k}\right)\right)a_k + \left(\frac{d^2\theta_z}{dx_i dx_j}\right)u\right.$$

$$\left.+\sum_k\left(\frac{d^2\theta_{p_k}}{dx_i dx_j}\right)v_k + \frac{d\theta_z}{dx_j}v_i + \frac{d\theta_z}{dx_i}v_j + \sum_{\alpha\beta}\frac{\partial}{\partial p_{\alpha\beta}}\left(\frac{d^2\theta}{dx_i dx_j}\right)c_{\alpha\beta}\right\}F_{c_{ij}}.$$

The equations for $F$ are now

$$(27)\quad \mathbf{S}[P_{jk}F_{A_i} + P_{ki}F_{A_j} + P_{ij}F_{A_k}] - a_k F_{c_{ij}} - a_i F_{c_{jk}} - a_j F_{c_{ik}} = 0 \qquad (i \neq j,\ j \neq k,\ k \neq i),$$

$$(28)\quad \mathbf{S}[P_{ij}F_{A_i} + P_{ii}F_{A_j}] - a_j F_{c_{ii}} - a_i F_{c_{ij}} = 0,$$

$$(29)\quad \mathbf{S}P_{ij}F_Z - uF_{c_{ij}} = 0,$$

$$(30)\quad \mathbf{S}P_{ij}F_{P_k} - v_k F_{c_{ii}} = 0,$$

$$(31)\quad \mathbf{S}\left[A_i F_{A_i} - P_i F_{P_i} - P_{ii}F_{P_{ii}} - \sum_{j=1}^{n} P_{ij}F_{P_{ij}}\right] + \mu F - a_i F_{a_i} + v_i F_{v_i} + c_{ii}F_{c_{ii}} + \sum_{j=1}^{m} c_{ij}F_{c_{ij}} = 0,$$

$$(32)\quad \mathbf{S}\left[A_i F_{A_j} - P_j F_{P_i} - \sum_{k=1}^{n} P_{kj}F_{P_{ki}} - P_{jj}F_{P_{ji}}\right] - a_j F_{a_i} + v_i F_{v_j} + c_{ij}F_{c_{jj}} + \sum_{k=1}^{m} c_{ki}F_{c_{kj}} = 0,$$

$$(33)\quad \mathbf{S}[P_i F_Z] - uF_{v_i} = 0,$$

$$(34)\quad \mathbf{S}A_i F_Z - uF_{a_i} = 0,$$

$$(35)\quad \mathbf{S}[A_j F_{P_i} + A_i F_{P_j}] - v_j F_{a_i} - v_i F_{a_j} = 0,$$

$$(36)\quad \mathbf{S}\left[ZF_Z + \sum_{i=1}^{n} P_i F_{P_i} + \sum_{\alpha\beta} P_{\alpha\beta}F_{P_{\alpha\beta}}\right] - \left(\sum_{\alpha\beta} c_{\alpha\beta}F_{c_{\alpha\beta}} + uF_u + \sum_{i=1}^{n} v_i F_{v_i}\right) = \lambda F \qquad (i, j, k, \alpha, \beta, = 1, \cdots, n).$$

We first consider the particular case when all the original expressions $f$ are of the first order.

In this case $P_{ij} = 0\ (i, j = 1, \cdots, n)$, and it may be readily seen that $F$ does not in general involve any of the magnitudes $c_{\alpha\beta}$.

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We have therefore the reduced system of equations for $F$:

$$\mathbf{S}\,[A_i F_{A_i} - P_i F_{P_i}] + uF - a_i F_{a_i} + v_i F_{v_i} = 0, \tag{37}$$

$$\mathbf{S}\,[A_i F_{A_j} - P_j F_{P_i}] + v_i F_{v_j} - a_j F_{a_i} = 0, \tag{38}$$

$$\mathbf{S} P_i F_Z - uF_{v_i} = 0, \tag{39}$$

$$\mathbf{S} A_i F_Z - uF_{a_i} = 0, \tag{40}$$

$$\mathbf{S}\,[A_j F_{P_i} + A_i F_{P_j}] - v_j F_{a_i} - v_i F_{a_j} = 0, \tag{41}$$

$$\mathbf{S}\left[ZF_Z + \sum_{i=1}^{n} P_i F_{P_i}\right] - uF_u - \sum_{i=1}^{n} v_i F_{v_i} = \lambda F \qquad (i, j = 1, \cdots, n). \tag{42}$$

From (39) and (40) we deduce that $F$ must be a function of $u$, $W$, $P_i$, $A_i$, where

$$W = uZ + \sum a_i A_i + \sum v_i P_i;$$

and the equations for $F$ are now

$$\mathbf{S}[A_i F_{A_i} - P_i F_{P_i}] + \mu F = 0, \tag{43}$$

$$\mathbf{S}\,[A_i F_{A_j} - P_j F_{P_i}] = 0, \tag{44}$$

$$\mathbf{S}\,[A_j F_{P_i} + A_i F_{P_j}] = 0, \tag{45}$$

$$\mathbf{S}\left[\sum_{i=1}^{n} P_i F_{P_i}\right] + uF_u = \lambda F \qquad (i, j, = 1, \cdots, n). \tag{46}$$

Hence the quantities $W$ are absolute invariants, and in addition there are the invariants given in the previous section. Also $u$ is an invariant.

The additional invariants obtained are readily interpreted. The function

$$u \equiv dz - \sum p_i dx_i,$$

is an invariant arising in connection with the contact transformation itself, and $W$ may easily be shown to be

$$df \equiv \frac{\partial f}{\partial z} dz + \sum_i \frac{\partial f}{\partial x_i} dx_i + \sum_i \frac{\partial f}{\partial p_i} dp_i.$$

Suppose that $u = 0$. Then unless the $r$-row determinants of the matrix

$$\left\| \begin{matrix} A_{1,1} & \cdots & A_{1,n} & P_{1,1} & \cdots & P_{1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{r,1} & & A_{r,n} & P_{r,n} & \cdots & P_{r,n} \end{matrix} \right\|$$

are all zero, and $F_Z$ is zero.

If we write $v_i = A_{r+1,i}$ and $a_i = -P_{r+1,i}$, the set of equations becomes the same as that given previously in which there were $(r+1)$ expressions and in

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which the variables $dz$, $dx_i$, $dp_i$, did not occur, the last equation only being different. The additional solutions are therefore $[f_i, f_{r+1}]$ $(i=1, \cdots, r)$, which are $df_1, df_2, \cdots, df_r$ subject to the condition that $u = 0$.

We now consider expressions $F$ of the second order. Suppose first that $u \neq 0$ and that $v_1, \cdots, v_n$ are not all zero.

Write $L$ for

$$uZ + \sum_i a_i A_i + \sum_i v_i P_i + \sum_{\alpha\beta} c_{\alpha\beta} P_{\alpha\beta},$$

then $F$ is readily shown to be a function of the variables $u$, $L$, $v_i$, $P_{ij}$, $A_i$ $(i, j = 1, \cdots, n)$, which satisfies the system of equations

$$\mathbf{S}[P_{jk} F_{A_i} + P_{ki} F_{A_j} + P_{ij} F_{A_k}] = 0 \quad (j \neq k, k \neq i, i \neq j),$$

$$\mathbf{S}[P_{ij} F_{A_i} + P_{ii} F_{A_j}] = 0,$$

$$\mathbf{S}\left[A_i F_{A_i} - P_{ii} F_{P_{ii}} - \sum_{j=1}^{n} P_{ij} F_{P_{ij}}\right] + \mu F + v_i F_{v_i} = 0,$$

$$\mathbf{S}\left[A_i F_{A_j} - P_{jj} F_{P_{jj}} - \sum_{k=1}^{n} P_{kj} F_{P_{kj}}\right] + v_i F_{v_j} = 0,$$

$$\mathbf{S}\left(\sum_{\alpha\beta} P_{\alpha\beta} F_{P_{\alpha\beta}}\right) + uF_u + \sum_{i=1}^{n} v_i F_{v_i} = \lambda F.$$

The magnitudes $L$ are therefore invariants. In addition $u$ is an invariant, and the remaining invariants are solutions of the given system.

$L$ is readily shown to be

$$df = \sum_i \frac{\partial f}{\partial x_i} dx_i + \frac{\partial f}{\partial z} dz + \sum_i \frac{\partial f}{\partial p_i} dp_i + \sum_{\alpha\beta} \frac{\partial f}{\partial p_{\alpha\beta}} dp_{\alpha\beta},$$

and it therefore admits of an immediate interpretation. There are also the invariants obtained in the preceding section which do not involve the magnitudes $v_i$ $(i = 1, \cdots, n)$ and the remaining invariants are those solutions of the set of equations last given which involve these magnitudes.

We see that when certain conditions given in the preceding section hold, the quantities $F_{A_i}$ $(i = 1, \cdots, n)$ are all zero.

There remains the system

$$\mathbf{S}\left(P_{ii} F_{P_{ii}} + \sum_{j=1}^{n} P_{ij} F_{P_{ij}}\right) - v_i F_{v_i} = \mu F,$$

$$\mathbf{S}\left(P_{jj} F_{P_{jj}} + \sum_{k=1}^{n} P_{kj} F_{P_{kj}}\right) - v_i F_{v_j} = 0,$$

$$\mathbf{S}\left(\sum_{\alpha\beta} P_{\alpha\beta} F_{P_{\alpha\beta}}\right) + \sum_{i=1}^{n} v_i F_{v_i} = \lambda F \quad (i, j, \alpha, \beta, = 1, \cdots, n).$$

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The solution of these equations is a function of the algebraic invariants and covariants of the $r$ quadratic forms

$$\sum_{ij} P_{ij} v_i v_j .$$

In addition, this function must be homogeneous in the variables $v$.

These forms

$$\sum_{ij} P_{ij} v_i v_j$$

are those which have been referred to earlier as the Characteristic Invariants of the expressions $f$.

Now suppose that $u$ and the quantities $v$ are all zero.

In this case $F_z$ and $F_{P_i}$ $(i = 1, \cdots, n)$ are all zero unless all the $r$-row determinants of the matrix

$$\left\| \begin{matrix} P_{1,11} & \cdots & P_{1,ij} & \cdots & P_{1,nn} \\ \vdots & & \vdots & & \vdots \\ P_{r,11} & \cdots & P_{r,ij} & \cdots & P_{r,nn} \end{matrix} \right\|$$

vanish.

We assume that these conditions are not satisfied, and therefore $F$ is a function of $a_i$, $A_i$, $P_{ij}$, $c_{ij}$ $(i, j = 1, \cdots, n)$.

The equations of types (27) and (28) are satisfied by the magnitudes

$$L \equiv \sum_i A_i a_i + \sum_{jk} P_{jk} c_{jk} .$$

It may readily be shown that these are the only independent solutions of this set of equations unless it is possible to make

$$\sum_{i=1}^{r} S_i B_i + AK = 0$$

identically, where any $n$ variables $y_1, y_2, \cdots, y_n$ are taken, and the magnitudes $B_i$ are linear functions of these variables, $K$ is a quadratic function of them, and

$$S_i \equiv \sum_{\alpha\beta} P_{i,\alpha\beta} y_\alpha y_\beta ,$$

$$A \equiv \sum_k a_k y_k .$$

This equation can be satisfied by making the linear functions all equal to $A$, multiplied by certain constant factors. This leads to solutions of our system of the type $L$.

If we assume that there is no other way of making the expression considered an identity, we see that the quantities $L$ are the only invariants involving the variables $A_k (k = 1, \cdots, n)$.

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It is readily seen that these expressions $L$ are equivalent to the expressions $df$, subject to the conditions

$$u = 0, \qquad v_k = 0 \qquad (k = 1, 2, \cdots, n).$$

It is easy to show that the remaining solutions of the system are the algebraic invariants of the system of quadratic forms $S_i$ and of the linear form $A$.

These forms $S$ are again the characteristic invariants of the expressions $f$; they have however, in this case, the magnitudes $y$ for independent variables.

Combining our results we have the following theorem:

*All invariants, of the restricted type considered of $r$ second order expressions $f$ in one dependent and $n$ independent variables are*

(1) *Expressions of the type $df$.*

(2) *Algebraic invariants and covariants of the quadratic forms*

$$\sum_{ij} \frac{\partial f}{\partial p_{ij}} v_i v_j,$$

*where the $v$'s are the variables, and*

$$v_k = dp_k - \sum_{i=1}^{n} p_{ki} dx_i,$$

*provided*

$$dz \neq \sum_{i=1}^{n} p_i dx_i,$$

*and the $v$'s are not zero.*

(3) *Algebraic invariants of the quadratic forms*

$$\sum_{ij} \frac{\partial f}{\partial p_{ij}} y_i y_j,$$

*and of the linear form*

$$\sum_{i=1}^{n} y_i dx_i,$$

*where the $y$'s are the variables, provided that*

$$dz = \sum_{i=1}^{n} p_i dx_i, \qquad dp_i = \sum_{k=1}^{n} p_{ik} dx_k.$$

In the above there are two sets of restrictions on the expressions $f$.

(1) The $r$-row determinants must not all be zero in the matrix

$$\left\| \begin{matrix} P_{1,11} & \cdots & P_{1,ij} & \cdots & P_{1,nn} \\ \vdots & & & & \\ P_{r,11} & \cdots & P_{r,ij} & \cdots & P_{r,nn} \end{matrix} \right\|.$$

(2) It must not be possible to satisfy the identity

$$\sum_{i=1}^{r} S_i B_i + AK \equiv 0,$$

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where $S$, $B$, $A$, $K$, are as previously defined, except by making the $B$'s all constant multiples of $A$.

The case when the second of the above restrictions is removed requires further consideration. It is obvious that it must be removed if $r$ is great enough, but the whole question will be left for future consideration. At present we content ourselves with a discussion of the case when $n$ is two.

For one expression $f$, $SB + AK \equiv 0$, provided that $S$ admits $A$ as a factor, since we are neglecting the possibility of $B$ being $\lambda A$, where $\lambda$ is a constant.

This gives $P_{11}a_2^2 - P_{12}a_1a_2 + P_{22}a_1^2 = 0$, as the condition for the existence of further integrals.

If this condition is satisfied, the equations of type (27) and (28) possess the two solutions

$$\alpha_1 = A_1a_1a_2 + P_{22}a_1c_{12} + P_{11}a_2c_{11},$$

$$\alpha_2 = A_2a_1a_2 + P_{11}a_2c_{12} + P_{22}a_1c_{11},$$

instead of the single one already given.

It may readily be shown that there are no new invariants, but if

$$I \equiv P_{11}a_2^2 - P_{12}a_1a_2 + P_{22}a_1^2 = 0,$$

then the two equations

$$\alpha_1 \equiv A_1a_1a_2 + P_{22}a_1c_{12} + P_{11}a_2c_{11} = 0,$$

$$\alpha_2 = A_2a_1a_2 + P_{11}a_2c_{12} + P_{22}a_1c_{22} = 0$$

are an invariant system.

We observe that $I$ is an invariant.

The equations $I = 0$, $\alpha_1 = 0$, $\alpha_2 = 0$, taken in conjunction with

$$dz = p_1dx_1 + p_2dx_2, \qquad dp_1 = p_{11}dx_1 + p_{12}dx_2, \qquad dp_2 = p_{12}dx_1 + p_{22}dx_2,$$

have an immediate and important interpretation in connection with the differential equation $f = 0$. They are precisely the equations for the characteristics of this differential equation.*

Now suppose that there are two expressions $f$. In this case there are four equations in seven variables of the types (27) and (28). They therefore possess in general three integrals. Two of these are already known, and are $df_1$, $df_2$.

The remaining one may be expressed as the determinant

$$\begin{vmatrix} \lambda_{1,1}, & \lambda_{1,2}, & 0 \\ K_{1,1}, & K_{1,2}, & I_1 \\ K_{1,2}, & K_{2,2}, & I_2 \end{vmatrix} \equiv \Delta,$$

* GOURSAT, *Equations aux dérivées partielles du second ordre*, vol. 2, p. 174.

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where

$$\lambda_1 = 2P_{11}a_2 - P_{12}a_1,$$
$$\lambda_2 = 2P_{22}a_1 - P_{12}a_2,$$
$$K_1 = a_1a_2A_2 + P_{11}a_2c_{12} + P_{22}a_1c_{22},$$
$$K_2 = a_1a_2A_1 + P_{22}a_1c_{12} + P_{11}a_2c_{11},$$
$$I = P_{22}a_1^2 - P_{12}a_1a_2 + P_{11}a_2^2.$$

It may easily be shown that $\Delta/a_1a_2$ satisfies the remaining system of equations, and therefore the complete system of integrals is $df_1$, $df_2$, $\Delta/a_1a_2$, and the invariants of the binary forms

$$(P_{1,11}, \tfrac{1}{2}P_{1,12}, P_{1,22} \between * )^2, \qquad (P_{2,11}, \tfrac{1}{2}P_{2,12}, P_{2,22} \between * )^2, \qquad (a_1, a_2 \between * ).$$

We omit for the present the interpretation of $\Delta$, and proceed to consider the case in which there are three expressions $f$.

In this case the equations of types (27) and (28) have the five functionally independent integrals $df_1$, $df_2$, $df_3$ and

$$H \equiv \begin{vmatrix} P_{1,11} & A_{1,1} & P_{1,22} \\ P_{2,11} & A_{2,1} & P_{2,22} \\ P_{3,11} & A_{3,1} & P_{3,22} \end{vmatrix} - \begin{vmatrix} A_{1,2} & P_{1,12} & P_{1,22} \\ A_{2,2} & P_{2,12} & P_{2,22} \\ A_{3,2} & P_{3,12} & P_{3,22} \end{vmatrix},$$

$$K \equiv \begin{vmatrix} P_{1,11} & A_{1,2} & P_{1,22} \\ P_{2,11} & A_{2,2} & P_{2,22} \\ P_{3,11} & A_{3,2} & P_{3,22} \end{vmatrix} - \begin{vmatrix} P_{1,11} & P_{1,12} & A_{1,1} \\ P_{2,11} & P_{2,12} & A_{2,1} \\ P_{3,11} & P_{3,12} & A_{3,1} \end{vmatrix}.$$

We substitute these integrals in the remaining equations, and the complete system of independent integrals of the equations thus obtained is the system of invariants of the binary forms

$$(P_{1,11}, \tfrac{1}{2}P_{1,12}, P_{1,22} \between * )^2,$$
$$(P_{2,11}, \tfrac{1}{2}P_{2,12}, P_{2,22} \between * )^2,$$
$$(P_{3,11}, \tfrac{1}{2}P_{3,12}, P_{3,22} \between * )^2,$$
$$(K, H \between * ),$$
$$(a_1, a_2 \between * ).$$

If we take the variables to be $a_2, -a_1$, we see that the solutions in question are the invariants and covariants of the binary forms

$$Ka_2 - Ha_1,$$
$$P_{1,11}a_2^2 - P_{1,12}a_1a_2 + P_{1,22}a_1^2,$$
$$P_{2,11}a_2^2 - P_{2,12}a_1a_2 + P_{2,22}a_1^2,$$
$$P_{3,11}a_2^2 - P_{3,12}a_1a_2 + P_{3,22}a_1^2.$$

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In addition to these there are the three solutions $df_1$, $df_2$, $df_3$, and these are all the functionally independent integrals of our type.

§4.

We shall consider to a small extent the more general type of expression $f$, that is to say the type in which there are more dependent variables than one.

Let there be $m$ dependent variables, $z_1, \cdots, z_m$, and as before let there be $n$ independent variables, $x_1, \cdots, x_n$. We use the same notation as on pp. 288, 289, and in addition $df/dx_\mu = A_\mu$.

In this case, the most general contact transformation possible may easily be shown to be a point transformation.

Let the expressions $f_\lambda$ be of the first order, that is to say, let them be functions of the variables $x_k$, $z_i$, $p_k^i$ $(i = 1, 2, \cdots, m;\ k = 1, 2, \cdots, n)$.

If $F$ is any first order invariant we have

$$\frac{dF}{dt} = \left(\mu_0 \frac{d\Omega_0}{dt} + \mu_1 \frac{d\Omega_1}{dt}\right) F,$$

where

$$\frac{d\Omega_0}{dt} = \sum_{k=1}^{n} \frac{\partial \xi_k}{\partial x_k} + \sum_{i=1}^{m} \frac{\partial \zeta_i}{\partial z_i},$$

$$\frac{d\Omega_1}{dt} = \sum_{i=1}^{m} \sum_{k=1}^{n} \frac{\partial \pi_k^i}{\partial p_k^i},$$

and $\mu_0$, $\mu_1$, have the meanings given on p. 288.

Expanding $dF/dt$ and equating to zero the coefficients of $\xi_k$, $\zeta_i$, $\partial\zeta_i/\partial x_\mu$, we see that the variables $x$, $z$, do not occur explicitly, and that the variables $p$, $x$, only occur through the variables $A$.

We assume $F$ to be a function of the variables $A$, $Z$, $P$, and

$$a_k \equiv dx_k,\ v_i \equiv dz_i - \sum_{k=1}^{n} p_k^i dx_k \qquad (k = 1, \cdots, n,\ i = 1, \cdots, m).$$

The values of the various increments involved are the following:

$$\frac{d}{dt} a_k = \sum_{\lambda=1}^{m} \frac{\partial \xi_k}{\partial z_\lambda} v_\lambda + \sum_{\mu=1}^{n} \frac{d\xi_k}{dx_\mu} a_\mu,$$

$$\frac{d}{dt} v_i = \sum_{\lambda=1}^{m} \frac{\partial \theta_i}{\partial z_\lambda} v_\lambda,$$

$$-\frac{dA_k}{dt} = \sum_{r=1}^{n} \frac{d\xi_r}{dx_k} A_r + \sum_{i=1}^{m} \sum_{\mu=1}^{n} P_\mu^i \frac{d^2\theta_i}{dx_\mu dx_k},$$

$$-\frac{dZ_\lambda}{dt} = \sum_{r=1}^{n} \frac{\partial \xi_r}{\partial z_\lambda} A_r + \sum_{s=1}^{m} \frac{\partial \theta_s}{\partial z_\lambda} Z_s + \sum_{i=1}^{m} \sum_{k=1}^{n} P_k^i \frac{d}{dx_k}\left(\frac{\partial \theta_i}{\partial z_\lambda}\right),$$

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$$-\frac{dP^h_k}{dt}=\sum_{i=1}^{m}P^i_k\frac{\partial\theta_i}{\partial z_h}-\sum_{\sigma=1}^{n}P^h_\sigma\frac{d\xi_k}{dx_\sigma}\qquad(h,\lambda=1,2,\cdots,m;\ k=1,2,\cdots,n).$$

Also

$$\mu_0\frac{d\Omega_0}{dt}+\mu_1\frac{d\Omega_1}{dt}=\mu\sum_{k=1}^{n}\frac{d\xi_k}{dx_k}+\nu\sum_{h=1}^{m}\frac{\partial\theta_h}{\partial z_h},$$

where $\mu$ and $\nu$ are constants depending on $\mu_0$ and $\mu_1$.

We substitute these values in the equation for $F$ and equate the coefficients of the various derivatives of $\xi$ and $\zeta$ to zero. We thus obtain the following system of equations for $F$:

$$a_\sigma F_{a_k}-\mathbf{S}A_kF_{A_\sigma}+\mathbf{S}\sum_{i=1}^{m}P^i_\sigma F_{P^i_k}=0\qquad(\sigma\neq k),\tag{47}$$

$$a_kF_{a_k}-\mathbf{S}A_kF_{A_k}+\mathbf{S}\sum_{i=1}^{m}P^i_kF_{P^i_k}=\mu F,\tag{48}$$

$$v_\lambda F_{a_k}-\mathbf{S}A_kF_{Z_\lambda}=0,\tag{49}$$

$$v_\lambda F_{v_i}-\mathbf{S}Z_iF_{Z_\lambda}-\mathbf{S}\sum_{k=1}^{n}P^i_kF_{P^\lambda_k}=0\qquad(\lambda\neq i),\tag{50}$$

$$v_iF_{v_i}-\mathbf{S}Z_iF_{Z_i}-\mathbf{S}\sum_{k=1}^{n}P^i_kF_{P^i_k}=\nu F,\tag{51}$$

$$-\mathbf{S}P^i_\mu F_{A_k}-\mathbf{S}P^i_kF_{A_\mu}=0,\tag{52}$$

$$-\mathbf{S}P^i_kF_{Z_\lambda}=0\tag{53}$$

$$(\sigma,k,\mu=1,\cdots,n);\ (i,\lambda,=1,\cdots,m).$$

The equations (49) give solutions of type

$$\Delta\equiv\sum_{k=1}^{n}a_kA_k+\sum_{i=1}^{m}v_iZ_i,$$

and the equations (53) then show that the $\Delta$'s cannot enter into $F$ unless the number $r$ of expressions $f_\lambda$ is greater than $mn$.

From equations (52) we deduce that the variables $A$ do not occur in $F$ unless

$$r>\tfrac{1}{2}m(n+1).$$

In the case when $m$ is unity, the equations are not all independent, and $r$ need not satisfy this last condition.

Suppose that $r$ is less than this number, then $F$ is a function of the variables $v_1,\cdots v_m,P^1_1,\cdots P^m_n$, which satisfies the equations

$$\mathbf{S}\sum_{i=1}^{m}P^i_\sigma F_{P^i_k}=0,\qquad(\sigma\neq k),$$

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$$\mathbf{S}\sum_{i=1}^{m} P_k^i F_{P_k^i} = \mu F,$$

$$\mathbf{S}\sum_{k=1}^{n} P_k^i F_{P_k^\lambda} - v_\lambda F_{v_i} = 0,$$

$$\mathbf{S}\sum_{k=1}^{n} P_k^i F_{P_k^i} - v_i F_{v_i} = -\nu F.$$

The last two equations show that $F$ must be an invariant or covariant of the linear forms

$$\sum_{i=1}^{m} P_{\lambda,k}^i v_i \qquad (k=1, 2, \cdots, n;\ \lambda=1, 2, \cdots, r).$$

The two first equations show that $F$ must at the same time be an invariant of the linear forms

$$\sum_{k=1}^{n} P_{\lambda,k}^i y_i \qquad (i=1, 2, \cdots, m;\ \lambda=1, 2, \cdots, r).$$

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# GROUPS OF ORDER $p^m$, WHICH CONTAIN CYCLIC SUBGROUPS OF ORDER $p^{m-3}$ *

BY

LEWIS IRVING NEIKIRK

## *Introduction.*

The groups of order $p^m$, which contain self-conjugate cyclic subgroups of orders $p^{m-1}$, and $p^{m-2}$ respectively, have been determined by BURNSIDE,† and the number of groups of order $p^m$, which contain cyclic non-self-conjugate subgroups of order $p^{m-2}$ has been given by MILLER.‡

Although in the present state of the theory, the actual tabulation of all groups of order $p^m$ is impracticable, it is of importance to carry the tabulation as far as may be possible. In this paper *all groups of order* $p^m$ ($p$ being an odd prime) *which contain cyclic subgroups of order* $p^{m-3}$ *and none of higher order* are determined. The method of treatment used is entirely abstract in character and, in virtue of its nature, it is possible in each case to give explicitly the generational equations of these groups. They are divided into three classes, and it will be shown that these classes correspond to the three partitions: $(m-3, 3)$, $(m-3, 2, 1)$ and $(m-3, 1, 1, 1)$, of $m$.

We denote by $G$ an abstract group $G$ of order $p^m$ containing operators of order $p^{m-3}$ and no operator of order greater than $p^{m-3}$. Let $P$ denote one of these operators of $G$ of order $p^{m-3}$. The $p^3$ power of every operator in $G$ is contained in the cyclic subgroup $\{P\}$, otherwise $G$ would be of order greater than $p^m$. The complete division into classes is effected by the following assumptions:

I. There is in $G$ at least one operator $Q_1$, such that $Q_1^{p^2}$ is not contained in $\{P\}$.

II. The $p^2$ power of every operator in $G$ is contained in $\{P\}$, and there is at least one operator $Q_1$, such that $Q_1^p$ is not contained in $\{P\}$.

III. The $p$th power of every operator in $G$ is contained in $\{P\}$.

The number of groups for Class I, Class II, and Class III, together with the total number, are given in the table below:

* Presented to the Society April, 25, 1903. Received for publication August 16, 1904, and February 25, 1905.

† *Theory of Groups of a Finite Order*, pp. 75–81.

‡ Transactions, vol. 2 (1901), p. 259, and vol. 3 (1902), p. 383.



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I.

| | I | $II_1$ | $II_2$ | $II_3$ | II | III | Total |
|---|---|---|---|---|---|---|---|
| $p>3$ $m>8$ | 9 | $20+p$ | $6+2p$ | $6+2p$ | $32+5p$ | 23 | $64+5p$ |
| $p>3$ $m=8$ | 8 | $20+p$ | $6+2p$ | $6+2p$ | $32+5p$ | 23 | $63+5p$ |
| $p>3$ $m=7$ | 6 | $20+p$ | $6+2p$ | $6+2p$ | $32+5p$ | 23 | $61+5p$ |
| $p=3$ $m>8$ | 9 | 23 | 12 | 12 | 47 | 16 | 72 |
| $p=3$ $m=8$ | 8 | 23 | 12 | 12 | 47 | 16 | 71 |
| $p=3$ $m=7$ | 6 | 23 | 12 | 12 | 47 | 16 | 69 |

The number of groups of order $p^m$, with cyclic subgroups of orders $p^m$, $p^{m-1}$, and $p^{m-2}$, are given in Table II:

II.

| | $p^m$ | $p^{m-1}$ | $p^{m-2}$ |
|---|---|---|---|
| $p>2$ $m>5$ | 1 | 2 * | 11 † |

The number of groups of order $p^\alpha$, $\alpha = 1, 2, 3, 4, 5$ are given in Table III:

III.

| | $\alpha=1$ | $\alpha=2$ ‡ | $\alpha=3$ ‡ | $\alpha=4$ ‡ | $\alpha=5$ § |
|---|---|---|---|---|---|
| $p=2$ | 1 | 2 | 5 | 14 | 51 |
| $p=3$ | 1 | 2 | 5 | 15 | 66 |
| $p=12k-1$ | 1 | 2 | 5 | 15 | $65+2p$ |
| $p=12k+5$ | 1 | 2 | 5 | 15 | $67+2p$ |
| $p=12k-5$ | 1 | 2 | 5 | 15 | $69+2p$ |
| $p=12k+1$ | 1 | 2 | 5 | 15 | $71+2p$ |

* BURNSIDE, *Theory of Groups*, Art. 65, p. 75.

† Ibid., Art. 66, p. 77; MILLER, Transactions, vol. 2 (1901), p. 259, and vol. 3 (1902), p. 383.

‡ YOUNG, *On the determination of groups whose order is a power of a prime*, American Journal of Mathematics, vol. 15 (1893), pp. 124–178. COLE and GLOVER, *On groups whose orders are the product of three prime factors*, American Journal of Mathematics, vol. 15 (1893), pp. 191–220. HÖLDER, *Die Gruppen der Ordnungen*, $p^3$, $pq^3$, $pqr$, $p^4$, Mathematische Annalen, vol. 43 (1893), pp. 301–412.

§ BAGNERA, *La composizione dei gruppi finiti il cui grado è la quinta potenza di un numero primo*, Annali di Matematica, vol. 3 (1898), pp. 137–228.



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I give in this paper the whole investigation of Class I, but only the results of Classes II, and III. The investigation as a whole will appear in the Publications of the University of Pennsylvania, Mathematics, no. 3.

*Class I.*

1. *General notations and relations.* — The group $G$ is generated by the two operators $P$ and $Q_1$. For brevity we set*

$$Q_1^a P^b Q_1^c P^d \ldots = [a, b, c, d, \ldots].$$

Then the operators of $G$ are given each uniquely in the form

$$[y, x] \qquad \begin{pmatrix} y = 0, 1, 2, \ldots, p^2 - 1 \\ x = 0, 1, 2, \ldots, p^{m-3} - 1 \end{pmatrix}.$$

We have the relation

$$(1) \qquad Q_1^{p^2} = P^{hp^2}.$$

There is in $G$, a subgroup $H_1$ of order $p^{m-2}$, which contains $\{P\}$ self-conjugately.† The subgroup $H_1$ is generated by $P$ and some operator $Q_1^y P^x$ of $G$; it then contains $Q_1^y$ and is therefore generated by $P$ and $Q_1^{p^2}$; it is also self-conjugate in $H_2 = \{Q_1^p, P\}$ of order $p^{m-1}$, and $H_2$ is self-conjugate in $G$.

From these considerations we have the equations

$$(2)\ddagger \qquad Q_1^{-p^2} P Q_1^{p^2} = P^{1+kp^{m-4}},$$

$$(3) \qquad Q_1^{-p} P Q_1^{p} = Q_1^{\beta p^2} P^{a_1},$$

$$(4) \qquad Q_1^{-1} P Q_1 = Q_1^{bp} P^{a_1}.$$

2. *Determination of $H_1$. Derivation of a formula for $[yp^2, x]^s$.* — From (2), by repeated multiplication we obtain

$$[-p^2, x, p^2] = [0, x(1 + kp^{m-4})];$$

and by a continued use of this equation we have

$$[-yp^2, x, yp^2] = [0, x(1 + kp^{m-4})^y] = [0, x(1 + kyp^{m-4})] \quad (m > 4),$$

and from this last equation,

$$(5) \qquad [yp^2, x]^s = \left[syp^2, x\left\{s + k\binom{s}{2} yp^{m-4}\right\}\right].$$

3. *Determination of $H_2$. Derivation of a formula for $[yp, x]^s$.* — It follows from (3) and (5) that

* With J. W. Young, *On a certain group of isomorphisms*, American Journal of Mathematics, vol. 25 (1903), p. 206.

† Burnside: *Theory of Groups*, art. 54, p. 64.

‡ Burnside: *Theory of Groups*, art. 56, p. 66.

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$$[-p^2, 1, p^2] = \left[\beta \frac{\alpha_1^p - 1}{\alpha_1 - 1} p^2, \alpha_1^p \left\{1 + \frac{\beta k}{2} \frac{\alpha_1^p - 1}{\alpha_1 - 1} p^{m-4}\right\}\right] \quad (m > 4).$$

Hence, by (2),

$$\beta \frac{\alpha_1^p - 1}{\alpha_1 - 1} p^2 \equiv 0 \pmod{p^3},$$

$$\alpha_1^p \left\{1 + \frac{\beta k}{2} \frac{\alpha_1^p - 1}{\alpha_1 - 1} p^{m-4}\right\} + \beta \frac{\alpha_1^p - 1}{\alpha_1 - 1} h p^2 \equiv 1 + k p^{m-4} \pmod{p^{m-3}}.$$

From these congruences, we have for $m > 6$

$$\alpha_1^p \equiv 1 \pmod{p^3}, \qquad \alpha_1 \equiv 1 \pmod{p^2},$$

and obtain, by setting

$$\alpha_1 = 1 + \alpha_2 p^2,$$

the congruence

$$\frac{(1 + \alpha_2 p^2)^p - 1}{\alpha_2 p^3} (\alpha_2 + h\beta) p^3 \equiv k p^{m-4} \pmod{p^{m-3}};$$

and so

$$(\alpha_2 + h\beta) p^3 \equiv 0 \pmod{p^{m-4}},$$

since

$$\frac{(1 + \alpha_2 p^2)^p - 1}{\alpha_2 p^3} \equiv 1 \pmod{p^2}.$$

From the last congruences

$$(\alpha_2 + h\beta) p^3 \equiv k p^{m-4} \pmod{p^{m-3}}. \tag{6}$$

Equation (3) is now replaced by

$$Q_1^{-p} P Q_1^p = Q_1^{\beta p^2} P^{1 + \alpha_2 p^2}. \tag{7}$$

From (7), (5), and (6)

$$[-yp, x, yp] = \left[\beta x y p^2, x\{1 + \alpha_2 y p^2\} + \beta k \binom{x}{2} y p^{m-4}\right].$$

A continued use of this equation gives

$$\begin{aligned} [yp, x]^s = \Big[ & syp + \beta \binom{s}{2} x y p^2, \\ & xs + \binom{s}{2} \left\{\alpha_2 x y p^2 + \beta k \binom{x}{2} y p^{m-4}\right\} + \beta k \binom{s}{3} x^2 y p^{m-4}\Big]. \end{aligned} \tag{8}$$

4. *Determination of $G$.*—From (4) and (8),

$$[-p, 1, p] = [Np, a_1^p + Mp^2].$$

From the above equation and (7),

$$a_1^p \equiv 1 \pmod{p^2}, \qquad a_1 \equiv 1 \pmod{p}.$$

Set

$$a_1 = 1 + a_2 p,$$

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and equation (4) becomes

$$Q_1^{-1} P Q_1 = Q_1^{bp} P^{1+a_2 p}. \tag{9}$$

From (9), (8) and (6),

$$[-p^2, 1, p^2] = \left[\frac{(1+a_2 p)^{p^2}-1}{a_2 p} bp,\ (1+a_2 p)^{p^2}\right],$$

and from (1) and (2),

$$\frac{(1+a_2 p)^{p^2}-1}{a_2 p} bp \equiv 0 \pmod{p^3},$$

$$(1+a_2 p)^{p^2} + bh\frac{(1+a_2 p)^{p^2}-1}{a_2 p} p \equiv 1 + kp^{m-4} \pmod{p^{m-3}}.$$

By a reduction similar to that used before,

$$(a_2 + bh)p^3 \equiv kp^{m-4} \pmod{p^{m-3}}. \tag{10}$$

The groups in this class are completely defined by (9), (1) and (10).

These defining relations may be presented in simpler form by a suitable choice of the second generator $Q_1$. From (9), (6), (8) and (10),

$$[1, x]^{p^3} = [p^3, xp^3] = [0, (x+h)p^3] \qquad (m>6),$$

and, if $x$ be so chosen that

$$x + h \equiv 0 \pmod{p^{m-6}},$$

$Q_1 P^x$ is an operator of order $p^3$ whose $p^2$ power is not contained in $\{P\}$. Let $Q_1 P^x = Q$. The group $G$ is generated by $Q$ and $P$, where

$$Q^{p^3} = 1, \qquad P^{p^{m-3}} = 1.$$

Placing $h = 0$ in (6) and (10) we find

$$\alpha_2 p^3 \equiv a_2 p^3 \equiv kp^{m-4} \pmod{p^{m-3}}.$$

Let $\alpha_2 = \alpha p^{m-7}$, and $a_2 = ap^{m-7}$. Equations (7) and (9) are now replaced by

$$\begin{aligned} Q^{-p} P Q^p &= Q^{\beta p^2} P^{1+\alpha p^{m-5}}, \\ Q^{-1} P Q &= Q^{bp} P^{1+ap^{m-6}}. \end{aligned} \tag{11}$$

As a direct result of the foregoing relations, the groups in this class correspond to the partition $(m-3, 3)$. From (11) we find

$$[-y, 1, y] = [byp, 1 + ayp^{m-6}] \qquad (m>8).*$$

* For $m=8$ it is necessary to add $a^2\binom{y}{2}p^4$ to the exponents of $P$ and for $m=7$ the terms $a(a+abp/2)\binom{y}{2}p^2 + a^3\binom{y}{3}p^3$ to the exponent of $P$, and the term $ab\binom{y}{2}p^2$ to the exponent of $Q$. The extra term $27ab^2k\binom{y}{3}$ is to be added to the exponent of $P$ for $m=7$ and $p=3$.

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It is important to notice that by placing $y = p$ and $p^2$ in the preceding equation we find that

$$b \equiv \beta \pmod{p},\ a \equiv \alpha \equiv k \pmod{p^3}\ (m > 7).*$$

A combination of the last equation with (8) yields

$$(12)\qquad \begin{aligned}[-y, x, y] = [&bxyp + b^2\tbinom{x}{2}yp^2,\\ &x(1 + ayp^{m-6}) + ab\tbinom{x}{2}yp^{m-5} + ab^2\tbinom{x}{3}yp^{m-4}] \quad (m > 8).\dagger\end{aligned}$$

From (12) we get

$$(13)\qquad \begin{aligned}[y, x]^s = [&ys + by\{(x + b\tbinom{x}{2}p)\tbinom{s}{2} + x\tbinom{s}{3}p\}p,\\ &xs + ay\{(x + b\tbinom{x}{2}p + b^2\tbinom{x}{3}p^2)\tbinom{s}{2}\\ &\quad + (bx^2p + 2b^2x\tbinom{x}{2}p^2)\tbinom{s}{3} + bx^2\tbinom{s}{4}p^2\}p^{m-6}] \quad (m > 8).\ddagger\end{aligned}$$

5. *Transformation of the Groups.*— The general group $G$ of Class I is specified, in accordance with the relations (2) (11) by two integers $a$, $b$ which (see (11)) are to be taken mod $p^3$, mod $p^2$, respectively. Accordingly setting

$$a = a_1p^\lambda,\ b = b_1p^\mu,$$

where

$$dv\,[a_1, p] = 1,\ dv\,[b_1, p] = 1 \qquad (\lambda = 0, 1, 2, 3;\ \mu = 0, 1, 2),$$

we have for the group $G = G(a, b) = G(a, b)(P, Q)$ the generational determination:

$$G(a, b): \begin{cases} Q^{-1}PQ = Q^{b_1p^\mu+1}P^{1+a_1p^{m+\lambda-6}}, \\ Q^{p^3} = 1,\ P^{p^{m-3}} = 1. \end{cases}$$

---

* For $m = 7$, $ap^2 - a^2p^3/2 \equiv ap^2 \pmod{p^4}$, $ap^3 \equiv kp^3 \pmod{p^4}$. For $m = 7$ and $p = 3$ the first of the above congruences has the extra terms $27(a^3 + ab\beta k)$ on the left side.

† For $m = 8$ it is necessary to add the term $a\binom{y}{2}xp^4$ to the exponent of $P$, and for $m = 7$ the terms $x\{a(a + abp/2)\binom{y}{2}p^2 + a^3\binom{y}{3}p^3\}$ to the exponent of $P$, with the extra term $27\,ab^2k\binom{y}{3}x$ for $p = 3$, and the term $ab\binom{y}{2}xp^2$ to the exponent of $Q$.

‡ For $m = 8$ it is necessary to add the term $\frac{1}{2}axy\binom{s}{2}[\frac{1}{3}y(2s-1) - 1]p^4$ to the exponent of $P$, and for $m = 7$ the terms

$$x\left\{\frac{a}{2}\left(a + \frac{ab}{2}p\right)\left(\frac{2s-1}{3}y - 1\right)\binom{s}{2}yp^2 + \frac{a^3}{3!}\left(\binom{s}{2}y^2 - (2s-1)y + 2\right)\binom{s}{2}yp^3 + \frac{a^2bxy^2}{2}\binom{s}{3}\frac{3s-1}{2}p^3 + \frac{a^2b}{2}\left(\frac{s(s-1)^2(s-4)}{4!}y - \binom{s}{3}\right)yp^3\right\},$$

with the extra terms

$$27\,abxy\left\{\frac{bk}{3!}\left[\binom{s}{2}y^2 - (2s-1)y + 2\right]\binom{s}{2} + x(b^2k + a^2)(2y^2 + 1)\binom{s}{3}\right\}$$

for $p = 3$, to the exponent of $P$, and the terms

$$\frac{ab}{2}\left\{\frac{2s-1}{3}y - 1\right\}\binom{s}{2}xyp^2$$

to the exponent of $Q$.

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Not all of these groups, however, are distinct. Suppose that

$$G(a, b)(P, Q) \sim G(a', b')(P', Q'),$$

by the correspondence

$$C = \begin{bmatrix} Q, & P \\ Q'_1, & P'_1 \end{bmatrix},$$

where

$$Q'_1 = Q'^{y'} P'^{x' p^{m-6}}, \text{ and } P'_1 = Q'^{y} P'^{x},$$

with $y'$ and $x$ prime to $p$.

Since

$$Q^{-1} P Q = Q^{bp} P^{1+ap^{m-6}},$$

then

$$Q_1'^{-1} P'_1 Q'_1 = Q_1'^{bp} P_1'^{+ap^{m-6}},$$

or in terms of $Q'$, and $P'$

$$[y + b'xy'p + b'^2\binom{x}{2}y'p^2,\ x(1 + a'y'p^{m-6}) + a'b'\binom{x}{2}y'p^{m-5}$$
$$+ a'b'^2\binom{x}{3}y'p^{m-4}] = [y + by'p,\ x + (ax + bx'p)p^{m-6}] \qquad (m > 8)$$

and

$$(14) \qquad by' \equiv b'xy' + b'^2\binom{x}{2}y'p \pmod{p^2},$$

$$(15) \qquad ax + bx'p \equiv a'y'x + a'b'\binom{x}{2}y'p + a'b'^2\binom{x}{3}y'p^2 \pmod{p^3}.$$

The necessary and sufficient condition for the simple isomorphism of these two groups $G(a, b)$ and $G(a', b')$ is, that the above congruences shall be consistent and admit of solution for $x$, $y$, $x'$ and $y'$. The congruences may be written

$$b_1 p^\mu \equiv b'_1 x p^{\mu'} + b_1'^2\binom{x}{2}p^{2\mu'+1} \pmod{p^2},$$

$$a_1 x p^\lambda + b_1 x' p^{\mu+1} \equiv y'\{a'_1 x p^{\lambda'} + a'_1 b'_1\binom{x}{2}p^{\lambda'+\mu'+1} + a'_1 b_1'^2\binom{x}{3}p^{\lambda'+2\mu'+2}\} \pmod{p^3}.$$

Since $dv\,[x, p] = 1$ the first congruence gives $\mu = \mu'$ and $x$ may always be so chosen that $b_1 = 1$.

We may choose $y'$ in the second congruence so that $\lambda = \lambda'$ and $a_1 = 1$ except for the cases $\lambda' \geqq \mu + 1 = \mu' + 1$ when we will so choose $x'$ that $\lambda = 3$.

The type groups of Class I for $m > 8$* are then given by

$$(\mathrm{I}) \qquad G(p^\lambda, p^\mu) : Q^{-1}PQ = Q^{p^{1+\mu}} P^{1+p^{m-6+\lambda}},\ Q^{p^3} = 1,\ P^{p^{m-3}} = 1,$$
$$\left(\begin{matrix} \mu = 0, 1, 2;\ \lambda = 0, 1, 2, \lambda \leqq \mu; \\ \mu = 0, 1, 2;\ \lambda = 3. \end{matrix}\right)$$

Of the above groups $G(p^\lambda, p^\mu)$ the groups for $\mu = 2$ have the cyclic sub-

* For $m = 8$ the additional term $ayp$ appears on the left side of congruence (14) and $G(1, p^2)$ and $G(1, p)$ becomes simply isomorphic. The extra terms appearing in congruence (15) do not effect the result. For $m = 7$ the additional term $ay$ appears on the left side of (14) and $G(1, 1)$, $G(1, p)$, and $G(1, p^2)$ become simply isomorphic, also $G(p, p)$ and $G(p, p^2)$.

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group $\{P\}$ self-conjugate, while the group $G(p^3, p^2)$ is the abelian group of type $(m-3, 3)$.

*Class II.*

1. *Sections.* — Class II is divided into three sections.
2. *Types.* — The types of Section I $(m > 5)$ are defined by

$$(\mathrm{II}_1)\begin{cases} Q^{-1}PQ = Q^{\beta h}P^{1+ap^{m-5}}, \\ R^{-1}PR = Q^{bp}P^{1+ap^{m-4}}, \\ R^{-1}QR = Q^{1+dp}P^{cp^{m-4}}, \\ R^p = Q^{p^2} = P^{p^{m-3}} = 1, \end{cases}$$

where the constants have the values given in the table below,

$(\mathrm{II}_1)$

| | $a$ | $b$ | $a$ | $\beta$ | $c$ | $d$ | | $a$ | $b$ | $a$ | $\beta$ | $c$ | $d$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 1 | 0 | 0 | 1 | 10 | 1 | 0 | $p$ | 0 | 0 | 0 |
| 2 | 0 | 0 | 1 | 0 | 0 | 0 | 11 | 0 | 0 | $p$ | 1 | 0 | 0 |
| 3 | 0 | 0 | 1 | 1 | 0 | 0 | 12 | 0 | 0 | $p$ | 0 | 1 | 0 |
| 4 | 0 | 0 | 1 | 0 | 1 | 0 | 13 | 1 | 0 | $p$ | 1 | 0 | 0 |
| 5 | 0 | 1 | 1 | 0 | $\kappa$ | 0 | 14 | 0 | 1 | $p$ | 0 | $\kappa$ | 0 |
| 6 | 0 | 0 | 1 | 1 | 1 | 0 | 15 | 0 | 0 | $p$ | 1 | 1 | 0 |
| 7 | $\omega$ | 0 | $p$ | 0 | 0 | 1 | 16 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 0 | 0 | 0 | 0 | 0 | 1 | 17 | 0 | 1 | 0 | 0 | 0 | 0 |
| 9 | 0 | 0 | $p$ | 0 | 0 | 0 | | | | | | | |

$\kappa = 0, 1$, and a non-residue (mod $p$),

$\omega = 0, 1, 2, \cdots, p-1$.

The types of Section 2 are defined by

$$(\mathrm{II}_2)\begin{cases} R^{-1}PR = P^{1+kp^{m-4}}, \\ Q^{-1}PQ = RP^{1+\epsilon p^{m-5}}, \\ Q^{-1}RQ = Q^{rp}RP^{\epsilon p^{m-4}}, \\ R^p = Q^{p^2} = P^{p^{m-3}} = 1, \end{cases}$$

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where the values of the constants are given in the following table:

$(\mathrm{II}_2)$

| | $k$ | $\epsilon$ | $c$ | $e$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 0 |
| 2 | 0 | 1 | 1 | 0 |
| 3 | 0 | 1 | 0 | $\omega$ |
| 4 | 0 | $p$ | $\omega$ | 0 |
| 5 | 0 | $p$ | 1 | 0 |
| 6 | 0 | $p$ | 0 | $\kappa$ |

$\kappa = 0, 1,$ and a non-residue $(\bmod p)$,

$\omega = 0, 1, 2, \cdots, p-1.$

The groups of Section 3 are defined by

$$(\mathrm{II}_3)\begin{cases} Q^{-1}PQ = PR^{\gamma 1+\epsilon p^{m-3}}, \\ Q^{-1}RQ = RP^{\epsilon p^{m-4}}, \\ R^{-1}PR = Q^p P, \\ R^p = Q^{p^2} = P^{p^{m-3}} = 1, \end{cases}$$

with the values $\gamma = 1$ and a non-residue $(\bmod p)$; $\epsilon = 1$, $e = 0, 1, 2, \cdots, p-1$, and $\epsilon = p$, $e = 0, 1,$ and a non-residue $(\bmod p)$, $2p + 6$ groups in all.

*Class III.*

*Types.* — The types of Class III are defined by

$$(\mathrm{III})\begin{cases} Q^{-1}PQ = P^{1+kp^{m-4}}, \\ R^{-1}PR = Q^\beta P, \\ R^{-1}QR = QP^{\alpha p^{m-4}}, \\ S^{-1}PS = R^\gamma Q^\delta P^{1+\epsilon p^{m-4}}, \\ S^{-1}QS = R^c Q, \\ S^{-1}RS = RP^{jp^{m-4}}, \end{cases}$$

where the constants have the values given in the table:

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(III)

| | $\alpha$ | $\beta$ | $c$ | $\gamma$ | $\delta$ | $k$ | $\varepsilon$ | $j$ | | $\alpha$ | $\beta$ | $c$ | $\gamma$ | $\delta$ | $k$ | $\varepsilon$ | $j$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 10 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 |
| 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 11 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| 3 | $\kappa$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 12 | $\kappa$ | 1 | 0 | 0 | 0 | 0 | 1 | 0 |
| 4* | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 13 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| 5 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 14* | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 |
| 6 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 15 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 |
| 7* | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 16* | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 1 |
| 8 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 17* | 0 | 1 | 0 | $\kappa$ | 0 | 1 | 0 | 1 |
| 9* | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | | | | | | | | | |

$\kappa = 0$, 1, and a non-residue (mod $p$).

* These groups for $p = 3$ ($k \neq 0$) are simply isomorphic with groups in Class II.

UNIVERSITY OF PENNSYLVANIA,
*February* 23, 1905.

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# ON THE INVARIANT SUBGROUPS OF PRIME INDEX*

BY

G. A. MILLER

The totality formed by all the operators of any group ($G$) which are common to all the invariant subgroups of prime index ($p$) constitutes a characteristic subgroup, and the corresponding quotient group is the abelian group of order $p^\lambda$ and of type $(1, 1, 1, \cdots)$.† The number of the invariant subgroups of index $p$ is therefore $p^\lambda - 1/p - 1$. The given totality includes all the operators of $G$ which are $p$th powers, and it is composed of such operators whenever $G$ is abelian. In this case $\lambda$ is clearly equal to the number of the invariants in the Sylow subgroup of order $p^m$ contained in $G$. This fact follows also directly from the theory of reciprocal groups, since $p^\lambda$ is the order of the subgroup generated by all the operators of order $p$ contained in $G$. For instance, the abelian group $G$ contains only one subgroup of index $p$ whenever its Sylow subgroup of order $p^m$, $m > 0$, is cyclic, it contains $p + 1$ such subgroups whenever this Sylow subgroup involves two invariants, $p^2 + p + 1$ whenever there are three invariants, etc.

When $G$ is non-abelian the determination of $\lambda$ is much more difficult. Its value can clearly not exceed the value of $\lambda$ for a Sylow subgroup of order $p^m$ contained in $G$, but it may be less. Hence it is of fundamental importance to determine the values of $\lambda$ for the groups of order $p^m$, and we shall assume that this is the order of $G$ in what follows. Instead of determining the value of $\lambda$ for given types of groups, it seems much more desirable to determine the possible types of groups when the value of $\lambda$ is given. This problem soon becomes very difficult. When $\lambda = 1$, $G$ is cyclic; and when $\lambda = m$, $G$ is the abelian group of type $(1, 1, 1, \cdots)$. These extreme cases relate to fundamental groups whose elementary properties are well known. The case when $\lambda = m - 1$ includes the Hamiltonian groups. All the groups which belong to this case have recently been determined.‡ The main object of the present paper is the study of an interesting category of groups which belong to the case when

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† BAUER, Nouvelles Annales, vol. 19 (1900), p. 509.

‡ Mathematische Annalen, vol. 60 (1905).



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$\lambda = m - 2$; viz., all those whose commutator subgroup is of order $p$. The two abelian groups which come under this case are of types $(3, 1, 1, \cdots)$ and $(2, 2, 1, 1, \cdots)$ respectively. In what follows it will be assumed that $G$ is non-abelian.

The following theorem is of considerable interest in itself and is proved here because it will be convenient to employ a special case of it in the study of the groups under consideration.

THEOREM. *If the commutator subgroup of a group of order $p^m$ is of order $p$, each of the operators which are common to every subgroup of index $p$ is invariant under the entire group.*

As an invariant subgroup of order $p$ is composed of invariant operators under a group of order $p^m$, it follows that all the commutators of the group in question are invariant. Hence each operator of the group contains at most $p$ conjugates. If an operator $(s)$ were not commutative with each of the operators which are common to every subgroup of index $p$, the subgroup of index $p$ composed of all the operators which are commutative with $s$ would not include all the given common operators. As this is impossible each of the common operators is invariant under the entire group.

Let $K$ represent the largest subgroup of $G$ which is common to all its subgroups of index $p$. Since $G$ contains just $1 + p + p^2 + \cdots + p^{m-3}$ subgroups of this index, the order of $K$ is $p^2$. We shall consider the groups in question in two sections, — the first will be devoted to those in which $K$ is cyclic and the second to those in which $K$ is non-cyclic. The commutator subgroup of order $p$ will be denoted by $C$ in both of these sections.

## § 1. *Determination of all the possible groups when $K$ is cyclic.*

Let $I$ represent the commutator quotient group of $G$, that is, $I$ is the quotient group of order $p^{m-1}$ corresponding to $C$. Since $I$ is abelian and contains just $1 + p + p^2 + \cdots + p^{m-3}$ subgroups of index $p$ it is of type $(2, 1, 1, \cdots)$. The subgroup of $G$ which corresponds to all the operators of $I$ whose orders do not exceed $p$ will be denoted by $H$. This subgroup is characteristic since it is composed of all the operators of $G$ whose orders are less than $p^3$. This result follows also directly from the fact that $H$ is composed of all the operators of $G$ whose $p$th powers are commutators of $G$.

Since $H$ is a characteristic subgroup of $G$ any two $G$'s must be distinct whenever their $H$'s are distinct. The converse is not necessarily true; that is, two $G$'s involving the same $H$ may be distinct groups as will be seen in what follows. Moreover, $H$ is one of the system of groups mentioned above in which $\lambda = m_1 - 1$; $p^{m_1}$ being the order of $H$. The value of $m_1$ is $m - 1$ as stated above. Hence $H$ may be regarded as known and it remains only to find the other possible operators of $G$. When $H$ is abelian it is simply isomorphic

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with $I$ and it is easy to prove that only one group ($G_1$) can involve such an $H$ and satisfy the conditions that $K$ is cyclic and that there are just $1 + p + p^2 + \cdots + p^{m-3}$ subgroups of index $p$, $p^m$ being the order of the group.

The group of cogredient isomorphisms of $G_1$ is of order $p^2$ since $C$ is of order $p$ and $H$ is abelian. The truth of this statement may be seen as follows. If $t$ is any operator of $G_1 - H$* it has just $p$ conjugates under $G_1$ and hence it is commutative with a subgroup of order $p^{m-2}$ contained in $H$. Since these $p^{m-2}$ operators are commutative with every operator of $H$ as well as with $t$ they are commutative with every operator of $G_1$, that is, the group of cogredient isomorphisms of $G_1$ is of order $p^2$. Since the $p^{m-2}$ invariant operators include $K$ they constitute an abelian group of type $(2, 1, 1, \cdots)$.

It is now very easy to prove that there could not be two distinct $G$'s involving the same abelian $H$. For in two such $G$'s the $H$'s could be made simply isomorphic in such a way that the $K$ of one $G$ would correspond to the $K$ of the other, and that the subgroups composed of the invariant operators of the two $H$'s would correspond. After this had been done, two operators of order $p^3$, one from each $G$, could be so chosen that their $p$th powers would correspond and that they would transform corresponding operators into corresponding operators. That is, the $G$'s would be simply isomorphic.†

When $H$ is non-abelian the considerations become a little more difficult. The results are as follows: *There are always just two $G$'s for a given $H$ except when $H$ involves only $p^2$ invariant operators; in this special case there is only one $G$ which involves a given $H$.* The main theorems which will be employed to arrive at these results are: The order of the group of cogredient isomorphisms of $G$ is an even power of $p$, and the order of the group of cogredient isomorphisms of $H$ is either the same as that of $G$ or it is the quotient obtained by dividing that of $G$ by $p^2$. This theorem is a direct consequence of the fact that the commutator subgroup of $G$ is of order $p$, and hence requires no proof here.‡ The other fundamental theorem is as follows: It is possible to make $H$ simply isomorphic with itself in such a way that a subgroup of index $p$ corresponds to any arbitrary subgroup of this index provided these two subgroups involve the same number of invariant operators and include invariant operators of order $p^2$. Moreover, the subgroups formed by the invariant operators can be made simply isomorphic in any arbitrary manner such that the commutator subgroup corresponds to itself.

The proof of this theorem is readily obtained by means of the following known properties of $H$. There is one and only one $H$ whose group of cogredient isomorphisms is of order $p^{2\alpha}$, $\alpha < (m-2)/2$; when $2\alpha < m - 3$ this $H$

* This symbol represents the operators of $G_1$ which are not also in $H$.

† Bulletin of the American Mathematical Society, vol. 3 (1897), p. 218.

‡ FITE, Transactions of the American Mathematical Society, vol. 3 (1902), p. 342.

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is the direct product of a group which has just $p^2$ invariant operators and an abelian group of type $(1, 1, \cdots)$. If a subgroup of index $p$ in $H$ includes all the invariant operators under $H$, the order of its group of cogredient isomorphisms is obtained by dividing the order of the group of cogredient isomorphisms of $H$ by $p^2$. When this condition is not satisfied the subgroup of index $p$ has the same group of cogredient isomorphisms as $H$ has.*

By employing these theorems it is easy to prove that there is only one $G$ which involves $H$ and has the same group of cogredient isomorphisms as $H$ has. For if there were two such $G$'s they could be obtained by adding successively to $H$ the two operators $t_1$, $t_2$ of order $p^3$. As the $p$th powers of these operators would be invariant under $H$ it would be possible to make $H$ simply isomorphic with itself in such a manner that $t_1^p$ would correspond to $t_2^p$. Hence the entire groups would be simply isomorphic; that is, there is only one such $G$.

When the group of cogredient isomorphisms of $G$ is not the same as that of $H$, it is necessary that $H$ include at least $p^3$ invariant operators. Whenever this condition is satisfied it is evident that a $G$ can be constructed. If there were two $G$'s involving the same $H$ they could again be constructed by adding to $H$ two operators of order $p^3$, which may be represented respectively by $t_1$, $t_2$. As these operators would be commutative with all the operators of a subgroup of index $p$ contained in $H$, and as $H$ could be made simply isomorphic with itself in such a manner that these subgroups would correspond to themselves, and, moreover, $t_1^p$ would correspond to $t_2^p$, it follows as before that there is only one such $G$. The results of this section may be summarized as follows:

When $H$ is given it is always possible to construct two $G$'s, provided $H$ contains more than $p^2$ invariant operators. When $H$ involves only $p^2$ invariant operators, only one $G$ is possible. One of these two systems is composed of all the groups of order $p^m$ which involve invariant operators of order $p^3$ and contain just $1 + p + p^2 + \cdots + p^{m-3}$ subgroups of index $p$. *There are just* $\frac{1}{2}(m-1)$ *such groups when* $m$ *is odd. When* $m$ *is even their number is* $\frac{1}{2}(m-2)$. The other system includes the same number of groups when $m$ is even, but it contains one less when $m$ is odd. That is, *the number of the groups which belong to this system is* $\frac{1}{2}(m-3)$ or $\frac{1}{2}(m-2)$, *as* $m$ *is odd or even.* This system is composed of all the possible groups of order $p^m$ which contain just $1 + p + p^2 + \cdots + p^{m-3}$ subgroups of index $p$ and are generated by operators of order $p^3$ without also including invariant operators of this order.

If a group belongs to the latter system and contains more than $p^2$ invariant operators it is the direct product of a $G$ which contains just $p^2$ invariant operators and an abelian group of type $(1, 1, \cdots)$. Similarly, if a group belongs to the former system and contains more than $p^3$ invariant operators it is the direct product of a $G$ containing just $p^3$ invariant operators and an abelian group of type $(1, 1, 1, \cdots)$.

*Mathematische Annalen, loc. cit.

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§ 2. *Determination of all the possible groups when $K$ is non-cyclic.*

The commutator quotient group $I$ is again the abelian group of type $(2, 1, 1, \cdots)$ and of order $p^{m-1}$. The characteristic subgroup $H$ corresponds to all the operators of $I$ whose orders do not exceed $p$ just as in the preceding section. All the operators of $G - H$ are of order $p^2$ while in the preceding section the corresponding operators were of order $p^3$. As all the operators of $H$ may be of order $p$ it may be the abelian group of type $(1, 1, 1, \cdots)$. If it is not this group it contains just $1 + p + p^2 + \cdots + p^{m-3}$ subgroups of index $p$ and hence it belongs to the known systems of groups which were mentioned above. The $p$th power of its operators are commutators of $G$.

Since any two invariant operators of $H$ which are of order $p$ without being commutators can be made to correspond in some simple isomorphism of $H$ with itself it follows that there is one and only one $G$ which has the same group of cogredient isomorphisms as the non-abelian $H$ contained in it. That is, every non-abelian $H$ which includes at least $p^2 - 1$ invariant operators of order $p$ is contained in one and only one $G$ whenever $G$ includes invariant operators not contained in $H$. As the $H$'s are known and these groups are obtained by extending $H$ by means of an operator of order $p^2$ whose $p$th power is in $H$ and which is commutative with every operator of $H$, this system of groups is completely determined. It therefore remains only to consider those $G$'s whose group of cogredient isomorphisms are larger than those of the $H$'s which they contain.

Every group in question may be constructed by extending some $H$ by means of an operator $t$ which is commutative with each of the operators of a subgroup $H'$ of index $p$ contained in $H$. As $t^p$ is contained in $H'$ and is commutative with every operator of $G$ without being a commutator of $G$ it follows that $H'$ includes at least $p^2 - 1$ invariant operators of order $p$. Since the order of the group of cogredient isomorphisms of $G$ is larger than that of $H$ the subgroup $H'$ cannot include all the invariant operators of $H$. Hence such an $H$ must involve at least $p^3$ invariant operators under $H$, and whenever an $H$ has this property a $G$ can evidently be constructed. Since two $G$'s must be distinct whenever their $H$'s are distinct and we found the conditions which $H$ must satisfy in order that a $G$ can be constructed, it remains only to determine how many $G$'s involve the same $H$. That is, if $H$ is given it is required to find all the possible operators of order $p^2$ which may be used to extend $H$ in such a way as to give rise to distinct groups.

Let $t_1$, $t_2$ be two operators which may be used to extend a fixed $H$ so as to give rise to two distinct $G$'s. Since it is possible to make $H$ simply isomorphic with itself in such a way that any invariant operator of order $p$ which is not a commutator can be made to correspond to any other operator having these properties, it is clear that the two $G$'s would be simply isomorphic if the $H'$ of the

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one could be made to correspond to the $H'$ of the other in some simple isomorphism of $H$ with itself. That is, our problem is reduced to finding the number of subgroups of $H$ which could be used as $H'$ but could not be made simply isomorphic in any simple isomorphism of $H$ with itself. In other words, the construction of all the possible groups in question is reduced to the determination of properties of a known system of groups.

As there are just three non-abelian $H$'s which have the same group of cogredient isomorphisms * our problem is reduced to the determination of all the $G$'s which involve these three distinct $H$'s. Two of these contain no invariant operators of order $p^2$ while the third contains invariant operators of this order. It is not difficult to see that there is only one $G$ which involves a given $H$ that does not include invariant operators of order $p^2$. For the $H'$ of such a $G$ cannot include invariant operators of order $p^2$ since it cannot involve all the invariant operators of $H$. Hence all the subgroups which could be used for $H'$ can be made to correspond in some simple isomorphism of $H$ with itself. That is, there is only one $G$ which involves such an $H$. It now remains only to consider the case when $H$ includes invariant operators of order $p^2$ and has a given group of cogredient isomorphisms.

When such an $H$ has more than $p^3$ invariant operators it follows directly from the known properties of $H$ that it includes three distinct subgroups of index $p$ which can be used as $H'$ since they have the same group of cogredient isomorphisms as $H$ has. These three $H$'s are simply isomorphic with the three possible $H$'s which have the same group of cogredient isomorphisms. When the $H$ under consideration has only $p^3$ invariant operators there are only two such $G$'s. Hence we have finally that *there are five $G$'s which have the same group of cogredient isomorphisms provided the order of this group is less than $p^{m-3}$. When this order is $p^{m-3}$ there are only four such $G$'s.*

As these results follow directly from the known properties of $H$ and from a known theorem in regard to simply isomorphic groups,† it seems unnecessary to enter upon further details. The special case when $H$ is abelian presents no difficulty as there is only one possible $G$ when $H$ is of type $(1, 1, 1, \cdots)$, and there are two $G$'s when $H$ is of type $(2, 1, 1, \cdots)$. The infinite systems of groups determined in this paper are of interest on account of their elementary properties and also in view of the fact that they are completely defined by the number of their subgroups of index $p$ and the order of the commutator subgroup. The close contact between this paper and the one in which the properties of every possible $H$ are determined should be noted. The latter will shortly appear in the Mathematische Annalen under the title: *An Extension of the Hamiltonian Groups.*

* Mathematische Annalen, loc. cit.

† Transactions of the American Mathematical Society, vol. 3 (1902), p. 342.

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# ON A GENERAL METHOD FOR TREATING TRANSMITTED MOTIONS AND ITS APPLICATION TO INDIRECT PERTURBATIONS*

BY

ERNEST WILLIAM BROWN

The mathematical treatment of any physical problem demands the construction of an ideal problem in which the conditions are different from those of the actual problem. It is assumed that the same general laws hold for the actual and ideal problems, but the complexity of the circumstances surrounding the former makes simplifications of some kind necessary in order that the analysis should not be unreasonably tedious. The ideal problem is therefore usually constructed by neglecting at first some of the influences which form a part of it but which are assumed to affect the results to a much smaller extent than those we retain. The simplified problem, which I call problem $A$, is then solved. The second step consists in finding what changes are necessary in the solution of problem $A$ when some or all of the neglected influences are included; this second problem I call $B$. The question under consideration here is the deduction of the solution of $B$ when that of $A$ has been found.

One of the methods for solving $B$ is that known as the *Variation of Arbitrary Constants*. In the deduction of the solution of $A$, arbitrary constants $a$ arise, and numerical values, depending on the given conditions of the problem, can be assigned to them. In order to obtain the solution of $B$, it is assumed that the solution of $A$ is still available provided that a different set of values be assigned to the constants $a$. But the new set of values will usually be different at different times and will therefore depend on the time; in other words, the constants $a$ become variables in problem $B$.

This last fact gives us another view of the relation between $A$ and $B$. Suppose that the position of the system which constitutes $B$ is defined by variables $x$, and that we desire to change from the variables $x$ to other variables $p$ in order to simplify the treatment of $B$. Then one $p$-set, namely, the $a$-set, can be obtained by supposing that the $x$-set and the $p$-set are connected by the same equations which connect the $x$-set and the $a$-set in the solution of problem $A$.

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Thus $A$ is no longer looked upon as a dynamical problem but as a collection of relations which give certain transformations useful for the treatment of $B$.

But we can go further than this. There is no special necessity imposed on us to use any of the $a$-sets thus found. We may be able to discover another set of variables, say a $p'$-set, allied to the $x$-set by relations different from those which connect the $x$-set and the $a$-set — relations suggested indeed by the latter and perhaps possessing similar properties, but possessing also additional properties more useful for the purpose in hand.

It is proposed to investigate here one such $p'$-set. The relations between the $x$- and $a$-sets usually contain a certain number of other constants $\alpha$ which were present in the original equations of problem $A$. It is assumed that the $x$- and $p'$-sets are connected by relations of the same *form* as those which connect the $x$- and $a$-sets, but that the $\alpha$ have given *variable* instead of given *constant* values assigned to them in the $p'$-set. Certain advantages are gained in the application of the method to the consideration of the 'indirect inequalities' in the motion of the moon, i. e., in the consideration of those perturbations of the lunar motion which arise from the action of a planet on the earth's motion and which are transmitted to the moon on account of the predominating influence of the earth on the moon. I have given some indications of this $p'$-set in a former paper.* Unfortunately, the main idea was obscured by the method adopted to develop it, and some statements concerning the applicability of it to the lunar problem require further limitations than those previously imposed. The present method of treatment also brings to light a theorem (art. 10 below) which I believe is new and which is of considerable importance in the discussion of the indirect inequalities.

1. *The method of Jacobi.* The equations of a dynamical system are supposed to have been expressed in the canonical form

$$(1) \qquad \frac{dx_i}{dt} = \frac{\partial \phi}{\partial y_i}, \qquad \frac{dy_i}{dt} = -\frac{\partial \phi}{\partial x_i} \qquad [\phi = \phi(x_i, y_i, t)],$$

where, to fix ideas, the $n$ variables $x_i$ may be thought as the coördinates defining the position of the system, the $n$ variables $y_i$ as the corresponding momenta, and $\phi$ as the sum of the kinetic and potential energies; these interpretations are nevertheless not necessary.

Let the variables† $x$, $y$ be changed to $2n$ new variables $p$, $q$, connected with $x$, $y$ by the relations

* *On the variation of the arbitrary and given constants in dynamical equations*, Transactions of the American Mathematical Society, vol. 4 (1903), pp. 333–350.

† The suffixes will be omitted whenever no confusion is thereby caused. In all cases $i$, $j$ take the values $1, 2, \cdots, n$.



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$$(2) \qquad y_i = \frac{\partial S}{\partial x_i}, \qquad q_i = \frac{\partial S}{\partial p_i} \qquad [S = S(x, p, t)],$$

then JACOBI has shown that the equations satisfied by the new variables are

$$(3) \qquad \frac{dq_i}{dt} = \frac{\partial}{\partial p_i}\left(\phi + \frac{\partial S}{\partial t}\right), \qquad \frac{dp_i}{dt} = -\frac{\partial}{\partial q_i}\left(\phi + \frac{\partial S}{\partial t}\right),$$

in which, after $\partial S/\partial t$ has been formed from $S$ expressed in its original form as a function of $t$ and the $x$, $p$, the function $\phi + \partial S/\partial t$ is supposed to be expressed in terms of $p$, $q$, $t$ by means of (2). No assumption is made regarding the form of $S$, but in order to secure the independence of the $x$, $y$ and also that of the $p$, $q$, it is necessary that the determinant

$$\left| \frac{\partial^2 S}{\partial x_i \partial p_j} \right| \qquad (i, j = 1, 2, \cdots, n),$$

shall not be zero.

Suppose that the $x$, $y$ receive two sets of independent arbitrary variations $\delta x$, $\delta y$ and $\delta' x$, $\delta' y$, inducing corresponding variations in the $p$, $q$. Equations (2) give

$$\Sigma_i(\delta x_i \delta' y_i - \delta y_i \delta' x_i) = \Sigma_i(\delta q_i \delta' p_i - \delta p_i \delta' q_i).$$

Whence, if the $x$, $y$ are expressed in terms of $t$ and the $p$, $q$, and if we adopt the usual notation:

$$(4) \qquad [\alpha, \beta] = \Sigma_i \left[ \frac{dy_i}{d\alpha}\frac{dx_i}{d\beta} - \frac{dx_i}{d\alpha}\frac{dy_i}{d\beta} \right],$$

we obtain the following relations:

$$(5) \qquad \begin{aligned} &[p_i, q_i] = 1, \qquad [p_i, q_j] = 0 \qquad (i \neq j), \\ &[p_i, p_j] = 0, \qquad [q_i, q_j] = 0 \qquad (i = \text{or} \neq j). \end{aligned}$$

We also have

$$\frac{d\phi}{dt} = \frac{\partial}{\partial t}\phi(x, y, t).$$

2. *Particular forms of* $S$. Suppose first that $S$ is so chosen as to satisfy the equation $\phi + \partial S/\partial t = 0$. The $p$, $q$ are then constants and they may be considered as the arbitraries of the solution. Such forms of $S$ are solutions, containing $n$ arbitrary constants exclusive of that additive to $S$, of the partial differential equation

$$\phi\left(x, \frac{\partial S}{\partial x}, t\right) + \frac{\partial S}{\partial t} = 0.$$

Next suppose that $S$ is so chosen as to satisfy the equation

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$$\phi + \frac{\partial S}{\partial t} = B, \tag{6}$$

where $B$ is a function of the $p$ only; this is a form much more convenient in many of the applications. Then the $p$ and therefore the $\dot{q}$ are constants. Let us put

$$p_i = c_i, \qquad q_i = w_i = b_i t + \epsilon_i, \qquad b_i = -\frac{\partial B}{\partial c_i}, \tag{7}$$

the arbitrary constants being $c$, $\epsilon$, and $B$ being a function of the $c$ only; at present $B$ is undetermined but it may be determined so as to satisfy future conditions as to the forms of $x$, $y$, when the latter are expressed in terms of $c$, $\epsilon$, $t$.

I shall suppose that this problem has been solved and that this second form for $S$ has been used so that the solution is expressed in terms of $c_i$, $w_i$, $t$, all of which are independent. On account of (2), we have,

$$y_i = \frac{\partial S}{\partial x_i}, \qquad w_i = \frac{\partial S}{\partial c_i} \qquad [S = S(x, c, t)], \tag{8}$$

$$\begin{aligned} [c_i, w_i] &= 1, \quad [c_i, w_j] = 0 \qquad (i \neq j), \\ [c_i, c_j] &= 0, \quad [w_i, w_j] = 0 \qquad (i = \text{or} \neq j). \end{aligned} \tag{9}$$

3. *The constants present in* $\phi$. Suppose that $\phi$ contains $m$ constants $\alpha_h$ to which numerical values have not been assigned. Some or all of these constants will be also present in every form of $S$. I propose to investigate in this article certain properties of $S$ and $\phi$ with relation to these constants. It is not necessary to suppose that the $\alpha_h$ are all the constants present in $\phi$. Such of them as are not specified in the $\alpha_h$ can be supposed to have had numerical values assigned to them, either explicit or implicit, so that they do not enter into the discussion.

In order to make the following results more general, I suppose that the $m$ constants $\alpha_h$ have been replaced by $m$ variables or constants $u_h$ connected with the $\alpha_h$ by the $m$ independent relations

$$f_k(u_h, \alpha_h, t) = 0 \qquad (h, k = 1, 2, \cdots, m),^* \tag{10}$$

so that the $u_h$ are functions of the $\alpha_h$ and $t$ only.

The function $S$ was originally a function of the $x, c, \alpha, t$ and it now becomes a function of the $x$, $c$, $u$, $t$; denote it by $S'$ when it is expressed in the latter form. We have

$$\frac{\partial S}{\partial x_i} = \frac{\partial S'}{\partial x_i}, \qquad \frac{\partial S}{\partial c_i} = \frac{\partial S'}{\partial c_i}, \qquad \frac{\partial S}{\partial t} = \Sigma_h \frac{\partial S'}{\partial u_h} \dot{u}_h + \frac{\partial S'}{\partial t}.$$

* The values $1, 2, \cdots, m$ are given to $h$, $k$ in all cases.

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Put

$$U_h = \frac{\partial S'}{\partial u_h}.$$

Then

$$\frac{dU_h}{dt} = \Sigma_i \frac{\partial^2 S'}{\partial u_h \partial x_i} \dot{x}_i + \Sigma_k \frac{\partial^2 S'}{\partial u_h \partial u_k} u_k + \frac{\partial^2 S'}{\partial u_h \partial t}. \tag{11}$$

Also denoting by $Q'$ the expression of any function $Q$ in terms of the $x$, $c$ $u$ and $t$, and putting

$$\frac{\partial' \phi}{\partial u_h} = \frac{\partial}{\partial u_h} \phi(x, y, u, t),$$

we obtain

$$\begin{aligned}
\frac{\partial' \phi}{\partial u_h} &= \frac{\partial \phi'}{\partial u_h} - \Sigma_i \frac{\partial \phi'}{\partial y_i} \frac{\partial y'_i}{\partial u_h} = -\frac{\partial}{\partial u_h}\left(\frac{\partial S}{\partial t} + B\right)' - \Sigma_i \dot{x}_i \frac{\partial^2 S'}{\partial u_h \partial x_i} \\
&= -\frac{\partial}{\partial u_h}\left(\Sigma_k \frac{\partial S'}{\partial u_k} \dot{u}_k + \frac{\partial S'}{\partial t} + B\right) - \Sigma_i \dot{x}_i \frac{\partial^2 S'}{\partial u_h \partial x_i} \\
&= -\Sigma_k \left(\frac{\partial^2 S'}{\partial u_h \partial u_k} \dot{u}_k + U_k \frac{\partial \dot{u}_k}{\partial u_h}\right) - \frac{\partial^2 S'}{\partial u_h \partial t} - \frac{\partial B}{\partial u_h} - \Sigma_i \frac{\partial^2 S'}{\partial u_h \partial x_i} \dot{x}_i.
\end{aligned}$$

Combining with (11) we find

$$\frac{dU_h}{dt} = -\frac{\partial' \phi}{\partial u_h} - \Sigma_k U_k \frac{\partial \dot{u}_k}{\partial u_h} - \frac{\partial B}{\partial u_h}, \tag{12}$$

an equation which enables us to find $U_h$ directly.

Again

$$U_h = \frac{\partial S'}{\partial u_h} = \frac{dS}{du_h} - \Sigma_j y_j \frac{dx_j}{du_h},$$

where $S$ and the $x$ are expressed in terms of the $c$, $w$, $u$, $t$. Hence if $U_h$ be similarly expressed,

$$\frac{dU_h}{dc_i} = \frac{d^2 S}{dc_i du_h} - \frac{d}{dc_i} \Sigma_j y_j \frac{dx_j}{du_h}. \tag{13}$$

But since the $c$, $u$, $w$ are independent,

$$\begin{aligned}
\frac{d^2 S}{dc_i du_h} &= \frac{d^2 S}{du_h dc_i} = \frac{d}{du_h}\left(\frac{\partial S}{\partial c_i} + \Sigma_j \frac{\partial S}{\partial x_j} \frac{dx_j}{dc_i}\right) \\
&= \frac{d}{du_h}\left(w_i + \Sigma_j y_j \frac{dx_j}{dc_i}\right) = \frac{d}{du_h} \Sigma_j y_j \frac{dx_j}{dc_i}.
\end{aligned}$$

Substituting in (13), performing the differentiations with respect to $u_h$, and remembering the notation (4), we find

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$$\frac{dU_h}{dc_i} = [u_h, c_i]. \tag{14}$$

In an exactly similar manner it can be shown that

$$\frac{dU_h}{dw_i} = [u_h, w_i], \tag{15}$$

$$\frac{dU_h}{du_k} - \frac{dU_k}{du_h} = [u_h, u_k].$$

The $U_h$ can therefore be determined directly from the expressions for $x, y$ when the solution has been completed.

4. *Two limitations.* In the remaining portion of this paper, I suppose

1°. That $t$ is not present explicitly in $\phi(x, y, u, t)$, so that if $t$ was present explicitly in the original expression for $\phi$, one or more of the $u$ must be variable.

2°. That $t$ is not present explicitly in the $x, y$ when the latter have been expressed in terms of the $c$, $w$, $u$, and $t$.

These assumptions do not very much limit the scope of the results and they render the exposition rather more simple.

We have now

$$\frac{\partial S'}{\partial t} = 0, \qquad \frac{\partial S}{\partial t} = \Sigma_h U_h \dot{u}_h, \qquad \phi + \Sigma_h U_h \dot{u}_h = -B. \tag{16}$$

5. *Variation of arbitrary constants.* The usual method consists in supposing that, after the equations (1) have been solved in terms of the $c$, $w$, and $t$, a new function $R(x, y, t)$ is added to $\phi$; account is taken of $R$ by supposing that the $x, y$ retain the same *forms* when expressed in terms of the $c$, $w$, and $t$, so that the $c$, $\epsilon$ are no longer constants. This amounts to nothing more than a transformation from the old variables $x, y$ to the new variables $c$, $w$, and the results of art. 1 can therefore be used.

In fact we now have

$$\Sigma_j \left( \frac{dx_i}{dc_j} \dot{c}_j + \frac{dx_i}{dw_j} \dot{w}_j + \frac{\partial x}{\partial t} \right) = \frac{\partial}{\partial y_i} (\phi + R),$$

$$\Sigma_j \left( \frac{dy_i}{dc_j} \dot{c}_j + \frac{dy_i}{dw_j} \dot{w}_j + \frac{\partial y}{\partial t} \right) = -\frac{\partial}{\partial x_i} (\phi + R),$$

which, on account of (9), give

$$\frac{dc_j}{dt} = \frac{\partial Q}{\partial w_j}, \qquad \frac{dw_j}{dt} = -\frac{\partial Q}{\partial c_i},$$

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where $Q = \phi + R + \partial S/\partial t$ is supposed to be expressed by means of (8) in terms of the $c$, $w$, and $t$. But $\phi$ being a function of the $x$, $y$, and $t$, and $\partial S/\partial t$ a function of the $x$, $c$, and $t$, and neither $\phi$ nor $\partial S/\partial t$ containing total derivatives with respect to $t$, $\phi + \partial S/\partial t$ must retain the same form whatever be the values given to the $c$, $w$. Hence we still have $\phi + \partial S/\partial t = -B$, a function of the $c$ only, and

$$Q = -B + R.$$

Suppose that $R$ is due to changing one of the constants $\alpha$ present in $\phi$ to a variable. Then

$$Q = -B + \frac{\partial'\phi}{\partial\alpha}\delta\alpha + \tfrac{1}{2}\frac{\partial'^2\phi}{\partial\alpha^2}(\delta\alpha)^2 + \cdots,$$

where $\delta\alpha$ is the variable part added to $\alpha$. The same method may be pursued, it is true, but if we follow it we take no account of the fact that the *form* of $\phi$ considered as a function of the $x$, $y$, $\alpha$, $t$ is unchanged, and it would seem as if this unchanged form should be made use of to simplify, if possible, the results. The new method of varying the constants which follows recognizes this fact and brings to light certain results which will be of value in the applications. As in art. 3, the $\alpha$ will be replaced by the $u$; the two conditions of art. 4 will also be imposed.

6. *Variation of arbitrary and given constants.* I suppose that the $x$, $y$ retain the same *forms* when the $u$ in $\phi$ are replaced by functions of the time different from those with which the original problem is solved, so that we replace not only the $c$, $w$ *but also the* $u$ by their new values in order to get the new values of $x$, $y$. The new values of $c$, $w$ will, therefore, be different in this method from those found by the method of art. 5. In fact we now have, since $t$ is present in the $x$, $y$ only through its presence in the $c$, $w$, $u$,

$$\sum_j\left(\frac{dx_i}{dc_j}\dot{c}_j + \frac{dx_i}{dw_j}\dot{w}_j\right) + \sum_h \frac{dx_i}{du_h}\dot{u}_h = \frac{\partial\phi}{\partial y_i},$$

$$\sum_j\left(\frac{dy_i}{dc_j}\dot{c}_j + \frac{dy_i}{dw_j}\dot{w}_j\right) + \sum_h \frac{dy_i}{du_h}\dot{u}_h = -\frac{\partial\phi}{dx_i}.$$

On account of (9), (14), (15), these give

$$(17)\quad \begin{aligned} \frac{dw_j}{dt} &= \frac{\partial\phi}{\partial c_j} - \sum_h [c_j, u_h]\dot{u}_h = \frac{\partial}{\partial c_j}(\phi + \sum_h U_h\dot{u}_h) = \frac{\partial Q}{\partial c_j}, \\ \frac{dc_j}{dt} &= -\frac{\partial\phi}{dw_j} + \sum_h [w_j, u_h]\dot{u}_h = -\frac{\partial}{\partial w_j}(\phi + \sum_h U_h\dot{u}_h) = -\frac{\partial Q}{\partial w_j}, \end{aligned}$$

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if $u_h$ be independent of $c_j$, $w_j$, and

$$Q = \phi + \Sigma_h U_h \dot{u}_h.$$

The equations are therefore still canonical.

Now when the $u$, $c$, $w$ have their original values, denoted by the index 0,

$$\phi^0 + \Sigma_h U_h^0 \dot{u}_h^0 = -B^0. \tag{18}$$

This equation is an identity after the $x$, $y$, $\dot{u}$, $U$ have been replaced by their values in terms of the $w$, $c$, $u$, and $t$, and we can, therefore, give to the $c$, $w$, $u$ any arbitrary variations, symbolically denoted by $\delta_1$, so that

$$\phi^0 + \delta_1\phi + \Sigma_h(U_h^0 + \delta_1 U_h)(\dot{u}_h^0 + \delta_1 \dot{u}_h) = -B^0 - \delta_1 B. \tag{19}$$

The actual variations in $Q$ are produced by adding portions, denoted symbolically by $\delta$, to the $c$, $w$, $u$, so that

$$Q = \phi^0 + \delta\phi + \Sigma_h(U_h + \delta U_h)\frac{d}{dt}(u_h^0 + \delta u_h). \tag{20}$$

As $\phi$, $U$, $B$, $\dot{u}$ are functions of the $c$, $w$, $u$ only and not of their derivatives with respect to $t$, we can take $\delta_1 = \delta$. But we cannot in general put $\delta_1 \dot{u} = d\delta u/dt$. For the identity (19) demands that $\dot{u}$ be first expressed in terms of $u$ by means of (10) and the variation then taken, while in $Q$ a portion $\delta u$ is to be added to $u$ and the derivative with respect to $t$ then formed. It was the failure to recognize this distinction that caused the error in my former paper.* The method there used implies the substitution of $-B$ for $Q$ under the false assumption that the derivative of $\delta u$ with respect to $t$ would be the same as $\delta\dot{u}$. The error is immediately evident in the case of those $u_h$ which were originally constants; for them, $\dot{u}_h = 0$ and, therefore, $\delta\dot{u}_h = 0$, while obviously $d\delta u/dt$ will only be zero if $\delta u$ be constant, and this will not generally be the case.

Combining (19), (20), with $\delta_1 = \delta$, we obtain

$$Q = -B + \Sigma_h U_h \left(\frac{d}{dt}\delta u_h - \delta\dot{u}_h\right), \tag{21}$$

the final form of $Q$.

I have also obtained this result, when squares and higher powers of $\delta$ are neglected, by direct transformation from the ordinary canonical equations of art. 5 above, making use of the relations given in art. 3.

7. There is, of course, no special necessity to confine this method to those variations which are produced by varying the given constants in $\phi$. Suppose

* Loc. cit., art. 17.

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that the present method is retained for the consideration of a function $R$ added to $\phi$, while the given constants in $\phi$ do not vary. We must then add to $Q$ the function

$$-\delta'\phi + R.$$

the first term of which is necessary to counterbalance the variations of the $u$ in $\phi$. But this proceeding appears to be of little or no value, as the results are more complicated than in the usual method. On the other hand the variations of the $u$ are now entirely arbitrary and we might be able to so choose them that the form of $Q$ is more simple or the canonical equations more easy to solve. What advantage may be gained will depend on the particular problem in hand and I shall not pursue this development further in the present paper.

8. *Case of the $u_h$ satisfying canonical equations.* Suppose that the $u$ are divided into two classes $c'_h$, $w'_h = b'_h t + \epsilon'_h$, where $b'_h = -\partial B'/\partial c'_h$. I suppose that in the initial problem, the $c'$, $\epsilon'$ are constants and that $B'$ is a function of the $c'$ only. Instead of using the symbols $U$, put

$$C'_h = \frac{\partial S'}{\partial c'_h}, \qquad W'_h = \frac{\partial S'}{\partial w'_h} \qquad [S' = S(x_i, c_i, c'_i, \pi'_i)],$$

so that

$$-\phi - B = \frac{\partial S}{\partial t} = \Sigma_h b'_h W'_h,$$

$$Q = -B + \Sigma_h C'_h \frac{d}{dt}\delta c'_h + \Sigma_h W'_h \left(\frac{d}{dt}\delta w'_h - \delta b'_h\right). \tag{22}$$

Further, I suppose that the variations of the $c'$, $w'$ are given by

$$\frac{dc'_h}{dt} = \frac{\partial}{\partial w'_h}(R' + B'), \qquad \frac{dw'_h}{dt} = -\frac{\partial}{\partial c'_h}(R' + B'),$$

or

$$\frac{d}{dt}\delta c'_h = \frac{\partial R'}{\partial w'_h}, \qquad \frac{d}{dt}\delta w'_h = -\frac{\partial R'}{\partial c'_h} + \delta b'_h.$$

Substituting in (22), we obtain

$$Q = -B + \Sigma_h \left(C'_h \frac{\partial R'}{\partial w'} - W'_h \frac{\partial R'}{\partial c'_h}\right), \tag{23}$$

where it is supposed, in accordance with the previous work, that $R'$ is independent of the $c$, $w$.

9. *Properties of the solution in the case of the lunar theory.* The application of these results to the lunar theory is immediate when the necessary limita-

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tions which constitute this particular case are imposed. For the sake of brevity, I shall not prove these limitations here but merely state those of them that are necessary for my purpose. They are as follows:

1°. The function $R'$ is the disturbing function due to a planet on the motion of the earth. It can be expanded as a sum of periodic terms and it has a small factor whose square can and will be neglected.

2°. A periodic term in $R'$ of period $\alpha$ years will give rise to periodic terms in the $\delta c'$ of the same period with a factor $\alpha$ in the coefficients: in the $\delta w'$ the factor $\alpha^2$ also occurs.

3°. It is supposed that the orbit of the earth lies in a fixed plane. There are then two angles $w'_h$, and three angles $w_i$.

4°. The $W'_h$ consist only of sums of cosines of multiples of the four differences of the five angles $w'_h$, $w_i$, the terms independent of $t$ being zero; the $C'_h$ consist of sines with similar arguments.

5°. The $c'_h$ are functions of the mean motion and of the eccentricity of the earth's orbit.

10. *The indirect planetary inequalities in the moon's motion.* There are very few sensible planetary inequalities in the moon's motion except when $\alpha$ is a number fairly large compared with unity; only the cases where $\alpha$ is large are treated here. It is further supposed that squares of the small factor which accompanies $R'$ are neglected. Then from (17), (23),

$$\frac{d}{dt}\delta c_i = \frac{\partial Q_1}{\partial w_i}, \qquad \frac{d}{dt}\delta w_i = -\frac{\partial Q_1}{\partial c_i} + \Sigma_j \frac{db_i}{dc_j}\delta c_j + \Sigma_h \frac{db_i}{dc'_h}\delta c'_h,$$

where

$$Q_1 = \Sigma_h \left( C'_h \frac{\partial R'}{\partial w'_h} - W'_h \frac{\partial R'}{\partial c'_h} \right).$$

If $R'$ contains a term of long period, $\alpha$ years, the factor $\alpha$ may be introduced in the $\delta c$, $\delta c'$ and therefore the factor $\alpha^2$ in the $\delta w$, $\delta w'$, and no higher power of $\alpha$ can be present. Hence the general

THEOREM. *When squares and higher powers of the ratio of the mass of a planet to that of the sun are neglected, the large factor $\alpha$ due to a long period inequality can never occur in the corresponding term in the moon's motion to a power higher than $\alpha^2$, even if its square is present in the corresponding inequality of the earth's motion.*

The importance of this result lies in the fact that it gives us a means of rejecting in advance many terms of long period, which under the old method would have to be examined in order to find out whether their coefficients were sensible in the moon's motion. The advantage of this method lies, to a great

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extent, in the theorem just stated and in the calculation of particular classes of inequalities; this will appear in the following article.

11. *Special classes of inequalities.* I divide the long-period inequalities into three classes for which certain results can be immediately deduced. The *order* of these refers to the power of $\alpha$ present. In the periodic inequalities it is supposed that at least one of the arguments belonging to the disturbing planet is present.

(*a*) *Inequalities containing the lunar arguments.* They are of two kinds. First, those produced by terms of moderately long period in the moon-earth-sun system combined with terms of nearly the same period in the earth-sun-planet system. The former terms have all quite small coefficients and there will be few of such inequalities with sensible coefficients. Second, those produced by combining terms of short period in the two systems. We shall generally be able to neglect the $\delta c'$, $\delta w'$, and the principal parts of the coefficients arise only in the $\delta w_i$ obtained from limiting the equations to

$$\frac{d}{dt}\delta c_i = \frac{\partial Q_1}{\partial w_i}, \qquad \frac{d}{dt}\delta w_i = \Sigma_j \frac{db_i}{dc_j}\delta c_j.$$

(*b*) *Periodic inequalities not containing the lunar arguments.* Here

$$\delta c_i = 0$$

and the $C'$, $W'$ are limited to those terms which have the arguments $w'_h$ only. The latter have all small coefficients and moreover the terms produced by $Q_1$ have only the factor $\alpha$. Hence it will be possible to obtain the chief parts of the coefficients by solving the equations

$$\frac{d}{dt}\delta w_i = \Sigma_h \frac{db_i}{dc'_h}\delta c'_h, \qquad c_i = \text{const.}$$

(*c*) *Non-periodic inequalities.* These again are of two classes. First, the terms containing $t$ to the first power only. These merely give constant additions to the mean motions of the lunar arguments. Second, the secular inequalities, which contain $t^2$. These latter are determined by

$$\delta c_i = 0, \qquad \frac{d}{dt}\delta w_i = \Sigma_h \frac{db_i}{dc'_h}\delta c'_h,$$

where $\delta c'_h$ is determined from the non-periodic term of $R'$. This is NEWCOMB'S theorem.* The theorem therefore only applies exactly, with the given limita-

* *Action of the Planets on the Moon*, Washington Astronomical Papers, vol. 5 (1894), p. 191.

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tions, to the secular inequalities, but it is approximately true for the greater part of the inequalities independent of the lunar arguments, while it is not applicable to inequalities containing the lunar arguments. In arts. 17–19 of my former paper (loc. cit. ante), it was stated that the theorem applied to all ndirect inequalities.

I have contented myself here with a brief outline of the chief results which the method furnishes for the lunar problem. There are numerous details and properties connected with the functions $C'$, $W'$ which might find a place. But these are of more interest in connection with a numerical calculation of the inequalities than of theoretical interest in the variation of the constants or even in the problem of three bodies. It therefore appears advisable to give them later with the publication of the details of a method for such an investigation.

HAVERFORD COLLEGE,
*April* 8, 1905.

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# ON HYPERCOMPLEX NUMBER SYSTEMS*

BY

L. E. DICKSON

*Generalization of the concept number system.*

1. The usual theory relates to systems of numbers $\sum_{i=1}^{n} a_i e_i$ in which the coördinates $a_i$ range independently over all real numbers or else over all ordinary complex numbers; for example, the real quaternion system, or the complex quaternion system. As an obvious generalization,† the coördinates may range independently over all the marks of any field $F$; for example, the rational quaternion system.

As a further generalization, the sets of coördinates $a_1, \cdots, a_n$ in the various numbers of a system may include only a part of the sets $b_1, \cdots, b_n$, each $b_i$ ranging independently over $F$; for example, the integral quaternion system. The various coördinates $a_1, \cdots, a_n$ need not have the same range; for example, the numbers

$$(a + 2b\sqrt{2})e_1 + (c + 4d\sqrt{2})e_2 \quad (a, b, c, d \text{ arbitrary integers})$$

form a closed system under addition, subtraction, and multiplication, subject to the associative law, if we set

$$e_1^2 = 2\sqrt{2}e_1 - 2e_2, \qquad e_1 e_2 = e_2 e_1 = e_1, \qquad e_2^2 = e_2.$$

If we make these generalizations on the coördinates, but retain the usual conception of the units $e_i$, we obtain only subsystems of the usual number systems, the case of modular fields being an exception. It is otherwise if we generalize our conception of the units themselves, freeing them from the restriction‡ of linear independence with respect to the set of all ordinary complex numbers, and assuming merely their linear independence with respect to the given field $F$.

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† DICKSON, *Definitions of a linear associative algebra by independent postulates*, Transactions, vol. 4 (1903), pp. 21–26.

‡ Retained implicitly by TABER, Transactions, vol. 5 (1904), p. 509.



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By this generalization we may regard an algebraic field * as a number system; in particular, all ordinary complex numbers form a number system * with respect to the set of all real numbers, the units being 1, $i$.

The question of further generalization is considered in § 5.

*Closed system of n-tuple elements with respect to a field F.*

2. A set of $n$ ordered marks $a_1, \cdots, a_n$ of $F$ will be called an *n-tuple element* $a$. The symbol $a = (a_1, \cdots, a_n)$ employed is purely positional, without functional connotation. Its definition implies that $a = b$ if and only if $a_1 = b_1, \cdots, a_n = b_n$.

A system of $n$-tuple elements $a$ in connection with $n^3$ fixed marks $\gamma_{ijk}$ of $F$ will be called a *closed system* if the following five postulates hold:

*Postulate* I. If $a$ and $b$ are any two elements of the system, then $s = (a_1 + b_1, \cdots, a_n + b_n)$ is an element of the system.

*Definition.* Addition of elements is defined by $a \oplus b = s$.

*Postulate* II. The element $0 = (0, \cdots, 0)$ occurs in the system.

*Postulate* III. If 0 occurs, then to any element $a$ of the system corresponds an element $a'$ of the system such that $a \oplus a' = 0$.

THEOREM. *The system is a commutative group under* $\oplus$.

*Postulate* IV. If $a$ and $b$ are any two elements of the system, then $p = (p_1, \cdots, p_n)$ is an element of the system, where

$$p_i = \sum_{j,k}^{1,\cdots,n} a_j b_k \gamma_{jki} \qquad (i = 1, \cdots, n).$$

*Definition.* Multiplication of elements is defined by $a \otimes b = p$.

*Postulate* V. The fixed marks $\gamma$ satisfy the relations

$$\sum_{j=1}^{n} \gamma_{rtj} \gamma_{jki} = \sum_{j=1}^{n} \gamma_{tkj} \gamma_{rji} \qquad (r, t, k, i = 1, \cdots, n).$$

THEOREM. *Multiplication is associative and distributive.*

To make the system $n$ *dimensional*, we add a sixth postulate:

*Postulate* VI. If $\tau_1, \cdots, \tau_n$ are marks of $F$ such that $\tau_1 a_1 + \cdots + \tau_n a_n = 0$ for every element $(a_1, \cdots, a_n)$ of the system, then $\tau_1 = 0, \cdots, \tau_n = 0$.

THEOREM. *The system contains n elements* $\epsilon_i = (a_{i1}, \cdots, a_{in})$, $i = 1, \cdots, n$, *such that* $|a_{ij}| \neq 0$.

Indeed, by postulate VI there occurs an element $\neq 0$. Hence if the theorem is false, we may assume that the system contains $\nu$ elements $\epsilon_1, \cdots, \epsilon_\nu$, $1 \leqq \nu < n$, such that not every determinant of order $\nu$ in the matrix

* For these, PEIRCE's theorem that all but one of the units of the first group can be assumed nilpotent evidently fails. My conception of number systems is therefore wider than TABER's.

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$$M_\nu \equiv \begin{bmatrix} a_{11} \cdots a_{1n} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{\nu 1} \cdots a_{\nu n} \end{bmatrix}$$

vanishes, while all of order $\nu + 1$ in $M_{\nu+1}$ vanish, $(a_{\nu+11}, \cdots, a_{\nu+1n})$ being an arbitrary element of the system. To fix the notations, let

$$D = |a_{ij}| \neq 0 \qquad (i, j = 1, \cdots, \nu).$$

The expansion of $|a_{ij}| = 0$ $(i, j = 1, \cdots, \nu + 1)$ gives a relation

$$\sum_{i=1}^{\nu+1} d_i a_{\nu+1i} = 0, \qquad d_{\nu+1} = D \neq 0,$$

in contradiction with postulate VI.

*Every n dimensional closed system a complex number system.*

3. This identification may be made formally by establishing a one-to-one correspondence between $a$ and $\sum a_i e_i$, where the $e$'s have the multiplication table $e_j e_k = \sum \gamma_{jki} e_i$.

The following method seems preferable. We define the product* $\rho a$ of a mark $\rho$ of $F$ and an element $a = (a_1, \cdots, a_n)$ to be $(\rho a_1, \cdots, \rho a_n)$. In terms of the $\epsilon_i$ (end of § 2) any element $a$ is expressible uniquely:

$$a \equiv \sum_{i=1}^{n} \rho_i \epsilon_i, \qquad a_j = \sum_{i=1}^{n} \rho_i a_{ij} \qquad (j = 1, \cdots, n),$$

the $\rho$'s belonging to $F$. We introduce new units $e_i$ obtained from the $\epsilon_i$ by applying the inverse of the transformation $(a_{ij})$:

$$\epsilon_i \equiv \sum_{j=1}^{n} a_{ij} e_j \qquad (i = 1, \cdots, n),$$

so that $e_1 = (1, 0, \cdots, 0)$; $\cdots$; $e_n = (0, 0, \cdots, 0, 1)$. Then

$$a \equiv \sum_{j=1}^{n} a_j e_j.$$

We note that the $e_i$ are not in general elements of the system; likewise for $(\rho_1, \cdots, \rho_n)$.

*On the independence of the postulates.*

4. If $F$ has a modulus $q$, postulates II and III are derivable from the others; in postulate III we may take $a_i' = (q - 1) a_i$. If $F$ is the field of integers

* The associative law $\sigma(\rho a) = (\sigma\rho) a$, and the distributive law $\rho(a \oplus b) = \rho a \oplus \rho b$ then follow from these laws for marks and from I. We would give the same definition for $a\rho$ if needed.

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modulo $q$, also postulate IV is redundant, since postulates I and VI insure the presence in the system of every element $a$, each $a_i$ an integer (§ 2). In fact, there exist integers $\rho_i$ such that

$$\sum_{i=1}^{n} \rho_i a_{ij} \equiv a_j \pmod{q} \qquad (j=1, \cdots, n).$$

If $n = 1$, $q = 2$ or $3$, postulate I is likewise redundant. If $n = 1$, postulate V is redundant. Aside from these special cases, postulates I–VI are independent, as shown by the following systems:

(I) Elements $0, (\pm 1, 0, \cdots, 0), \cdots, (0, \cdots, 0, \pm 1)$; each $\gamma_{ijk} = 0$.

(II) $F$ non-modular; elements $a$, with each $a_i$ an arbitrary positive integer; each $\gamma_{ijk} = 1$.

(III) Set (II) with element 0 added; each $\gamma_{ijk} = 0$.

(IV) $F \neq GF[q]$; elements $a$, with each $a_i$ an arbitrary integral mark; each $\gamma_{ijk} = 0$ except $\gamma_{111}$, while $\gamma_{111} \neq$ integer.*

(V) Elements $a$ with each $a_i$ arbitrary in $F$; $n > 1$; $\gamma_{112} = \gamma_{221} = 1$, the remaining $\gamma_{ijk}$ all zero. [Postulate V fails for $r = t = i = 1$, $k = 2$.]

(VI) Elements $a$ with $a_2 = \cdots = a_n = 0$, $a_1$ arbitrary; $n > 1$; each $\gamma_{ijk} = 0$. An example for $n \geqq 1$ is the system with the single element 0; each $\gamma = 0$.

*Further generalization of the concept number system.*

5. The simplest example of a number system the coördinates of whose elements do not all belong to a field is furnished by a reducible system, the coördinates of the numbers of one subsystem belonging to a field of modulus $p$, those of a second subsystem to a field either without modulus or with modulus $\neq p$. To extend the definition (§ 2) to include this case, we take as elements $(a_1, \cdots, a_n)$, where $a_1, \cdots, a_{n_1}$ belong to a field $F_1$; $a_{n_1+1}, \cdots, a_{n_1+n_2}$ to $F_2$; etc. A similar change is to be made in postulate VI. Further, a $\gamma_{jki}$ vanishes unless $j$, $k$, $i$ belong to a single set of subscripts $1, \cdots, n_1$; etc.

Inversely, if in all the numbers $a_1 e_1 + \cdots + a_n e_n$ of a given system, $a_1, \cdots, a_{n_1}$ are marks of a field $F_1$ of modulus $p$; $a_{n_1+1}, \cdots, a_{n_1+n_2}$ are marks of a field $F_2, \cdots$, where each $F_i$ $(i > 1)$ is either without modulus or with modulus $\neq p$, then the given system is reducible, the units of the subsystems being $e_1, \cdots, e_{n_1}$; $e_{n_1+1}, \cdots, e_{n_1+n_2}$; etc. Proof will be given here only for the special case† in which each of the fields $F_i$ $(i > 1)$ has a modulus $q_i$. Let $1 \leqq j \leqq n_1$, $n_1 + 1 \leqq k \leqq n_1 + n_2$. Then

$$p(e_j e_k) = (pe_j) e_k = 0, \qquad q_2(e_j e_k) = e_j(e_k q_2) = 0.$$

* For $n = 2$ we may use the elements $A_1 + A_2 j$; $A_r = a_r + \alpha_r i$, $i^2 = j^2 = -1$, $ji = -ij$; viz. the real quaternion system expressed in two units $1, j$, but without $\gamma$'s satisfying IV.

† Noted by Mr. WEDDERBURN, who gave a different proof.

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Since $p$ and $q_2$ are distinct primes, there exist integers $P$ and $Q$ such that $Pp + Qq_2 = 1$. Hence $e_j e_k = 0$.

It remains to examine the restriction that the coördinates of the numbers of a system (or of a subsystem of a reducible system) shall belong to a field. One would surely retain the assumption that the aggregate of these coördinates can be extended by adjunction to an aggregate $A$ forming a commutative group under an operation called addition and such that a second operation called multiplication, obeying the associative and distributive laws, can be performed within $A$. We retain the commutative law for multiplication and assume that a product vanishes only when one factor vanishes, as otherwise in either case the aggregate of coördinates would be fully as complicated as a number system. Under these assumptions the aggregate forms part of a field. *

THE UNIVERSITY OF CHICAGO,
*February* 5, 1905.

* See KÖNIG, *Algebraischen Grösen*, 1903, pp. 8–9.

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# A THEOREM ON FINITE ALGEBRAS*

BY

J. H. MACLAGAN-WEDDERBURN†

FROBENIUS ‡ and C. S. PEIRCE § have shown that, in the domain of all real numbers, the only linear associative algebras every number of which, except zero, possesses an inverse, are quaternions and its subalgebras, and also that in the complex domain no algebra (except field F(·)) has that property. In the present paper it is shown that the Galois field is the only algebra of this type which possesses a finite number of elements.

## § 1.

Since addition is commutative in a linear associative algebra, it may be shown, as in the Galois field theory,‖ that for any number $x$ of the algebra there exists a prime integer $p$ for which $p$ times $x$ is zero and further that $p$ is the same for every $x$. It follows that, in the group formed by the numbers of the algebra under addition, every element is of period $p$ and therefore the order of the group is of the form $p^n$ where $n$ is some positive integer. The numbers $\neq 0$ of the algebra thus form a group $F$, of order $p^n - 1$, under the operation of multiplication. The self-conjugate elements of $F$, together with the zero element, form a Galois field. For, if $y_1$ and $y_2$ are self-conjugate elements of $F$ and $x$ is any element,

$$(y_1 + y_2)x = y_1x + y_2x = xy_1 + xy_2 = x(y_1 + y_2),$$

i. e., the self-conjugate elements are closed under addition, as well as under multiplication, and hence with 0 form a Galois field. Since the identity is a self-conjugate element of $F$ this Galois field always exists. If the order of the Galois field is $p^m$, the order of the subgroup of $F$, composed of its self-conjugate elements, is of order $p^m - 1$. The subgroup will be denoted by $G$ and the corresponding Galois field by $GF[p^m]$.

Let $x_1$ be any element $\neq 0$ of the algebra; then there are exactly $p^m$ distinct

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‡ G. FROBENIUS, Crelle, Bd. 84 (1878), p. 59.

§ C. S. PEIRCE, American Journal of Mathematics, vol. 4 (1881), p. 225.

‖ As developed, for instance, in Professor DICKSON'S *Linear Groups*, p. 9.



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numbers of the form $\xi_1 x_1$, where $\xi_1$ is a mark of $GF[p^m]$. If $x_2$ is any element not included in this set, there are $p^{2m}$ distinct numbers of the form $\xi_1 x_1 + \xi_2 x_2$. Similarily if $x_3$ is any number, which is not of the form $\xi_1 x_1 + \xi_2 x_2$, there are $p^{3m}$ numbers of the form $\xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3$, and so on.

We can evidently enumerate in this way all the numbers of the algebra and hence we can find say $s$ numbers $x_\alpha (\alpha = 1, 2, \cdots, s)$ such that any number $x$ can be expressed uniquely in the form

$$x = \sum_{\alpha=1}^{s} \xi_\alpha x_\alpha$$

where $\xi_\alpha (\alpha = 1, 2, \cdots, s)$ are marks of $GF[p^m]$. The order of $F$ is then $p^{ms} - 1$.

§ 2.

Let $x_a$ be any number of the algebra which does not lie in $GF[p^m]$. Then, if $y_1$ and $y_2$ are any two numbers commutative with $x_a$, $y_1 + y_2$ and $y_1 y_2$ are also commutative with $x_a$ and hence the set of all numbers commutative with $x_a$ forms a subalgebra. The group, formed by the numbers of this algebra under multiplication, will be denoted by $F_{x_a}$. Since $F_{x_a}$ contains $G$, we find as in § 1 that its order is of the form $p^{ms_a} - 1$, where $s_a$ is some positive integer. Hence on dividing the elements of $F$ into conjugate classes we get

$$(1) \qquad p^n - 1 = p^m - 1 + \sum_{a=1}^{t} k_a \frac{p^n - 1}{p^{ms_a} - 1}.$$

This shows that, if the least common multiple of the $s_a (a = 1, 2, \cdots, t)$ is $s'$, $p^m - 1$ is divisible by $(p^n - 1)/(p^{ms'} - 1)$. Therefore

$$(p^m - 1)(p^{ms'} - 1) = l(p^n - 1).$$

Reducing this modulo $p^m$ we see that $l$ must have the form $kp^m - 1$ $(k > 0)$. Since $ms' \leqq n$, we have $k = 1$ and $ms' = n = ms$.

§ 3.

It follows from the theory of hypercomplex numbers, that there is an equation of lowest degree,

$$(2) \qquad f(x) \equiv x^r + a_1 x^{r-1} + a_2 x^{r-2} + \cdots + a_{r-1} = 0,$$

with coefficients in $GF[p^m]$, which is satisfied identically by any given number $x$ of the algebra, irrespective of any special relation between the coördinates of $x$, except the condition that they lie in $GF[p^m]$. Further, there is at least one element of the algebra which satisfies no similar equation of lower degree. Indeed, (2) states that $x^{r-1}, x^{r-2}, \cdots, x^0$ are linearly independent with respect to

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$GF[p^m]$, and the condition of independence can evidently be put in a form which states that certain determinants, whose elements are rational integral functions of the coördinates of $x$, do not all vanish identically. Hence there must be some set of values of the coördinates for which $x^{r-1}, x^{r-2}, \cdots, x^0$ are independent and hence the particular $x$ so obtained can satisfy no equation of lower degree than $r$. (2) is called the characteristic or identical equation, while the equation of lowest degree satisfied by a given $x$ is called its reduced * equation.

$f(x)$ is irreducible in $GF[p^m]$; for, were it reducible neither factor would possess an inverse, contrary to our hypotheses. Similarily, the reduced equation of a given number is irreducible.

For any given $x$ all the roots of its characteristic equation $f(x) = 0$ are roots of its reduced equation † $\phi(x) = 0$, both being regarded as ordinary equations in the $GF[p^m]$, and therefore, since $\phi(x)$ is irreducible, $f(x)$ is a power of $\phi(x)$ and the degree of the reduced equation is a divisor of $r$.

We now assume that $F_{x_a}$ is abelian for every $x_a$ which does not lie in the $GF[p^m]$. It is shown below that the general case can be made to depend on this simpler one. Under this assumption $F_{x_a}$ is the multiplicative group of a Galois field and is therefore cyclic.‡ If $x_a$ is chosen as the generator of $F_{x_a}$ and if $s_a$ is the degree of the reduced equation of $x_a$, there are exactly $p^{ms_a} - 1$ different rational functions of $x$ with coefficients in $GF[p^m]$; hence the order of $F_{x_a}$ is $p^{ms_a} - 1$. Now it was shown above that, for some $x_a$, $s_a = r$ and also that each $s_a$ is a factor of $r$. Hence $r$ is the least common multiple of $s_1, s_2, \cdots$ and therefore $r = s$. But $ms = n$. Hence $F_{x_a}$ is identical with $F$, i. e., the algebra is a Galois field.

Suppose now that $F$ is not abelian. Then for some $x$, not contained in $G$, $F_x$ is not abelian. Similarly there must be some element of $F_x$, which is not self-conjugate in $F_x$ and is such that the group of those elements of $F_x$ which are commutative with it, is not abelian; and so on. We could then deduce in this way a series of groups of decreasing order, no member of the series being abelian. This is however impossible since each group of necessity contains $G$ and the order of $F$ is finite. Hence $F$ must be abelian.

## § 4.

The same result can be deduced as follows without the aid of the theory of the characteristic equation. It was shown in § 2 that, if $F$ is not a field, its elements can be arranged in subgroups the orders of whose multiplicative groups

* FROBENIUS, loc. cit.

† Cf. FROBENIUS, loc. cit.; E. WEYR, Monatshefte für Mathematik und Physik, vol. 1 (1890), p. 163.

‡ Cf. DICKSON, loc. cit., p. 13.

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are of the form $p^{ms_a} - 1$. It follows then from SYLOW's theorem, that every prime divisor of $p^n - 1$ is also a divisor of some $p^{ms_a} - 1$ $(ms_a < n)$. This is however only possible in the following two cases:* (i) $p = 2$, $n = 6$; (ii) $p = 2^k - 1$, $n = 2$.

In case (i) we have $m = 1$ since the only divisors of $n$ are 2 and 3, which are relatively prime, hence from (1), § 2,

$$2^6 - 1 = 1 + x_1 \frac{2^6 - 1}{2^3 - 1} + x_2 \frac{2^6 - 1}{2^2 - 1}$$

where $x_1$ and $x_2$ are integers not both zero. This gives $62 = 9x_1 + 21x_2$, an equation which cannot be satisfied by integers. Hence this case cannot occur. Case (ii) is evidently inadmissable since $n = 2$ is a prime.

## § 5.

The following proof is perhaps more direct. If all the elements of the algebra are multiplied successively by any one element except zero it is easily seen from the distributive law that they are permuted among themselves in such a way as to leave their additive relations unchanged, i. e., each such operation gives an isomorphism of the additive group with itself.† It follows that in the group of isomorphisms of the additive group there are two subgroups simply isomorphic with the multiplicative group; namely, one obtained by left and the other by right handed multiplication.

Each operation of one of these subgroups is commutative with every operation of the other, and it is easily seen that their greatest common subgroup corresponds to the set of the self-conjugate elements of $F$ and is therefore of order $p^m - 1$. These two subgroups then generate a subgroup of order $(p^n - 1)^2/(p^m - 1)$ of the group of isomorphisms. Since the additive group is an abelian group of type $(1, 1, \cdots, 1)$, its group of isomorphisms is the general linear homogeneous group $GLH(n, p)$ of order $(p^n - 1)(p^n - p)\cdots(p^n - p^{n-1})$, an expression which is divisible by $(p^n - 1)^2/(p^m - 1)$ only in the two special cases mentioned above, unless $m = n$. Having excluded these two cases above, we must have $m = n$. Therefore $F$ is abelian.‡

THE UNIVERSITY OF CHICAGO,
*March* 31, 1905.

* BIRKHOFF and VANDIVER, *On the Integral Divisors of $a^n - b^n$*, Annals of Mathematics, vol. 5 (1904), p. 177; L. E. DICKSON, *On the Cyclotomic Function*, American Mathematical Monthly, April, 1905.

† I am indebted to Professor E. H. MOORE for calling my attention to this point of view.

‡ Professor DICKSON has deduced the same result from the theory of canonical forms.

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# THE RELATION OF THE PRINCIPLES OF LOGIC TO THE FOUNDATIONS OF GEOMETRY*

BY

JOSIAH ROYCE

## INTRODUCTION.

In the year 1886, in the Philosophical Transactions of the Royal Society, Mr. A. B. KEMPE published *A Memoir on the Theory of Mathematical Form*, in which, amongst other matters, he discussed the fundamental conceptions both of symbolic logic and of geometry. The ideas there indicated were further developed, by Mr. KEMPE, in an extended paper *On the Relation between the Logical Theory of Classes and the Geometrical Theory of Points*, in the Proceedings of the London Mathematical Society for 1890. Despite the close attention that has since then been devoted to the study of the foundations of geometry, Mr. KEMPE's views have remained almost unnoticed. They concern, however, certain matters which recent research enables us to regard with increasing interest. I have been led, therefore, to attempt a restatement of KEMPE's logical-geometrical theory. The restatement has led me to conceptions which, although implied in those which Mr. KEMPE emphasizes, present a number of aspects which I believe to be novel, so that a considerable part of the present research follows a path of its own. My introductory words will indicate the nature of KEMPE's contribution to the problem of the foundations of geometry, the kind of task which his work has set before me, and my own main interest in preparing this paper.

The fundamental ordinal relation of geometry is the relation which can be, at pleasure, described as the triadic relation "between," or as an asymmetrical, transitive dyadic relation, such as "before," or "antecedent to," or "sequent to." Essentially the same relation is at the root of all serial order, and on this basis the logic of such order has lately been elaborately discussed by Mr. BERTRAND RUSSELL, in his *Principles of Mathematics*.

The axioms of geometry, as Dr. VEBLEN has stated them (Transactions of the American Mathematical Society, July, 1904), consist (1) of

* Presented to the Society April 29, 1905. Received for publication May 5, 1905.



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assertions characterizing the "between" relation, and duly restricting the application of this relation so far as the "lines" of geometry are concerned, and (2) of existential propositions defining certain entities that shall possess the relation. A similar prominence of asymmetrical transitive relations appears in Dr. HUNTINGTON's various *Sets of Postulates* for numbers, groups, etc. (Ibid., January and April, 1905).

The algebra of logic may be viewed (as Dr. HUNTINGTON, following Mr. PEIRCE and SCHROEDER, has lately afresh shown in detail), as depending upon the relation of inclusion or subsumption, sometimes symbolized by $\prec$. This relation is dyadic and transitive, and may be either symmetrical or unsymmetrical. Upon the basis of this one relation we can define the various operations of formal logic, such as logical multiplication and addition. If the relation $\prec$ is in a given instance symmetrical, it ensures what is commonly viewed as the "uniqueness" of an entity. That is: $a \prec a$; and if $a \prec b$, while $b \prec a$, than $b = a$ (see Dr. HUNTINGTON's paper of July, 1904, in these Transactions, for a fuller statement of the various results of these considerations). The relation $a \prec b$, in so far as it obtains between non-equivalent elements, may serve to define linear series: $a \prec b \prec c \prec d$, etc.; where $a \prec c$, and $a \prec d$. In such a series $c$ may obviously be said to lie "between" $b$ and $d$, and the analogy to the geometrical relation "between" is in so far plain. "Dense," and in fact, continuous linear series of the subsumption type can be conceived after the analogy of point series. But on the other hand, a system of logical classes differs, with respect to linear relations, from a system of points on a line in *two* very notable ways: —

(1) If $a \prec b \prec d$, and if it is also true that $a \prec c \prec d$, any one of the three relations $b \prec c$, $c \prec b$, $c = b$, is indeed *possible;* but, in case of the logical entities, it is also possible that $b$ and $c$ are such that no one of these relations actually holds between these two. Thus, Siberia is included within the Russian Empire, which itself may be viewed as included within the "Eurasian" continent. And Siberia is also included in Asia, which may also be regarded as included within the "Eurasian" continent. These subsumptions are transitive, and in so far linear in their type. Yet the Russian Empire and Asia do not form a pair possessing the relation $\prec$, read in either sense.

(2) If $a \prec b \prec c \prec d$, and if, also, $i \prec b \prec c \prec j$, the relations of $i$ and $a$, of $j$ and $d$ are similarly left indeterminate. These relations need not be directly expressible in terms of $\prec$ at all. That is, nothing in the logical relations forbids linear series (whether dense, or continuous or not) to have two or more "points," i. e., elements, in common, while any number of the other elements of the series remain entirely distinct. The logical lines, as Mr. KEMPE observes, may intersect any number of times.

For this very reason, however, the system of logical entities may be viewed

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simply as much more general and inclusive than the system of the points of space. And thus it becomes possible to regard a given space-form as *a selection from amongst the entities present in a system that exemplifies the logical relation* $-\!\!<$. That is: One may view the points of a space as a select set of logical elements, chosen, for instance, from a given "universe of discourse." This thought, whose possible fruitfulness for the logical development of the foundations of geometry I regard as highly notable, is the essential thought at the basis of Mr. KEMPE's paper of 1890, cited at the outset of this introduction.

The reason why such a thought seems promising is this: The relations amongst logical entities are, in any case, the most fundamental relations that we know. Experience shows us in the outer world those ordinal space-relations which geometry generalizes in the concept of "between." But our own thinking processes show us the meaning of the logical relation $-\!\!<$. The latter relation, then, is more suited to be the basis for a theory of the logic of an exact science, in case we can only so define and restrict its application that our ideal geometrical relations can come to be viewed as special instances of those forms which we can develop by the use of pure logic.

Mr. KEMPE's procedure, in the paper of 1890, is, in bare outline, as follows: He sets out, not by assuming the ordinary algebra of logic, but by defining, through postulates, a purely abstract set of entities called by him the "base-system," and a relation which may be viewed as a generalized "between." The latter is the relation which, in its most general form, is characteristic of what KEMPE himself calls, later in his paper, "flat collections" of any number the elements of the "base-system." But the relation first appears as a *triadic* relation, and is so characterized in the postulates. KEMPE uses the notation: $ac \cdot b$, to mean the assertion: "$a$, $b$, and $c$, form a 'linear triad,' with $b$ between $a$ and $c$." So far the expressions used resemble those for Dr. VEBLEN's generalized relation "in the order $abc$." But KEMPE's linear triad has these fundamental properties: (1) "If $ab \cdot c$, and $a = b$, then $c = a = b$." (2) "If $a=b$, then $ac \cdot b$ and $bc \cdot a$, whatever entity of the system $c$ may be."* In other words, KEMPE permits the "between" relation to hold where the related elements are, for all the purposes of the operations of the system, identical; and then he defines the distinctness of elements by means of a restriction of the relations that are permitted to hold in triads of distinct elements. The result is that the "between" relation becomes Dr. VEBLEN's "in the order," whenever the elements are all distinct.

The other properties of the "between" relation which are in question for

* I vary a little the order of Mr. KEMPE's statement of his principles. The relation = is defined by Mr. KEMPE only in a very highly abstract form which I need not here attempt to discuss. Geometrically interpreted, if this relation holds between points, these become identical points.

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Mr. KEMPE, are obtained by him through assuming two forms of triadic "transversal" propositions as fundamental postulates, viz.: *

I. If two linear triads, $ap \cdot b$ and $cp \cdot d$, exist, such that (as indicated by the notation), $b$ is between $a$ and $p$, and $d$ is between $c$ and $p$, then there exists an entity, $q$, which lies, in a linear triad, between $a$ and $d$, and, in another linear triad, between $b$ and $c$.

II. If, in the linear triads $ab \cdot p$ and $cp \cdot d$ (as indicated by the notation), $p$ lies between $a$ and $b$, and $d$ between $c$ and $p$, then $q$ exists such that $q$ lies, in a linear triad, so that $d$ is between $a$ and $q$, while, in another linear triad, $q$ is between $b$ and $c$.

If one interprets these assertions as relating to points in space, they become assertions obviously relating, respectively, to the diagrams following. But, as

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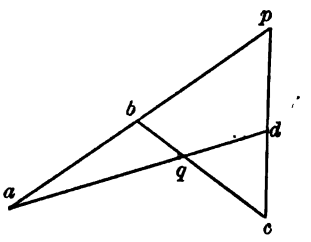


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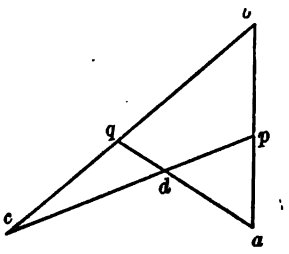


they are stated at the outset of Mr. KEMPE's paper, these principles have no specification beyond what the general properties of the linear triad, as just defined, predetermine.

One other existential proposition Mr. KEMPE uses as his *fifth* fundamental principle. This is simply the proposition that any entity belongs to the base-system whose presence there is not inconsistent with the four other principles, —a proposition which of course formally renders the two existential principles, here numbered I and II, superfluous; and which leaves the account of the "base system" inevitably somewhat unsatisfactory.

Mr. KEMPE now proceeds upon this basis, to show, by a decidedly original, although necessarily intricate procedure, that the elements of the base system, as thus defined, possess the properties and relations of a system of logical classes, or of other entities subject to the algebra of logic. In other words, he develops the entire algebra of logic, including the definitions and properties of the operations of logical multiplication and logical addition, without any other assumptions than those simple properties of the "between" relation which have just been stated. The proofs given are such as to apply to any finite number of the elements. Mr. KEMPE leaves, however, some doubt as to infinite collections.

* I vary slightly Mr. Kempe's mode of enunciating these existential propositions at the outset of his paper.

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Highly instructive observations are incidental to this development. The system of logical entities appears as possessing a thoroughly symmetrical structure. The "zero"-element and the "universe"-element have no essential distinction from any other similarly related pair of "obverse" elements. Negatives, in general appear as "obverses," because of the symmetrical contrast of their respective relations to the remainder of the system. All the fundamental relations of logic appear as triadic rather than as dyadic. But upon this triadic basis, polyadic relations also develop — the relations of KEMPE's "Flat-collections." These collections, thus named by reason of their resemblance to the various possible configurations of points in an $n$-dimensional space — "on a line," "on a plane," "in a three dimensional space," etc. — are Mr. KEMPE's means of relating the purely logical to the geometrical entities.

The junction of his principles with the regular algebra of logic once completed, although leaving certain doubts as to the application of his proof to infinite sets, KEMPE proceeds to the geometrical application. By (1) selecting certain "linear sets" of the elements of the base system; and (2) selecting from these sets those which conform to a new principle (here for the first time introduced into the essay), namely, to the principle that any two of the elements of a selected linear set shall determine the whole linear set to be selected, KEMPE is in possession of a system of foundations for a geometry of a "flat space" of $n$ dimensions. The further development of such a geometry is indeed merely sketched in the paper in question. But since the "triangle-transversal" axiom is provided for by the initial principles of the system, and since, by the selection of the linear sets of elements, the ordinary properties of the geometrical "between," and the axiom as to the determination of a line by any two of its points are now also secure, KEMPE's result, although only indicated in his text, is in the main clear. *A space of n dimensions is a select class or set of elements which themselves are entities in a logical "field."* The selection of the entities of a given space is arbitrary; and so the space-forms whose entities are selected may be varied in any way whatever which is consistent with the triangle-transversal-axiom, and with the properties of the generalized between-relation. The problem of the continuity of the geometric sets is only very generally treated, and is not solved.

The wide outlook thus suggested into the theory of space-forms certainly deserves to be better considered than KEMPE's treatment of the subject has so far been considered by mathematicians. For me, however, as a student of philosophy, a still further interest attaches to those results which I have thus suggested, an interest which my mathematical colleagues may also share.

The problem of the foundations of geometry is only a part of that general problem regarding the fundamental concepts of the exact sciences which is now so widely studied. KEMPE's research suggests that, since metrical relations, and

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therefore (as KEMPE himself briefly indicated in his *Theory of Mathematical Form*, § 309), the whole algebra of ordinary quantity, can be reduced, in any system of three or more dimensions, to a series of propositions based upon purely ordinal relations, — the entire system of the relationships of the exact sciences stands *in a much closer connection with the simple principles of symbolic logic* than has thus far been generally recognized.

Mr. BERTRAND RUSSELL, using very different methods, insists, indeed, in general, upon the closeness of such a connection. But the distinction between the "logic of relations," and the older "logic of classes," and of "propositions," — a distinction which Mr. BERTRAND RUSSELL in his *Principles of Mathematics* regards as something quite fundamental, seems to me to become, in the light of KEMPE's research, a distinction probably quite superficial. Hence to my mind, Mr. KEMPE's theory goes far deeper than Mr. RUSSELL's. Give us a system of entities of the types of logical classes, and we shall find that their relations (all statable in terms of KEMPE's "between"), are already (quite apart from a separate "logic of relations"), certainly as rich as the totality of the relations studied in geometry, and are, for reasons upon which KEMPE has dwelt, probably as rich as the totality of the relations known to the exact sciences, at least so far as the latter have yet been developed. The bare prospect of such a result deserves a careful consideration, in case one takes interest in the unification of the categories of science. KEMPE's theory promises such an unification.

The present memoir proposes to contribute towards a more precise statement of the theory thus outlined. At the basis of my own discussion, I place, however not KEMPE's "between" relation, but another fundamental relation of symbolic logic which has the interest of being absolutely *symmetrical*, while, when it obtains amongst $n$ entities, it permits (upon the basis of certain simple existential propositions), the definition of the properties of KEMPE's "flat collections," and so the definition of asymmetrical relations of a very high degree of complexity. This change of starting point is the prime novelty of the present discussion, as contrasted with KEMPE's.

The *contrast between symmetrical and unsymmetrical relations* seems, to the ordinary view, absolute. Mr. RUSSELL, in his late volume, so treats it. Geometry, and the ordinary algebra of quantity (as these subjects are usually treated), seem to depend on regarding the distinction as quite fundamental. In symbolic logic, however, as Mrs. LADD-FRANKLIN long ago pointed out (in her paper on the algebra of logic in the volume called "*Studies in Logic by members of the Johns Hopkins University*," Boston, 1883), a "symmetrical copula," namely that of "inconsistency," or of "opposition," can be made to accomplish all the work of the ordinary unsymmetrical copula $\prec$. In other words, if I have otherwise defined the meaning of "not," the statement "$x$ is inconsistent with not-$y$," means the same as "$x$ implies $y$." The copula in the former case is

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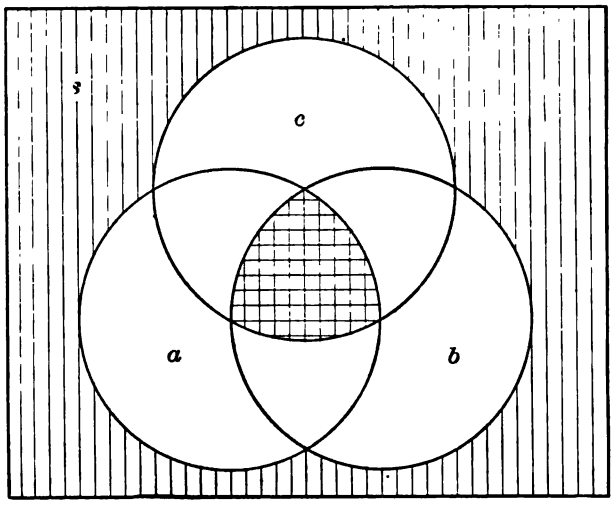
symmetrical, in the latter unsymmetrical. The former expression makes explicit the "relative product" (as it is called by PEIRCE and RUSSELL) of *two* symmetrical relations (viz., "opposes" and "not"). This "relative product" is, itself, indeed an unsymmetrical relation. But the constituents of this product are symmetrical. This already suggests how asymmetry may be definable in terms of symmetry.

Using as my suggestion some brief observations of KEMPE (in §§ 75–82, of his paper in the Proceedings of the London Mathematical Society), I have therefore chosen to define, by postulates, at the outset of my discussion, a symmetrical relation which I may call "the $O$-relation." This relation is essentially polyadic, and applies at once to any number of terms greater than one. In logical terms, this is the relation in which (if we were talking of the possible chances open to one who had to decide upon a course of action) any set of *exhaustive but, in their entirety, inconsistent choices* would




stand to one another. It is also the relation in which the members of any set of areas stand to one another when there is no area (except the "zero"-area) which is common to *all* the areas of the set at once, while together these areas exhaust some larger surface (which therefore resembles, in its relation to them, a logical "universe"). Thus if, in the annexed diagram, the surface $s$ contains three circles, $a$, $b$, and $c$, and if we then agree to disregard, or to view as stricken out or destroyed, the here shaded portions of the diagram, the circles $a$, $b$, and $c$ have then only the stricken-out or "zero" area in common, while together they exhaust what we thus permit to remain of the surface $s$. In consequence, $a$, $b$, and $c$ here form what KEMPE calls "an obverse collection," and what I call, in this paper, an "$O$-collection."

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If *two* objects stand in the *O*-relation each to the other, then these two are related as a pair of "negative" or contradictory classes, or statements, are related to one another. But any number or multitude of objects, in case such are otherwise permitted to exist in a given system, may stand, not in pairs, but as a whole collection, in this relation, and will then constitute what I call an "*O*-collection." The name that I give to the relation is derived from the close analogy of such collections, even when they contain more than two elements, to KEMPE's pairs of "obverse elements." But I do not myself wish to call the *O*-collections, with KEMPE, "obverse collections," because, as will be seen, I find it convenient to use an expression in my text (where I speak of "mutually obverse" collections), in a way that would make such usage confusing. Hence I read the expression "*O*-collection," simply as written.

In my text the *O*-relation is entirely freed from dependence upon all such examples as the ones just used, and is defined solely by postulates, and is to be taken solely as it is there defined. The fact however that it is, in its relational properties, identical with the "yes-no" relation—the earliest exact relation defined by the human mind—is, in this introduction, important. *For what I am in the end to show is that all the serial and other ordinal relations known to logic and to geometry, and all the operations known to both, so far as they are pure exact sciences, are ultimately reducible to assertions that certain entities do, while certain entities do not, stand to one another in the perfectly symmetrical O-relation.*

MS.

My procedure differs from KEMPE's, not only in making this wholly symmetrical relation, instead of KEMPE's "between," fundamental, but also in the existential principles which I assume. KEMPE's "transversal" axioms form with me a theorem, proved late in the discussion. My own existential principles have to be wide enough to provide for the "continuity" of the system, or, rather, for its inclusion of infinitely numerous continuous systems, and definite enough to make the system of the entities to which the logical calculus is applicable a determinate manifold, inclusive of the points of a space of $n$-dimensions. The usual treatment of the algebra of logic provides only for arbitrarily determined sets of $2^n$ or of $2^{2^n}$ entities in a given logical system. KEMPE calls any such selected set a "full set." KEMPE's further postulate, however, calling for "all entities" consistent with the formal laws, is itself indefinite. In seeking adequate postulates I have been led to two observations which, although in themselves fairly obvious, seem to me to be new, viz., (1) a relation is here shown between the existence of logical sums and products and the general theory of limits and of continuity; and (2) a general definition of the pairs of elements which I have called "conjugate resultants"*

* This concept of "conjugate resultants" is generalized from KEMPE's own generalization of his "unsymmetrical resultants" in § 28 of the essay of 1890. My use of the concept differs in many ways from his.

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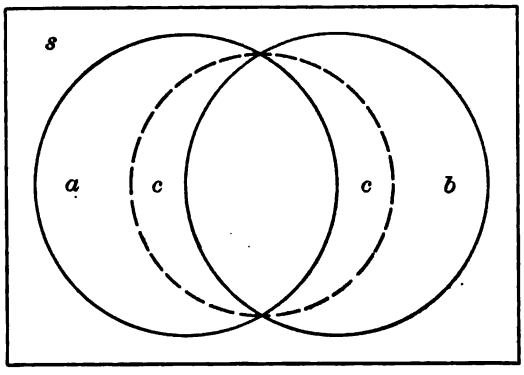
is made centrally important. The algebra of logic, so far as I know, has not hitherto been brought into definite relations with the problem of the continuum. This is one of the things that I here accomplish. This undertaking involves proving all the principles of logic so as to make them applicable to infinite sets of entities at once. This also I have here done.

KEMPE's "linear triad" of elements is represented, in any logical system of classes, by the classes, or areas $a$, $b$, and $c$, which stand in the relation which is represented in the adjoining diagram by the closed figures so lettered. Any area $c$ which includes the common part of $a$ and $b$, and which is included within their logical sum, is, in KEMPE's phrase, such that "$c$ is between $a$ and $b$." I hereafter symbolize this relation, in my own way, as $F(c|ab)$. The relation in question is called by me the $F$-relation, because it is that characteristic of




KEMPE's "flat collections." The $F$-relation, so long as "obverses" or "negatives" exist, follows immediately from, and is equivalent to, an $O$-relation. For, in the diagram if $s$ is the total surface in which $a$, $b$, and $c$ are included, then when "$c$ is between $a$ and $b$," "$a$, $b$, and $\bar{c}$ ($\bar{c}$ being the obverse of $c$) constitute an $O$-collection," or "are in the $O$-relation."

The outcome of our discussion will show that, while logical relations can be indifferently stated as $O$-relations, or stated as $F$-relations, or (when once addition, multiplication, and negation have been defined) can be stated in terms of equivalence, the $F$-relations are the only natural means of expressing the geometrical ordinal relations. This difference, however, between the logical and the geometrical entities, is due to the simple fact that (as KEMPE points out), when geometrical sets are considered, the obverses of the elements of any set are excluded from that set, so that the obverses may be viewed as ideal elements of the geometry in question. In fundamental meaning all these relations spring from a common root.

If "$m$ is between $b$ and $c$," I sometimes call $m$ "mediator" or again, on occasion, "resultant" of $b$ and $c$. I extend the term "resultant" to include

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the case where a single element stands in an $F$-relation to any collection of elements.

The axioms or principles assumed at the outset of my discussion, in § 19, are verifiable for a collection of areas all of which are included within a given area, if the $O$-relation is interpreted as, for the sake of illustration, I have just done. The consistency of these axioms is thus secured from the start. For the verification of "Principle VI" of my set of principles, see § 118 of my text. For KEMPE's term "base-system," I substitute "the system $\Sigma$."

The considerable length of the discussion may be justified by the importance, (1) Of a development of the principles of logic solely in terms of a symmetrical polyadic relation; and (2) Of the need of supplementing KEMPE's results by a theory of the continuity of the "base-system."

## THE SYSTEM $\Sigma$, AND THE $O$-COLLECTIONS.

### CHAPTER I. DEFINITIONS AND PRINCIPLES.

1. The system $\Sigma$, whose structure we are to consider in what follows, consists of certain "elements," which we shall regard, in the present discussion, as simple and homogeneous. As symbols for these elements, we shall employ the small letters of the alphabet: $a, b, \cdots, i, j, \cdots, x, y$, etc. In many cases, for the sake of distinguishing one element of a set from others, we shall need subscript marks; and for these too we shall nearly always employ small letters, or if that be convenient, numbers, thus: $a_k, b_i, \cdots, x_1; x_2, p_v$, etc. It is to be noted however, that, unless the contrary is especially indicated, these subscripts are *merely* convenient distinguishing marks; so that the numbers when used as subscripts will, in general, *not possess any ordinal meaning*, but will be used merely as tags. The few exceptions to this rule will explain themselves.

2. The elements thus symbolized may be viewed either singly, or in their collections. A collection of elements will usually be in question, in what follows, as a sort of complex or secondary unit. We shall apply predicates to collections when they are viewed as such complex units, shall compare collections, combine them into larger collections, make partitions of them into the partial collections of which larger collections are composed, classify collections, etc. A collection may consist of a single element of $\Sigma$, and is then called a monad, or a monad-collection. A collection of two elements is called a pair, of three a triad, of four a tetrad, of $n$ elements an $n$-ad. But a collection may consist of an infinite multitude of elements. And, in fact, whenever our statements, and whenever the conditions imposed in the course of a given investigation, do not set definite limits to the multitude of the elements that belong to such collections as are at any time in question, it is always to be understood that the collections

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of which we then speak are actually permitted to consist not only of any finite number, but of any multitude whatever of elements.

3. Collections in which a single element, such as $b$, or $x$, is viewed as occurring repeatedly, are to be regarded as permissible objects for our consideration; and if we define the number or the multitude of elements in such a collection, or if, for any other reason, we have to treat all the various elements of a given collection in various ways, then, for the purposes of the count, or of the other individual treatment of the elements of any collection, the various repetitions of a given element are to be treated as, in so far, distinct members of the collection in question. Empty, or "zero" collections will not be considered in the present discussion.

4. In order to symbolize a collection without indicating, by the mere symbol, any assertion except the assertion that the collection consists of certain elements, we shall write the symbols for the elements in question, separated by commas, and in a parenthesis. Thus the symbol $(a, b)$ denotes "the pair which consists of the elements $a$ and $b$." The symbol $(x, x)$ denotes "the pair consisting of $x$ and of $x$ repeated." The symbol $(x, x, y, y, r)$, denotes "the collection consisting of $x$ and of $x$ repeated, and of $y$ and of $y$ repeated, and of $r$." The symbol $(x_1, x_2, \cdots, x_r, \cdots)$ denotes "the collection consisting of $x_1$, of $x_2$, and of an unspecified multitude of other elements, each of which is symbolized by $x$ written with some subscript." In such a case, if no restriction of the multitude of the elements is stated, this multitude of the elements of the collection need not be limited to that of the whole numbers; and the use of whole numbers as subscripts is then of no special significance, beyond that of the convenience of such subscript-symbols. Other subscript-symbols would be equally possible, and may, upon occasion, be used.

5. In many cases, we shall need to symbolize a collection without at the moment designating any of the single elements of which it consists. In such case we shall use Greek letters, $\alpha$, $\beta$, $\gamma$, $\delta$, etc. (and, in a few cases, the capital Greek letters also), as symbols for *entire collections*. Thus, the symbol $\alpha$ means "the collection designated as $\alpha$, consisting of elements which are not hereby further specified." In such a case, the collection $\alpha$ may be, in fact, a perfectly determinate collection, and the symbol $\alpha$ will then be merely a convenient abbreviation. In other cases, $\alpha$ may stand for an unspecified instance of some class of collections; the members of the collection at any time in question being left, by the conditions of the discussion, to be otherwise determined. A collection $\alpha$ or $\beta$ or $\delta$ may be unrestricted as to the multitude of its elements, or may be a monad, a pair, a triad, etc., according as the conditions of a given statement permit or determine.

6. A frequent operation, in our discussion, will be the adjunction of elements to an already given collection, or to elements already under consideration. The

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symbol $(\alpha, x)$ will mean "the collection consisting of the collection $\alpha$ of elements together with the element $x$, adjoined thereto." The symbol $(\alpha, \beta, \gamma)$ will mean "the collection formed by adjoining to the collection $\alpha$ all the elements of the collections $\beta$ and $\gamma$." Greek letters, as symbols for whole collections, and small letters $a$, $x$, $m$, etc., as symbols for elements, may thus be combined in the same expression in order to indicate what adjunctions are at any time in question.

7. When we speak simply of "a collection" of the elements of $\Sigma$, without further specification of the character of this collection, *the order in which the elements of the collection are named, or otherwise indicated, or in which they stand in the collection, is wholly indifferent.* A collection, in such a general case, *is determined wholly by the fact that certain elements do, while certain elements do not belong to it.* The arrangement of the elements within the collection will concern us only in case the definition of a given type of collections, or the conditions of a given problem, expressly require us to take note of such arrangement.

8. Fundamental, in our discussion of the properties of the system $\Sigma$, is a classification of the collections of elements into those which are, and those which are not what we shall call "$O$-collections." The $O$-collections form a class of collections whose fundamental properties we define by the laws hereafter stated. The symbol $O(xyz\cdots)$ is to be read as the statement: "The elements $x$, $y$, $z$, etc., taken together, constitute an $O$-collection." The symbol $O(\alpha)$ is to be read as the statement: "The collection $\alpha$ is an $O$-collection." The symbol $O(\beta\gamma)$ is to be read as the assertion that "The total collection formed out of the collections $\beta$ and $\gamma$ is an $O$-collection." At pleasure we shall also use the abbreviated form of expression: "the collection $O(xyz\cdots)$," meaning "the collection such that the assertion $O(xyz\cdots)$ is true." The symbol $O(\alpha x)$ means that "the collection formed by adjoining the element $x$ to the collection $\alpha$ is an $O$-collection." The symbols $O(\alpha\beta\gamma\delta)$, $O(xy\alpha pq\beta jr\delta)$, etc., are to be read so as to assert that whatever total collection of elements and collections is indicated by the letters enclosed in the parentheses, is an $O$-collection.

9. If a collection is *not* an $O$-collection, the fact may have to be separately asserted. We propose in the cases where, for the sake of conciseness such usage is advisable, to call the class of all those collections which are *not* $O$-collections, $E$-collections. The symbol $E(\alpha)$ may be read at pleasure as the assertion: "The collection $\alpha$ is *not* an $O$-collection," or again "is an $E$-collection." Correspondingly we read the symbols: $E(xy)$ (where we also assert that "$x$ and $y$ form an $E$-pair," or "do not form an $O$-pair"); $E(x\alpha)$; $E(xy\beta\gamma)$, etc. If $\alpha$ is not an $E$-collection, then $O(\alpha)$.

In general, we shall speak of "$O$-pairs," "$O$-triads," "$O$-tetrads," "$O$-$n$-ads," and of "$E$-pairs," "$E$-triads," etc., wherever our collections, whether $E$-col-

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lections or $O$-collections, are restricted as to the number of elements in a way to which we wish to call attention.

10. When we simply assert $O(\alpha)$ or $E(\beta)$ of any collection, *the order in which the elements of an O-collection or of an E-collection are named or considered is indifferent.* The elements in question in such cases, simply *do* or *do not* belong to the collections in question without regard to the order in which the elements stand.

*Equivalent elements.*

11. In case two elements are such that *each of them can be substituted for the other in every O-collection in which that other occurs*, while leaving the collection in question still an $O$-collection, then these two elements are said to be *equivalent each to the other* or to be *mutually equivalent elements.* But if there exists an $O$-collection into which either of these elements enters, while the other cannot be substituted for the first, in that collection, without altering it into an $E$-collection, then the two elements are not equivalent. The equation $x = y$ means, therefore, in the present discussion *not* that $x$ and $y$ are identical, but simply that "either $x$ or $y$ may be substituted for the other in any $O$-collection wherein that other occurs, while the substitution leaves the collection in question still an $O$-collection." In case it were possible that *neither* $x$ nor $y$ formed a member of *any* $O$-collection, this definition would imply that they were then also equivalent. The usual properties of the relation of equivalence obviously follow from this definition: viz., $x = x$; if $x = y$, then $y = x$; if $x$, $y$, and $z$ are such that $x = y$, and $y = z$, then $x = z$. It is plain that, if $x = y$, either $x$ or $y$ can be substituted for the other in any $E$-collection in which that other occurs. For if $x = y$, it is, by definition, impossible that $E(x\alpha)$ while $O(y\alpha)$; hence, if $E(x\alpha)$, $E(y\alpha)$ follows; and the converse is also obvious from the definition of equivalence.

12. As just pointed out, equivalent elements need not be identical. Hence, although the assertion: "$a$ is not equivalent to $b$," obviously implies the assertion: "$a$ is not the same element as $b$," — these two assertions must still be carefully distinguished, since the second of them does not imply the first. For the assertion "$a$ is not equivalent to $b$," we shall use the symbol $a \neq b$. This means that "there exists at least one $O$-collection into which one of these elements enters, while, if the other is substituted for the first in that collection, the collection in question becomes an $E$-collection."

*Mutually obverse elements and collections.*

13. If two elements of $\Sigma$, say $p$ and $q$, are such that $O(pq)$, then $p$ and $q$ are said to be *mutually obverse elements*, or *obverses*, each of the other. Mutually obverse elements are then such elements as together form an $O$-pair. If the assertion $O(xx)$ were true of any element, that element would be an obverse of itself. A given element may possess various obverses.

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14. If a set of $O$-pairs exists, such that, for certain existent elements of $\Sigma$, $O(ab)$, $O(cd)$, $O(ef)$, $O(pq)$, etc. (this set possessing either finite or infinite multitude), and if we suppose a collection $\alpha$ made up by the selection of one member, and of one member only, from each of these pairs, and if a collection $\beta$ is supposed to be made up out of all the remaining members of the pairs, the collections $\alpha$ and $\beta$ are said to be mutually *obverse collections*.

15. In consequence of the formation of the collections $\alpha$ and $\beta$ just defined, certain repetitions of elements may occur in $\alpha$, or in $\beta$, or in both. In such a case, just as in the cases mentioned above, in 3, the various repetitions of any given element are to be regarded as distinct members of the collections $\alpha$ and $\beta$ in question. Thus, if the $O$-pair $O(ab)$ is repeated, then, out of the two pairs $O(ab)$ and $O(ab)$, treated, for the purpose in hand, as distinct pairs, we can form, according to the procedure defined in 14, the collections $(a, a)$, and $(b, b)$. These collections are hereupon to be regarded, by virtue of our definition, as mutually obverse collections. Again, if the pairs $O(pq)$, $O(xy)$, $O(mn)$, $O(xk)$, and $O(mq)$, are given, then the collections $(p, x, m, x, m)$, and $(q, y, n, k, q)$, are mutually obverse collections, as are also any two collections that can be formed out of these same pairs by any permissible variation of the procedure defined in 14. The order in which the members of each of the mutually obverse collections are written, is again indifferent.

16. Suppose a collection $\delta$ is first given. Let each element of this collection be such that it can be made to form an element of a pair of elements which (whether the elements of this pair are, or are not, repetitions of those present in other pairs), is distinct from the pair of which any other element of $\delta$ is a member. If each of the pairs thus formed is an $O$-pair, then the collection $\epsilon$, consisting of all the remaining elements of the pairs in question, is an obverse of the collection $\delta$, while $\delta$ is an obverse of $\epsilon$. Thus, if $\delta$ is the collection $(x, m, k, l, l)$, and if $O(xy)$, $O(mj)$, $O(kn)$, $O(lr)$ and $O(ls)$; and if $\epsilon$ is the collection $(y, j, n, r, s)$, then the collections $\delta$ and $\epsilon$ are mutually obverse collections.

*Complements and resultants.*

17. In case an element $q$ exists such that, for a given collection $\beta$, $O(\beta q)$ is true, the element $q$ is called a *complement* of $\beta$. In case $q$ and $r$ exist such that, for a given collection $\beta$, $O(\beta q)$ is true, while, at the same time $O(qr)$, then $r$ is called a *resultant* of $\beta$.

18. The properties of those collections which may be formed of the elements of $\Sigma$ are, in the main, properties determined by the existence of equivalent and of non-equivalent elements, of obverses and of resultants, together with the existence of certain laws and principles which hold valid for the system.

19. To the statement of these principles or "postulates" we now proceed. They may be classified under two heads. They are: (1) General laws to which

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all $O$-collections, in case such exist, are to conform; and (2) Principles requiring, either conditionally or unconditionally, the existence of certain elements, and of certain collections.

(1) *Laws to which all O-collections conform.*

I. If $O(\alpha)$, then $O(\alpha\gamma)$, whatever collection $\gamma$ may be.

II. If, whatever element $b_n$ of $\beta$ be considered, $O(\delta b_n)$, and if $O(\beta)$ is also true, then $O(\delta)$.

(2) *Principles requiring the existence of elements of* $\Sigma$.

III. There exists at least one element of $\Sigma$.

IV. If an element $x$ of $\Sigma$ exists, then $y$ exists such that $x \neq y$.

V. Whatever pair $(p, q)$ exists, such that $p \neq q$, $r$ also exists such that, while both $O(rp)$ and $O(rq)$ are false, $O(pqr)$ is true.

VI. If $w$ exists such that $O(\vartheta w)$, then $v$ also exists such that $O(\vartheta v)$, and such, too, that, whatever element $t_n$ of $\vartheta$ be considered, $O(vwt_n)$.

20. These principles may be restated, with less use of symbols, thus:

I. An $O$-collection remains an $O$-collection, whatever elements or collections may be adjoined to it.

II. If a collection $\beta$, consisting wholly of elements which are complements of a collection $\delta$, is an $O$-collection, then $\delta$ itself is an $O$-collection.

III and IV (in combination). The system $\Sigma$ contains at least one pair of mutually non-equivalent elements.

V. If any pair of mutually non-equivalent elements is given, a third element of $\Sigma$ exists which forms an $O$-pair with *neither* of the elements of this pair, but which is such that the three elements in question together constitute an $O$-triad.

VI. If there exists any complement of a given collection $\vartheta$, then, if $w$ be such a complement, there exists a complement of $\vartheta$, viz. $v$, such that every element of $\vartheta$ is a complement of the pair $(v, w)$.

At the close of the introduction a system $\Sigma$ which conforms to all the foregoing principles, has been already pointed out.

## Chapter II. Elementary Consequences of the Principles.

21. *The elimination of obverses.* If any collection $\alpha$ is such that $x$ and $y$ exist such that $O(\alpha x)$ and $O(\alpha y)$, while $O(xy)$, then $O(\alpha)$. This follows directly from principle II, if the pair $(x, y)$ be viewed as the collection $\beta$ of that principle.

22. *The correspondence of mutually obverse O-collections.* If any collection $\pi$ is such that $O(\pi)$, and if a collection $\rho$ is a collection which is an obverse collection of the collection $\pi$, then $O(\rho)$. For let $p_n$ be any element of $\pi$. Then in $\rho$ there exists (by the definition of mutually obverse collections, as given

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in 14, 15, 16), some element $r$ such that $O(p_{\mu}r)$. By principle I, we may adjoin to the $O$-pair, $O(p_{\mu}r)$, all the remaining elements of $\rho$ besides $r$, and the thus enlarged collection will still remain an $O$-collection; so that $O(p_{\mu}\rho)$. As an analogous result holds of every other element, $p_{\nu}$, of $\pi$, without exception, it appears that $\pi$ consists entirely of elements each of which is a complement of the collection $\rho$. Since, however, $O(\pi)$, by principle II, $O(\rho)$.

23. *The elimination of common elements.* If $\eta$ and $\vartheta$ are mutually obverse collections; and if $x$ exists such that $O(\beta x)$ and $O(\vartheta x)$, while $y$ exists such that $O(xy)$, then $O(\beta\eta)$. For, by 22, from $O(\vartheta x)$ follows $O(\eta y)$, since, by adjoining $x$ to $\vartheta$, and $y$ to $\eta$, we form the two mutually obverse collections $(x, \vartheta)$ and $(\eta, y)$. By adjunction, in accordance with principle I, from $O(\beta x)$ follows $O(\beta\eta x)$ and from $O(\eta y)$ follows $O(\beta\eta y)$. Since $O(xy)$, there follows, from $O(\beta\eta x)$ and $O(\beta\eta y)$, by principle II, as explained in 21, $O(\beta\eta)$.

24. *The elimination of partial collections.* If $\epsilon$ consists solely of elements which are complements of a collection $\lambda$, if $\delta$ and $\gamma$ are mutually obverse collections, and if $O(\delta\epsilon)$, then $O(\gamma\lambda)$. For $\epsilon$ by hypothesis consists of elements which are complements of $\lambda$. Let $e$ be, then, an element of $\epsilon$. Then $O(e\lambda)$, and hence, by adjunction (principle I), $O(e\gamma\lambda)$. Furthermore, $\delta$ consists wholly of elements which are complements of $\gamma$. Hence if $d$ is an element of $\delta$, $O(d\gamma)$ and hence, by adjunction (principle I), $O(d\gamma\lambda)$. Any element of $\epsilon$, and also any element of $\delta$, is thus a complement of $(\gamma\lambda)$. Hence the $O$-collection $O(\delta\epsilon)$ consists entirely of elements which are complements of the collection $(\gamma, \lambda)$. Hence the collection $(\gamma, \lambda)$ is itself an $O$-collection, by principle II. If $\epsilon$ reduces to a single element, $e$, then the hypotheses above stated reduce to $O(\delta e)$ and $O(\lambda e)$, while $\delta$ and $\gamma$ are mutually obverse collections, and the result then becomes identical with that of 23. But if $\delta$ reduces to a single element $d$ and $\gamma$ to an obverse element $c$ such that $O(cd)$, then the result is that if $\epsilon$ consists solely of elements which are complements of a collection $\lambda$, and if $d$ is such that $c$ exists such that $O(cd)$, while $O(d\epsilon)$, then $O(c\lambda)$.

25. The operations of the reduction of collections through the elimination of elements and of partial collections as explained in the foregoing, will be found to be of fundamental significance throughout our procedure in what follows.

26. *The existence of obverse elements.* By virtue of principles III and IV, there exists $(x, y)$ such that $x \neq y$. Since $x \neq y$, it follows from the definition of equivalence (11) that there exists at least one $O$-collection into which one of these elements, say $x$, enters; while in that collection (whether $y$ is also a member of the collection in question or not), $y$ cannot be substituted for $x$ without changing the collection into an $E$-collection. Let $\alpha$ be the collection thus characterized. Then, by hypothesis, $O(\alpha)$.

If $\alpha$ contains all the elements of $\Sigma$, occurring either once each, or in any multitude of repetitions, then a collection exists which contains all of the ele-

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ments of $\Sigma$, and which is an $O$-collection. If $\alpha$ does not contain all of the elements of $\Sigma$, nevertheless, by principle I, since $O(\alpha)$, all of the elements of $\Sigma$ which do not appear in $\alpha$ may be adjoined to $\alpha$, and the resulting collection, say $\vartheta$, will be an $O$-collection.

Hence, in any case, there exists a collection which contains every element of $\Sigma$ (each element occurring in that collection either once only, or else repeatedly), while this collection, say $\vartheta$, is an $O$-collection.

27. Since $\vartheta$ is such that $O(\vartheta)$, every element of $\Sigma$ is a complement of $\vartheta$, by principle I. Let $w$ be any element of $\Sigma$. Since $w$ is a complement of $\vartheta$, it follows by principle VI, that $v$ exists, such that, whatever element $x$ of $\Sigma$, or of $\vartheta$, be chosen, $O(xvw)$. Since every element of $\vartheta$ is thus a complement of the pair $(v, w)$, while $O(\vartheta)$, it follows by principle II, that $O(vw)$ is true. By adjoining to the pair $(v, w)$ all the elements of $\Sigma$ which do not belong to this pair, we now have $O(\Sigma)$, an assertion according to which each element of $\Sigma$ is supposed to appear once, and without repetition, in the $O$-collection in question. Since $O(vw)$, $v$ and $w$ are mutually obverse elements.

Since any element whatever of $\Sigma$ may be taken instead of $w$, while, each time, an element would be found to take the place here occupied by $v$, we have so far two results:

(1) The system $\Sigma$, taken in its entirety, is an $O$-collection.

(2) Every element of $\Sigma$ possesses at least one obverse.

28. A fundamental property of all pairs of mutually obverse elements hereupon comes to our notice, and is as follows: Let $b$ be any element. Let $q$ and $r$ be two obverses of $b$, so that $O(qb)$ and $O(rb)$. Hereupon let $\gamma$ be any collection such that $O(\gamma q)$; that is, let the collection $O(\gamma q)$ be any $O$-collection into which $q$, one of the obverses of $b$, enters. Then, by adjunction (principle I), we have, since $O(br)$, $O(br\gamma)$; and, since $O(\gamma q)$, $O(qr\gamma)$ (wherein we may of course change, as we here do, at pleasure, the order in which the members of the $O$-collection are written). The collection $(r, \gamma)$ is thus such that, if either of the members of the $O$-pair $O(bq)$ be separately adjoined to it, the resulting enlarged collection is each time an $O$-collection. Hence, by principle II, $O(r\gamma)$. It thus appears that, whatever the collection $\gamma$ may be, if $O(q\gamma)$, $O(r\gamma)$ follows. By a precisely analogous reasoning we could show that if $\gamma$ is such that $O(r\gamma)$, $O(q\gamma)$ follows. Hence the two obverses of $b$ here in question, viz., $q$ and $r$, are such that either of them may be substituted for the other in any $O$-collection in which that other occurs, while still leaving that collection an $O$-collection. Hence by the definition of equivalence $q = r$. As the reasoning thus used applies to any two obverses of the same element $b$, whatever $b$ is, we have, as a result, the proposition that *any two obverses of the same element are mutually equivalent elements.* That is, again, if $q$ is an obverse of $b$, and $b$ is an obverse of $r$, then $q = r$.



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29. If $x = y$, while $O(qx)$ and $O(ry)$, then, by the definition of equivalence, $O(rx)$, and from $O(qx)$ and $O(rx)$ follows, by 28, the proposition $q = r$. Hence *all the obverses of equivalent elements are equivalent.*

30. If $u \neq v$, while $O(nu)$ and $O(mv)$, then $m \neq n$. For if $m = n$, then, by 29, $u = v$. Hence *the obverses of non-equivalent elements are themselves non-equivalent.*

31. It thus appears that all the obverses of the same element, or of equivalent elements, are mutually equivalent, and that non-equivalent elements cannot possess mutually equivalent obverses, still less the same obverse. Accordingly, since equivalence here means capacity for mutual substitution in $O$-collections, we may henceforth let a *single one* of the obverses of a given element represent the whole class of these obverses, for all the purposes involved in the present discussion of $O$-collections. This uniquely selected representative of all the obverses of any element $x$, we shall henceforth regard, therefore, as *the* obverse of $x$, and as equivalent to the obverse of any element equivalent to $x$. We shall symbolize this single representative of all these obverses by $\bar{x}$, or, in general, by writing a bar above the symbol of the element of which at any time we define the obverse. $\bar{x}$ cannot be equivalent to the obverse of any element which is not equivalent to $x$. As the unique representative of the obverses of $\bar{x}$ we may hereupon take an element symbolized by $\bar{\bar{x}}$.

32. By definition, and by 31, $x = \bar{\bar{x}}$; and $\bar{\bar{x}}$ will henceforth be so chosen as to be identically the *same* element as $x$. The operation of obversion (that is, of finding, for any element $x$, the unique representative, $\bar{x}$, of the class of elements any one of which forms, with $x$, an $O$-pair), hereupon becomes an entirely univocal operation. This operation, if once repeated, is so defined as to be an operation which restores to us the original element.

33. When one collection, $\delta$, is an obverse of another collection, $\epsilon$ (see 14, 15, 16), each of these collections, by the substitution of the equivalent elements (in case such substitution is required for the purpose), may be made into a collection consisting wholly of the unique representatives of the obverses of the various members of the other collection. An obverse of the collection $\delta$, thus reduced to the form of a collection of the unique representative obverses of the elements of $\delta$, shall henceforth be symbolized, in our discussion, by $\bar{\delta}$. By the symbol $\bar{\delta}$ we mean, therefore, a certain chosen unique representative of all those collections any one of which is an obverse of the collection $\delta$. The collection $\bar{\delta}$ may then be so chosen as to be identical with $\delta$.

34. If $a = b$, then $O(a\bar{b})$ and $O(b\bar{a})$. For $O(a\bar{a})$ by definition (13, 31). Hence, since $a = b$, and since, by the definition of equivalence, we can accordingly substitute $b$ for $a$ in $O(a\bar{a})$, we have $O(b\bar{a})$. And since $O(b\bar{b})$, we have, by the substitution of $a$ for $b$, $O(a\bar{b})$. On the other hand, if either $O(a\bar{b})$, or $O(b\bar{a})$, is known to be true, then, by the definition of obverses, $a$ is

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an obverse of $\bar{b}$, or $b$ is an obverse of $\bar{a}$, as the case may be. But in either case, since obverses of the same element or equivalent elements are equivalent, $\bar{a} = \bar{b}$, and $a = b$.

35. A fundamental characteristic of the system $\Sigma$ is, further, the fact that: *No monad is an O-collection.* For if $O(x)$, then, by adjunction (principle I), $O(xx)$. By the same principle, if $O(x)$, $O(xq)$, whatever element $q$ may be. Hence any monad $q$ is such that whether $x$, or an obverse of $x$ (namely $x$ itself), be adjoined to $q$, always $O(xq)$. Hence, by principle II, $O(q)$. Therefore if a *single* element $x$ exists such that $O(x)$, *every* element of $\Sigma$, as for instance $q$, is such that $O(q)$. Hence, by principle I, whatever collection of elements be adjoined to any element $q$ of $\Sigma$, the resulting collection is an $O$-collection. Hence (by 11), since all possible collections are thus $O$-collections, *all* the elements of $\Sigma$ are mutually equivalent. But this contradicts principle IV. Hence it is impossible that any monad $x$ should exist such that $O(x)$. Every monad, therefore, is an $E$-collection.

36. It will be convenient, at this point, to restate the theorems of 21–24 in the notation for obverse elements which has now been adopted:

(1) If $O(\alpha x)$ and $O(\alpha\bar{x})$, then $O(\alpha)$ (see 21).

(2) If $O(\pi)$, then $O(\bar{\pi})$ (see 22).

(3) If $O(\beta x)$ and $O(\vartheta x)$, then $O(\beta\bar{\vartheta})$ (see 23).

(4) If $O(\delta\epsilon)$, while $\lambda$ exists such that $O(\lambda e_n)$ for every element $e_n$ of $\epsilon$, then $O(\bar{\delta}\lambda)$ (see 24).

(5) If $O(d\epsilon)$, while $\lambda$ exists such that $O(\lambda e_n)$ for every element $e_n$ of $\epsilon$, then $O(\bar{d}\lambda)$ (see 24).

37. Any repetitions of an element which occur in an $O$-collection may be stricken out, so that the element in question occurs but a single time; and the resulting collection will still be an $O$-collection. For suppose $\alpha$ to be a collection consisting wholly of the element $a$, repeated any multitude of times. And suppose $\beta$ to be such that $O(\alpha\beta)$. From $O(\alpha\beta)$ follows by 36 (2), $O(\bar{\alpha}\bar{\beta})$. If any element, either of $\bar{\alpha}$, or of $\bar{\beta}$, be adjoined to the collection $(a, \beta)$ (which consists of $a$, occurring only once, with $\beta$ adjoined), it is plain that the resulting collection will be enlarged so as to constitute an $O$-collection. For $O(a\bar{a}\beta)$; and, if $b_n$ be any element of $\beta$, and $\bar{b}_n$ the obverse of this element $O(a\beta\bar{b}_n)$. The collection $(\bar{\alpha}\bar{\beta})$, consisting entirely of complements of $(a, \beta)$, is thus an $O$-collection. Hence $O(a\beta)$, by principle II.

38. It further follows that, if $\beta$ itself is also a collection consisting solely of $a$ repeated, and if $O(\alpha\beta)$ then the collection $O(\alpha\beta)$ which now consists solely of $a$ repeated, can be reduced to $O(aa)$, and hence to $O(a)$. But $O(a)$ is impossible by 35. Hence no collection consisting solely of repetitions of a single element can be an $O$-collection. Every such collection must be an $E$-collection.

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It still further follows that, *no element is equivalent to its own obverse.* For since $O(x\bar{x})$, it follows, by the definition of equivalence, that, if $x = \bar{x}$, $O(xx)$, Hence by 37, $O(x)$. But this, by 35, is impossible.

39. If $a$ and $b$ are such that the mutually obverse pairs $(\bar{a}, b)$ and $(a, \bar{b})$ are such that $O(\bar{a}bx)$ and $O(a\bar{b}x)$, then by 21 and 22, $O(\bar{a}b)$. Hence, by 34, $a = b$.

40. If any collection $\delta$ is such that $O(\delta)$ is false, so that $E(\delta)$ is true, it follows, from principle I, that if $\epsilon$ is any collection such that $\epsilon$ consists wholly of elements which belong to the collection $\delta$, while $\epsilon$ does not include all of these elements, then $O(\epsilon)$ is false. For if $O(\epsilon)$, then, since $\delta$ may be formed from $\epsilon$ by adjoining to $\epsilon$ certain elements, principle I would require that $O(\delta)$ should be true. Whatever elements, therefore, we omit from an $E$-collection, $\delta$, the remaining elements form an $E$-collection. Or, in other words, if a collection is an $E$-collection, all possible partial collections that can be formed by selecting some of its elements, and omitting others, are also $E$-collections, so that if $\epsilon$ is any such partial collection, $O(\epsilon)$ is false.

41. If $E(a\bar{b})$, then $a \neq b$. For if $a = b$, then, by 34, $O(a\bar{b})$, which contradicts the hypothesis $E(a\bar{b})$. And, on the other hand, if $a$ and $b$ are such that $a \neq b$, then $E(a\bar{b})$, for if not, then $O(a\bar{b})$, and, therefore, by 34, $a = b$, which contradicts the hypothesis $a \neq b$. Thus then, if two elements, $x$ and $y$, form an $E$ pair, the obverse of either of these elements is not equivalent to the other element; i. e., $x \neq \bar{y}$; and $\bar{x} \neq y$. Plainly, furthermore, if $E(x\bar{y})$, then $E(\bar{x}y)$, and conversely, again, if $x$ exists such that either $E(a\bar{b}x)$ or $E(\bar{a}bx)$ is true, then, by 40, either $E(a\bar{b})$, or else $E(\bar{a}b)$ is true. But as we have just seen each of the assertions: $E(a\bar{b})$ and $E(\bar{a}b)$, implies the other, and also implies $a \neq b$. In the same way, if $E(xyz)$, then $E(xy)$, $E(yz)$ and $E(xz)$. Hence $x \neq \bar{y}$, $y \neq \bar{z}$, etc. In general, if $E(\alpha)$, and if $x$ and $y$ are any two of the elements of $\alpha$, then, by 40, $E(xy)$, and hence $x \neq \bar{y}$. This is also immediately evident from principle I, and from the definition of obverses. For if $x = \bar{y}$, then $O(xy)$, and then any collection into which both $x$ and $y$ enter is, by principle I, an $O$-collection.

42. If $E(\alpha)$, and if $x$ be any element whatever, then either $E(\alpha x)$ or $E(\alpha\bar{x})$ must be true. For if neither of these assertions is true, then $O(\alpha x)$ and $O(\alpha\bar{x})$; and then, by principle II, $O(\alpha)$.

43. By 35 every monad is an $E$-collection. Let $x$ and $y$ be any two elements of the system $\Sigma$. Each of these elements possesses an obverse. Since $E(x)$, by 42 either $E(xy)$ or $E(x\bar{y})$ is true; and since $E(y)$, either $E(xy)$ or $E(\bar{x}y)$ is true.

## Chapter III. The $F$-Collections.

44. If two collections, $\beta$ and $\eta$, are such that $O(\beta\bar{\eta})$, then the collections $\beta$ and $\eta$ stand to each other in a relation which we shall also, at pleasure, express

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by the symbol $F(\beta|\eta)$, wherein the symbols $\beta$ and $\eta$ appear with a short vertical line between them. This symbol then, in the first place, expresses precisely the *same* facts that are expressed by the symbol $O(\beta\bar{\eta})$. That is the symbol $F(\beta|\eta)$ may at pleasure be read as the assertion: "The collection consisting of $\beta$ taken together with the collection $\eta$ (the collection which is the obverse of $\bar{\eta}$), constitutes in its totality, an $O$-collection." Since, by 22 and 33, $O(\beta\bar{\eta})$ implies $O(\bar{\beta}\eta)$, and conversely, the symbol $F(\beta|\eta)$ could equally well be read: — "The collection $(\bar{\beta}, \eta)$ is an $O$-collection." But the symbol $F(\beta|\eta)$ is especially intended to emphasize the fact that, when $O(\bar{\beta}\eta)$, and consequently when $O(\beta\bar{\eta})$, the collections $\beta$ and $\eta$ stand to each other in a relation which is *mediated by the existence of their respective obverse collections.* $\beta$ and $\eta$ are then collections such that each, if adjoined to the obverse of the other collection, unites with that obverse, to constitute a total collection that is an $O$-collection. Expressing this fact with a primary reference to $\beta$ and $\eta$, instead of to $\beta$ and $\bar{\eta}$, or to $\bar{\beta}$ and $\eta$, the symbol $F(\beta|\eta)$ may now be read as the assertion: "The collection $\beta$ forms, *with* the collection $\eta$, a *determinate F-collection.*" The special significance of the adjective *determinate* will appear below. The vertical line is intended as a sort of punctuation mark, to indicate the distinction between the two collections in question.

45. If $O(\gamma)$, and if hereupon $\gamma$ be in *any* way exhaustively divided into two "partitions," that is, into two mutually exclusive collections of elements $\delta$ and $\epsilon$, such that $O(\delta, \epsilon)$ is the same collection as $\gamma$, it is plain from the foregoing that $F(\delta|\bar{\epsilon})$. So the same $O$-collection makes possible various different assertions in terms of determinate $F$-collections. If $\gamma$ is a collection of unrestricted multitude, the multitude of the possible assertions in terms of $F$-collections becomes also unrestricted.

46. The rule for transforming our assertions so that what are explicitly defined as $O$-collections shall appear in the form of explicitly designated and determinate $F$-collections, is consequently as follows: If the assertion $O(\gamma)$ is given, and if we are to express this as an assertion regarding some determinate $F$-collection, then we choose at random any partial collection $\delta$ of the elements of $\gamma$. Let $\bar{\epsilon}$ be the collection which is the obverse of the collection $\epsilon$, where $\epsilon$ consists of all the remaining members of $\gamma$, not included in $\delta$. Write $F(\delta|\bar{\epsilon})$, or, at pleasure, $F(\bar{\epsilon}|\delta)$. That is, put $\delta$ on one side and $\bar{\epsilon}$ on the other side of the vertical. The way in which the partial collections included in the parenthesis are placed, in so far as these two collections are merely considered with respect to their succeeding or preceding the vertical, is then capable of transposition at pleasure. The resulting expression is to be read, as above defined, and as an assertion in terms of a determinate $F$-collection. Instead of $F(\bar{\epsilon}|\delta)$, we can equally well write $F(\epsilon|\bar{\delta})$, or $F(\bar{\delta}|\epsilon)$.

47. The rule for the inverse operations transforming an assertion regard-

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ing a determinate $F$-collection into an assertion regarding an $O$-collection is now obvious. If the assertion $F(\alpha|\beta)$ is given, then we first take the collection $\bar{\alpha}$, or, at pleasure $\bar{\beta}$; that is, we take the obverse collection corresponding that collection which stands on one chosen side of the vertical; and then we combine this obverse collection with that collection which stands on the other side of the vertical. Hereupon we write $O(\bar{\alpha}\beta)$ or $O(\beta\bar{\alpha})$, or $O(\bar{\beta}\alpha)$ or $O(\alpha\bar{\beta})$, at pleasure.

48. In case expressions such as $F(\alpha\beta x|\gamma\delta\bar{y}z)$, or similarly complex symbols appear, we read in this way: "The collections $\alpha$ and $\beta$, together with the element $x$, constitute a collection which, taken as a whole, forms an $F$-collection with a collection consisting of the partial collections $\gamma$ and $\delta$, and of the elements $\bar{y}$ and $z$." The expression just set down asserts the same as is asserted by the symbol $O(\alpha\beta x\bar{\gamma}\delta y\bar{z})$, or as is asserted by the symbol $O(\bar{\alpha}\bar{\beta}\bar{x}\gamma\delta\bar{y}z)$.

49. If a collection, $\lambda$, consists of some finite number, $n$, of elements, it is one of a set or group of $2^n$ collections which can be formed from the given collection by first leaving that collection unchanged, and by then transforming, in every possible way, one, two, three, $\cdots$ and finally all of the $n$ elements of $\lambda$ into their respective obverses. If one of these $2^n$ collections is an $O$-collection, e. g., if the original collection $\lambda$ is an $O$-collection; then the collection $\bar{\lambda}$, which is one of the set of $2^n$ collections, is also an $O$-collection. All of the other collections of the set are hereby required to be determinate $F$-collections. But in symbolizing these determinate $F$-collections, the arrangement of the elements is no longer wholly indifferent. One must, in each case, set upon one side of the vertical all of those elements which, in any one of the transformed collections, are obverses of elements of $\lambda$; upon the other side one must place all those elements which are identical with elements of $\lambda$. One is then to set the rearranged collection within a parenthesis, and is to write $F$ before this parenthesis. In each case one thus asserts the same fact as is asserted by $O(\lambda)$; but does so, each time, with a different stress upon the partial collections whose relations to one another are thus pointed out. By means of the determinate $F$-collections, one thus analyzes, in a particular way, the various aspects of the meaning of the assertion $O(\lambda)$. Yet each one of the determinate $F$-collections points back, infallibly, to the same pair of $O$-collections; and also predetermines the constitution of all the other determinate $F$-collections of the same set; so that, in thus emphasizing various aspects of the meaning of the assertion $O(\lambda)$, one still never loses the power to return from one aspect, thus emphasized, to any or to all the other aspects of the same assertion. The determinate $F$-collections thus defined may be grouped in $(2^n - 2)/2$ pairs; since, in general, if $F(\alpha|\beta)$, $F(\bar{\alpha}|\bar{\beta})$ is true. When we enumerate the set of determinate $F$-collections, it is sufficient to name one of each pair.

50. *Indeterminate F-collections.* It is, however, occasionally convenient to

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express simply the assertion that there exists *some* $O$-collection $O(\bar{\kappa}\lambda)$, such that the collection $(\kappa, \lambda)$ is precisely the same collection as a given collection $\eta$, that is, such that $(\kappa, \lambda)$ stands for a partition of $\eta$; while we nevertheless leave it entirely undetermined *what one* of the possible partitions of the collection $\eta$ it is with regard to which this assertion holds true. In this case we may write simply $F(\eta)$, a symbol which we read as the assertion: "The collection $\eta$ is an (indeterminate) $F$-collection." An indeterminate $F$-collection may prove, when its determination is specified, to be any one of the determinate $F$-collections which correspond to the possible partitions of $\eta$. Thus, if $\eta$ is the same collection as $(\kappa, \lambda)$ and if $F(\kappa|\lambda)$ is true, $F(\eta)$ is true; while if we merely know that $F(\eta)$ is true, we know that some one of the assertions $F(\kappa|\lambda)$, $F(\kappa'|\lambda')$, etc., is true—where $(\kappa, \lambda)(\kappa', \lambda')$, etc., are various possible partitions of the single collection $\eta$. In the same way, if $F(\eta)$, some one of the assertions $O(\bar{\kappa}\lambda)$, $O(\bar{\kappa}'\lambda')$, etc., is true.

If, setting out from the assertion $O(\gamma)$, we consider some possible partition of the collection $\gamma$, say $(\delta, \epsilon)$, and then, instead of writing, as above (45), $F(\delta|\bar{\epsilon})$ we write simply $F(\delta\bar{\epsilon})$, we surrender some of the information conveyed in the original assertion $O(\gamma)$, as well as in the assertion $F(\delta|\bar{\epsilon})$. For it now no longer appears what determinate $F$-collection corresponds to the indeterminate $F$-collection $F(\delta\bar{\epsilon})$; and the latter assertion tells us only that some one of the possible partitions of the collection $(\delta, \bar{\epsilon})$ is such that, if it is made (e. g., the partition $(\delta_1, \delta_2, \bar{\epsilon}_1, \bar{\epsilon}_2)$, wherein $(\delta_1, \delta_2)$ is the same collection as $\delta$, and $(\bar{\epsilon}_1, \bar{\epsilon}_2)$ the same collection as $\bar{\epsilon}$)—then $F(\delta_1\epsilon_1|\delta_2\bar{\epsilon}_2)$, so that $O(\delta_1\bar{\epsilon}_1\bar{\delta}_2\epsilon_2)$. It is plain that the indeterminate $F$-collections occur in pairs. If $F(\alpha)$, then $F(\bar{\alpha})$.

51. An example will serve to distinguish more clearly the kinds of information conveyed by the assertion that a collection is a determinate and by the assertion that this collection is an indeterminate $F$-collection. Let $F(ab|cd)$ be true. This is equivalent to asserting $O(ab\bar{c}\bar{d})$. Any one of the possible collections which can be formed by transforming $(a, b, c, d)$ through the substitution of the obverse of one or of more of its elements, is then also a determinate $F$-collection. Thus the assertion $F(ab|cd)$ requires:

For the collection $(a, b, c, \bar{d})$ the assertion $F(ab\bar{d}|c)$;
For the collection $(a, b, \bar{c}, d)$ the assertion $F(ab\bar{c}|d)$;
For the collection $(a, \bar{b}, c, \bar{d})$ the assertion $F(a\bar{d}|c\bar{b})$;
For the collection $(a, \bar{b}, \bar{c}, \bar{d})$ the assertion $F(a\bar{c}\bar{d}|\bar{b})$;

and so on; while each of these assertions implies, and is implied by $O(ab\bar{c}\bar{d})$.

The assertion $F(abcd)$ does not necessarily imply any one of the foregoing determinate assertions. It indicates that, of the $2^n - 1$ or 15 possible collections other than $(a, b, c, d)$ in the set of collections producible from $(a, b, c, d)$ through the substitution of obverses, some one, and consequently (since $O(\alpha)$ implies $O(\bar{\alpha})$), some *pair* of collections, must be $O$-collections. It would

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depend upon this pair to determine how the vertical lines which define the determinate $F$-collections ought to be distributed in each of the various cases which hereupon arise. Some one of these possible distributions express the truth in each case; but the assertion $F(abcd)$ does not tell us which one this is. Thus, if $F(abcd)$, and if we then consider the collection $(a, \bar{b}, c, \bar{d})$, it is possible that:

$F(a|\bar{b}c\bar{d})$, i. e., $O(ab\bar{c}d)$, and $O(\bar{a}\bar{b}cd)$; or $F(a\bar{b}c|\bar{d})$, i. e., $O(\bar{a}bcd)$, and $O(\bar{a}b\bar{c}\bar{d})$; or $F(a\bar{d}|\bar{b}c)$, i. e., $O(\bar{a}\bar{b}cd)$ and $O(ab\bar{c}\bar{d})$, and so on for all the other cases.

The assertion $F(abcd)$ requires some pair of these alternative $O$-assertions or some corresponding pair of the $F$-assertions, to be true, but does not specify what pair in any of the cases in question.

The indeterminate $F$-collections, like the $O$-collections, are perfectly symmetrical. In case of a pair $(a, b)$, the alternative pairs of assertions:—

$$\left\{ (1) \begin{array}{c} O(ab) \\ O(\bar{a}\bar{b}) \end{array} \middle| (2) \begin{array}{c} O(a\bar{b}) \\ O(\bar{a}b) \end{array} \right\}$$

are such that if the pair (1) of assertions are both true, the assertions (2) are both of them false. Hence $F(ab)$ can mean only that $O(\bar{a}b)$ and $O(a\bar{b})$ are both of them true; while $F(a\bar{b})$ means that both $O(ab)$ and $O(\bar{a}\bar{b})$ are true. Hence, in case of pairs of elements, the distinction between determinate and indeterminate $F$-collections vanishes: and the assertion $F(ab)$ is perfectly determinate.

*Elementary properties of the F-collections: operations and transformations.*

52. A number of elementary properties of $F$-collections, and a survey of certain ways in which they may be transformed, may now be readily obtained from the already established properties of the $O$-collections.

(1) If $F(xy)$ then, since $O(x\bar{y})$, $x = y$. See 34.

(2) If $F(ax|b)$ and $F(bx|a)$, then $a = b$. For $O(a\bar{b}x)$ and $O(\bar{a}bx)$. See 39.

(3) If $F(\eta)$, then $F(\eta\gamma)$ where $\gamma$ is any collection whatever (Principle I).

(4) If $F(\eta)$, then $F(\bar{\eta})$, as was already observed in connection with the definition of the $F$-collections. If $F(\beta|\eta)$, then $F(\bar{\beta}|\bar{\eta})$.

(5) If $F(\eta|x)$ and $F(\eta|\bar{x})$, then $O(\eta)$, since $O(\eta\bar{x})$ and $O(\eta x)$. The question may then arise whether $F(\eta)$ is in a given case, also true. It is here first obvious that $F(\eta)$ does not follow from $F(\eta|x)$ and $F(\eta|\bar{x})$. For instance, if $F(ab|x)$ and $F(ab|\bar{x})$, then $O(ab)$. But, if $O(ab)$, $a = \bar{b}$; while if $F(ab)$, $a = b$ (by (1) of the present paragraph). By 38, however, if $O(ab)$, it is impossible that $a = b$. Hence, if $F(ab|x)$ and $F(ab|\bar{x})$,

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$F(ab)$ is false. On the other hand, if $O(\alpha)$, then by adjunction, $O(\alpha x\bar{y})$ and $O(\alpha xy)$. Hence, in this case, $F(\alpha x|y)$ and $F(\alpha x|\bar{y})$. But in this case also since $O(\alpha)$, $O(\alpha\bar{x})$. And so $F(\alpha|x)$, i. e. $F(\alpha x)$, is likewise true. If $F(\eta|x)$ and $F(\eta|\bar{x})$, $F(\eta)$ is accordingly possible; but does not follow herefrom.

53. Any determinate $F$-collection remains such when transformed according to the following rule: *Substitute* for any element, or for any collection, which stands upon one side of the vertical, *the obverse* of that element or collection, *transfer* the obverse in question to the other side of the vertical, being careful to retain, as the result of the transfer, at least one element on each side of the vertical. Thus, if $F(x\beta|y\delta)$, then $F(\beta|\bar{x}y\delta)$, $F(\bar{y}\beta x|\delta)$, $F(x|\bar{\beta}y\delta)$, $F(\beta\bar{y}\bar{\delta}|\bar{x})$, etc., are all of them true. This is obvious, because all these expressions mean the same as $O(\bar{x}\bar{\beta}y\delta)$ or as $O(x\beta\bar{y}\bar{\delta})$.

This is called transformation by transfer. If the elements are transferred by this rule except that *all* the elements are permitted at the end to stand upon *one* side of the vertical, the vertical can then be omitted; but the resulting collection must be regarded as thus transformed into an $O$-collection.

54. If $F(x\beta|\bar{x}\gamma)$ then $F(\beta|\bar{x}\gamma)$ and $F(x\beta|\gamma)$. If $F(x\beta|\bar{\beta}\gamma)$, then $F(x\beta|\gamma)$ and $F(x|\bar{\beta}\gamma)$. That is, if the obverse of an element or of a collection, which stands on *one* side of the vertical, itself stands on the *other* side of the vertical, then either of the two mutually obverse collections or elements may be stricken out (by 37). For if $F(x\beta|\bar{x}\gamma)$, then $O(x\beta x\bar{\gamma})$, and so $O(\beta x\bar{\gamma})$. Hence $F(\beta x|\gamma)$ and $F(\beta|\bar{x}\gamma)$. This is called a transformation by means of the omission of superfluous obverses; and the procedure obviously applies to collections as well as to elements.

55. If $F(\alpha|\beta)$, and if all the elements of $\alpha$ are mutually equivalent, and all the elements of $\beta$ are mutually equivalent, then all the elements of the collection are mutually equivalent.

For $O(\alpha\bar{\beta})$. Let $a$ be one of the elements of $\alpha$ and $b$ of $\beta$. Then, by 36, $O(\alpha\bar{\beta})$ reduces to $O(a\bar{b})$, whence follows, by 34, $a = b$.

It is now obvious that any repetitions of an element which occur upon one side of the vertical in a determinate $F$-collection may be stricken out; and also that, by virtue of principle I, any element may be added to that collection which stands upon either side of the vertical, so that, if $F(\alpha|\beta)$, $F(\alpha y|\beta)$ and $F(\alpha|\beta y)$, where $y$ is any element.

*Elimination-theorems for F-collections.*

56. If two collections $\beta$ and $\delta$ are such that there exists a collection $\pi$ such that $F(\delta|\pi)$, while, for every member $y_r$ of the collection $\pi$, $F(y_r|\beta)$, then $F(\beta|\delta)$.

For, if $F(\delta|\pi)$, then $O(\delta\bar{\pi})$. And if $F(y_r|\beta)$, then, for every member

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$\bar{y}_r$ of the collection $\bar{\pi}$ it is true that $O(\bar{y}_r\beta)$. Substituting for the symbol $\bar{\pi}$ the symbol $\epsilon$, and for $\bar{y}_r$ (the representative symbol for any member of $\bar{\pi}$), the symbol $x_r$ (as the representative symbol for any member of the collection now called $\epsilon$), we have $\delta$ and $\beta$ such that there exists a collection $\epsilon$ such that $O(\delta\epsilon)$, while, for every member $x_r$ of $\epsilon$, $O(x_r\beta)$. Hence, by 36, and by virtue of the properties of $O$-collections pointed out in 24, $O(\delta\bar{\beta})$. Hence $F(\delta|\beta)$ or $F(\beta|\delta)$.

57. In case the collection $\delta$ reduces to the single element $d$, the theorem assumes the following form:

If any collection $\beta$ forms a determinate $F$-collection with every member of $\pi$, separately considered, while the collection $\pi$ taken as an entirety, is such as to form a determinate $F$-collection with an element $d$, then $\beta$ forms a determinate $F$-collection with $d$.

This theorem permits the elimination of $\pi$, in case $F(d|\pi)$, and in case the set of determinate $F$-collections $F(y_r|\beta)$ is given, where $y_r$ is a variable for which every element of $\pi$ may separately be substituted.

58. If $\pi$ reduces to a single member $y$, we have the result of 56 reduced to the form:

If two collections $\beta$ and $\delta$ are such that there exists an element $y$ such that $F(y|\beta)$ and $F(y|\delta)$, then $F(\beta|\delta)$.

This last result furnishes a means for the direct elimination of an element $y$ common to two determinate $F$-collections, in case $y$ stands alone, on one side of the vertical, in each collection. Here too we deal with a type of *transitivity* whose consequences are of great importance.

59. If $F(x\beta|y\gamma)$, and $F(y\beta|z\gamma)$, then $F(x\beta|z\gamma)$. For, by transfer (53), from $F(x\beta|y\gamma)$ follows $F(x\beta\bar{\gamma}|y)$. And from $F(y\beta|z\gamma)$ follows $F(\bar{\beta}z\gamma|y)$. Hence, by 58, we can eliminate $y$, and thus we obtain $F(x\beta\bar{\gamma}|\bar{\beta}z\gamma)$. By 54, we may hereupon transform this $F$-collection by striking out $\bar{\beta}$ from the right side of the vertical (since $\beta$ occurs on the left side), and $\bar{\gamma}$ from the left side (since $\gamma$ occurs on the right side). We thus obtain $F(x\beta|z\gamma)$, which was to be proved.

The transformations and the type of elimination here used are typical of the methods which are to be employed in considering and in combining $F$-collections. These methods correspond to the adjunctions and eliminations already used in case of $O$-collections.

60. The result of 59 is the principal theorem relating to the transitivity of the relations involved in $F$-collections. Its importance justifies a proof directly in terms of the properties of $O$-collections.

If, namely, $F(x\beta|y\gamma)$, then $O(x\beta\bar{y}\bar{\gamma})$. From this follows, by adjunction, $O(x\beta\bar{\gamma}\bar{y}\bar{z})$. If $F(y\beta|z\gamma)$, then $O(y\beta\bar{z}\bar{\gamma})$. From this follows, by adjunction, $O(x\beta\bar{\gamma}y\bar{z})$. From the two $O$-pentads, thus formed through adjunction, follows, by principle II, $O(x\beta\bar{\gamma}\bar{z})$. Hence $F(x\beta|\gamma z)$.

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61. The hypothesis $F(x\beta|y\gamma)$ sets $x$ in a determinate dyadic relation $R$ to $y$; and this relation is unsymmetrical, since $y$ does not stand in this relation to $x$. The hypothesis $F(y\beta|z\gamma)$ sets $y$ in this same relation $R$ to $z$. The conclusion sets $x$ in the relation $R$ to $z$. The relation in question is therefore transitive. But the unsymmetrical transitive dyadic relation in question has been entirely derived from the wholly symmetrical relations defined by the $O$-collections.

62. If $\beta$, $\gamma$, and $\delta$ are given collections, and if $y$ and $e$ are given elements, such that $F(y\beta|\delta)$ and $F(e\delta|\beta)$, while $F(e|y\gamma)$, then $F(\gamma\delta|\beta)$.

For by transfer (53), from $F(y\beta|\delta)$ and $F(e|y\gamma)$ follow the two assertions $F(y|\delta\bar{\beta})$ and $F(y|e\bar{\gamma})$. By 58, it follows that $F(\delta\bar{\beta}|e\bar{\gamma})$; whence follows again by transfer $F(\bar{\beta}\gamma\delta|e)$. From $F(e\delta|\beta)$ follows, by transfer, $F(e|\beta\bar{\delta})$. From this and $F(\delta\bar{\beta}\gamma|e)$ follows, by 58, $F(\beta\bar{\delta}|\gamma\delta\bar{\beta})$. By omission of superfluous obverses (54), we obtain $F(\beta\bar{\delta}|\gamma)$ and $F(\beta|\gamma\delta)$.

63. If $F(x|\beta)$ and $F(y|\beta)$, while $F(xy|\delta)$ then $F(\delta|\beta)$.

This follows directly from 56, in case the collection $\pi$ of that theorem reduces to the pair $(x, y)$.

64. If $\alpha$, $\delta$, $\epsilon$ are collections, and $b$ and $c$ are elements, such that (1) $F(b\alpha|\delta)$ and (2) $F(c\alpha|\delta)$, while (3) $F(bc|\epsilon)$, then $F(\delta|\alpha\epsilon)$.

For by (1) $F(b\alpha|\delta)$ while by (2), $F(\alpha|\delta\bar{c})$. Let $b = y$, and $c = \bar{e}$. Then $F(y\alpha|\delta)$, while $F(\alpha|\delta e)$; and, by (3), $F(y|\epsilon e)$, i. e., $F(e|y\bar{\epsilon})$.

Hence, by 62, $F(\bar{\epsilon}\delta|\alpha)$. Whence follows $F(\alpha\epsilon|\delta)$. That is, if two collections $\alpha$ and $\delta$ are such that if either $b$ or $c$ be separately adjoined to $\alpha$, the resulting collection forms an $F$-collection with $\delta$, and if the pair $(b, c)$ forms an $F$-collection with $\epsilon$, then if $\epsilon$ itself be adjoined to $\alpha$, the resulting collection forms an $F$-collection with $\delta$.

## Chapter IV. $F$-triads, mediators and antecedents.

65. What elements exist in the system $\Sigma$ we have as yet but very imperfectly investigated. Yet before we proceed to this investigation, it will prove convenient to outline the general character of the order which is possible in the system $\Sigma$, so far as we have yet developed this order. The fact that given elements do or do not belong to a certain $O$-collection, or do or do not constitute an $O$-collection, is one which appears directly to establish no sort of order amongst the elements of $\Sigma$. The relation in which various elements stand to one another when they belong to the same $O$-collection, is so far absolutely *symmetrical*, and nothing can be said of one member of such a collection which is not asserted of everyone of the others, so far as this collection is concerned.

But the fact that every element of $\Sigma$ possesses an obverse, enables one to establish relations between certain elements, or sets of elements, relations which are due to the further fact that given elements may enter into $O$-collections with the

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*obverses* of certain other elements. The consequence of this is that $F$-collections are definable. Let us provisionally assume that a large variety of $F$-collections exist in $\Sigma$.

66. $F$-collections, if indeterminate, are, like $O$-collections, of a wholly symmetrical structure, and their members are so far undistinguished from one another. But the members of a determinate $F$-collection are no longer so symmetrically disposed. It is indeed true that the order in which the elements in the set of an $F$-collection on each side of the vertical are considered, is indifferent. Nor does it make any difference which set is written before or after the vertical. But if $F(\beta|\gamma)$, then, in general, any element $x$ of $\beta$ is related to any element $y$ of $\gamma$ in a way which is *not* reciprocated. For $x$ is related to $y$ as that element which, in combination with some collection $\beta'$ of companion elements, forms an $F$-collection with $y$, when $y$ is combined with a collection $\gamma'$ of elements ($\beta'$ being the collection of the other elements of $\beta$ besides $x$, $\gamma'$ being the collection of the other elements of $\gamma$ besides $y$). This relation, if read in the other direction, changes, in general, its character, and so is an unsymmetrical relation.

But, as 59 has shown us, this unsymmetrical dyadic relationship is *transitive*. In terms of this relation certain sets of the elements may be *ordered* and so arranged in series like points on a line.

67. The fundamental form of such series becomes manifest if we pay attention to those cases of the much more general theorems regarding $F$-collections, which appear as special results if we consider only *triads* of elements.

In this case if, for example, $F(ac|b)$ or $F(ca|b)$, that is, if $O(ac\bar{b})$, we shall call $b$ "the *mediator* between $a$ and $c$," or, where that is more convenient, the "mediator of the pair $(a, c)$." The "mediator" of a pair is accordingly an element whose obverse forms an $O$-triad when adjoined to that pair. If $O(pqr)$, then $F(pq|\bar{r})$, so that the obverse of any member of an $O$-triad is the mediator of the pair formed by the other elements of that triad.

The relation of the mediator to the elements which it mediates may be treated at pleasure either as a triadic or as a dyadic relation. In order to treat it as a dyadic relation we may take account of the fact that, if $F(ac|b)$, $b$ is in a certain relation to $a$ *with respect to* $c$, and is in a certain relation to $c$ *with respect to* $a$. This aspect of the matter may become especially important in case we deal with a number of triads in which the mediators are any elements whatever, while all the pairs mediated have a common element $y$. Such pairs appear if $F(ym|q)$, $F(yn|r)$, $F(yo|s)$, etc. In all such cases $q$ has the same relation to $m$ that $r$ has to $n$, and that $s$ has to $o$, etc., since $q$ is in a given relation to $m$ *with respect to* $y$; and the same holds true in the other instances in question.

68. If $F(yq|p)$, we may, whenever that is convenient, first select one element of the pair $(q, y)$, say the element $y$, and thereupon say that $p$ is in a relation

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to $q$ which we shall call the relation: "*antecedent of $q$ with respect to $y$.*" The element $y$ we may hereupon call the "*origin*" from which the relation is defined or reckoned. Equally, if we choose the element $q$ as our origin, we can say that $p$ is in the relation to $y$ of being "an antecedent of $y$ with respect to $q$". We shall symbolize the assertion "$p$ is antecedent of $q$ with respect to $y$," by the expression $p \prec_y q$ or $q \succ_y p$; and we can also at pleasure read either of these expressions thus: "$q$ is a consequent of $p$ with respect to $y$." This expression means precisely the same as the expression $F(yq|p)$, or as the expression $O(yq\bar{p})$. Expressing the facts in the new way, as involving the relation of an antecedent to a consequent, has merely the advantage of bringing out certain aspects of the situation which will be conveniently expressible in terms of a dyadic relation — a relation which, as already pointed out, will prove to be unsymmetrical and transitive and, therefore, useful for the definition of serial order amongst certain specially selected elements of $\Sigma$. The equivalence of meaning of the three expressions: $O(qy\bar{p})$, $F(qy|p)$, and $p \prec_y q$, enables us at once to see how superficial is the difference between symmetrical and unsymmetrical relations. All that any one of these three expressions asserts is that the two perfectly symmetrical $O$-collections $O(qyx)$ and $O(xp)$ (where $x = \bar{p}$), both exist as collections of the elements of $\Sigma$.

69. It is of course expressly true that the relation of antecedent to consequent, or of consequent to antecedent, has meaning *only* with reference to a given origin. It is this origin which gives "*sense*" to the pair $(pq)$ in the expression $p \prec_y q$. On the other hand, if a question arises as to whether the "sense," or asymmetry, of the dyadic relation here in question, is a fundamental fact, or is unanalyzable — this question is answered in advance by our derivation of the whole asymmetry from the perfectly symmetrical properties which characterize the various members of any $O$-collection.

The system $\Sigma$, as we shall hereafter see, includes elements whose relations are precisely the ones which are of the most fundamental importance in all the exact sciences. The customary procedure of these sciences may be said, in the main, to involve the definition of these relations in terms of the relation of antecedent and consequent. Wherever a linear series is in question, wherever an origin of coördinates is employed, wherever "cause and effect," "ground and consequence," orientation in space or direction of tendency in time are in question, the dyadic asymmetrical relations involved are essentially the same as the relation here symbolized by $p \prec_y q$.

This expression, then, is due to certain of our best established practical instincts and to some of our best fixed intellectual habits. Yet it is not the only expression for the relations involved. It is in several respects inferior to the more direct expression in terms of $O$-relations. The range of its efficacy as an expression will become clearer hereafter. When, in fact, we attempt to

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describe the relations of the system $\Sigma$ merely in terms of the antecedent-consequent relation, we not only limit ourselves to an arbitrary choice of origin, but miss the power to survey at a glance relations of more than a dyadic, or triadic character.

*Properties of the relation of mediator, antecedent, and consequent.*

70. If $F(ab|c)$, and $a = b$, then $c = a = b$. For this is a special case of the principle proved in 55.* Furthermore, if $p \prec_y q$ and $q \prec_y p$, then $p = q$, since this is but another expression of 52 (2). Consequently, if $p \neq q$, and $p \prec_y q$, then $q \prec_y p$ is false. The relation of antecedent to consequent, if it obtains at all between a pair of non-equivalent elements, is therefore in that case inconvertible; hence the relation becomes totally asymmetrical so soon as it is confined to pairs of non-equivalent elements.

71. If $F(a|by)$ and $F(b|cy)$, then $F(a|cy)$ and $F(b|ac)$. | If $a \prec_y b$ and $b \prec_y c$, then $a \prec_y c$; and $b \prec_a c$.

The proof is as in 59. Thus from $F(b|a\bar{y})$, and $F(b|cy)$ follows $F(a\bar{y}|cy)$ and $F(a|cy)$. From $F(y|a\bar{b})$ and $F(y|b\bar{c})$ follows $F(a\bar{b}|b\bar{c})$; and consequently $F(b|ac)$. The form of the theorem stated on the right is simply a direct translation into the symbolism of the relation of antecedent and consequent. The latter relation is thus shown to be transitive.

72. If $F(x|ab)$ and $F(x|bc)$, while $F(b|ac)$, then $x = b$. For since $F(x|bc)$ while $F(b|ca)$, we have $F(c|x\bar{b})$ and $F(c|b\bar{a})$. Whence follows $F(x|b\bar{a})$; that is $F(b|ax)$. From $F(x|ab)$ and $F(b|ax)$ follows, by 52 (2) $x = b$.

In other words, if $x \neq b$, and $b$ is a mediator of the pair $(a, c)$, it is impossible that $x$ should be at once a mediator of the two pairs $(a, b)$ and $(b, c)$.

73. If $F(x|ab)$ and $F(y|ab)$ and $F(d|xy)$, then $F(d|ab)$. | If $x \prec_a b$ and $y \prec_a b$ and also $d \prec_x y$, then $d \prec_a b$.

This follows directly from 63 and expresses the fact that a mediator of two elements which, with respect to a given origin, are antecedents of the same element, is itself an antecedent of that element with respect to the same origin.

74. If $F(yb|d)$, $F(yc|e)$ and $F(ed|b)$, then $F(cd|b)$. | If $d \prec_y b$ and $e \prec_y c$, while $b \prec_d e$, then $b \prec_d c$.

This is a special case of the proposition proved in 62. If two pairs, $(b, y)$ and $(c, y)$, have a common element $y$, and if each pair forms a determinate $F$-triad when a term $d$ (in one case), or $e$ (in the other case) is set on the opposite side of the vertical, and if the member $b$ of the one pair is a mediator of $e$ and $d$, while $d$ is the third member of the $F$-triad in which $b$ occurs, then $b$ is a mediator of $d$ and of $c$, where $c$ is the remaining member of the other pair $(c, y)$.

* This theorem is used by KEMPE as a fundamental principle in defining the relation here called that of Mediator.

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Otherwise: If, with respect to a common origin $y$, the elements $d$ and $e$ are, respectively, antecedents of the elements $b$ and $c$, so that $d$ is antecedent of $b$ and $e$ of $c$, then, if $b$ is a mediator of $d$ and $e$, $b$ is also a mediator of $c$ and $d$.

75. If $F(ab|d)$, $F(ac|d)$ and $F(bc|e)$, then $F(ae|d)$. | If $d \prec_a b$ and $d \prec_a c$ and $e \prec_b c$, then $d \prec_a e$.

This follows from 64.

That is, whatever element is mediator of two pairs which have an element in common, is mediator of the pair composed of the common element and of any mediator of the pair, formed by the elements which, belonging to the original pairs are not common to them. Otherwise, whatever element is, with respect to a given origin, an antecedent of each of a pair of elements, is antecedent of any mediator of this pair.

76. Any element $a$ is a mediator between any element $y$ and the obverse of $y$; and any element $y$ is a mediator between itself and any other element $a$.

For $O(\bar{a}y\bar{y})$ and $O(ay\bar{y})$.

77. For any origin $y$, it is true, that $y$ is an antecedent of itself and of every other element, including $\bar{y}$; while any element $a$ is an antecedent of itself and of $\bar{y}$.

For, $O(\bar{y}ya)$, $O(\bar{y}yy)$ and $O(\bar{y}y\bar{y})$, while $O(\bar{a}ay)$ and $O(\bar{a}y\bar{y})$.

78. If $a \prec_y b$, then $b \prec_y \bar{a}$.

For $O(\bar{a}by)$, and hence $F(\bar{b}|\bar{a}y)$.

79. Whatever element is, with respect to a given origin $y$, an antecedent of every member of an $O$-collection except one, is also an antecedent of the obverse of this excepted element.

Let $\pi$ be any collection, and $e$ such an element that $O(\pi e)$. Then, by this hypothesis $F(\pi|\bar{e})$. Let $q$ be an element such that $F(q|x_r y)$ is true of every element $x_r$ of $\pi$ so that $q \prec_y x_r$. Then, by transfer, $F(q\bar{y}|x_r)$. Hence the collection $(q, \bar{y})$ forms a determinate $F$-collection, when set on one side of the vertical, with *any* element of the determinate $F$-collection $F(\pi|\bar{e})$ on the other side of the vertical, except the element $\bar{e}$. Hence, by 57, it is *also* true that $F(q\bar{y}|\bar{e})$. Hence $F(q|y\bar{e})$. Hence $q \prec_y \bar{e}$.

80. If, with respect to $y$, $q$ is an antecedent of *every* member of a given $O$-collection, then $q$ is equivalent to $y$. For $q$ is in any case an antecedent, with respect to the origin $y$, of every member of the collection except any member $e$. Hence $q \prec_y \bar{e}$. But, by the present hypothesis, $q \prec_y e$ is *also* true. Hence $O(\bar{q}ye)$ and $O(\bar{q}y\bar{e})$. Hence $O(\bar{q}y)$. Hence $q = y$.

81. Whatever element is, with respect to a given origin, a consequent of every member of an $O$-collection except one, is also a consequent of the obverse of this excepted element.

Suppose $\pi$ such that $F(\pi|\bar{e})$ as before. And suppose that, for every element $x_r$ of $\pi$, $x_r \prec_y q$, so that $F(x_r|yq)$. By the same reasoning as that of the last theorem $F(\bar{e}|qy)$; and hence $\bar{e} \prec_y q$.

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82. If $q$ is a consequent of every member of the collection, then it is also true that $e \prec_y q$; whence $O(qye)$ and $O(qy\bar{e})$. Hence $O(yq)$, and consequently $q = \bar{y}$.

## Chapter V. The existence of elements. chains. resultants. conjugate pairs of resultants. the algebra of logic.

83. In discussing, in the foregoing, the most characteristic relations of the elements of $\Sigma$, we have tacitly and provisionally assumed that elements may exist in sufficient variety to exemplify these relations. Our existential principles have so far been used mainly to establish the existence of pairs of obverse and of non-equivalent elements. We must now proceed to survey, more in detail, the actual structure of the system $\Sigma$.

84. By principles III and IV, the system $\Sigma$ contains a pair of non-equivalent elements, say $(xy)$. By 26, the pair $(\bar{x}, \bar{y})$ also exists. It is however in so far possible that $x = \bar{y}$, and hence that $y = \bar{x}$. In that case the members of the pair $(x, y)$ become respectively equivalent to the members of the pair $(\bar{y}, \bar{x})$, or, again, to the members of the pair $(x, \bar{x})$. Were the system $\Sigma$ to consist *merely* of the single pair of mutually non-equivalent and mutually obverse elements, $(x, \bar{x})$, principles I, II, III, IV, and VI would all of them be true of the system as thus restricted. For $O(x\bar{x})$; and if, to this collection, we add any collection $\gamma$, consisting either of $x$, or of $\bar{x}$, repeated any multitude of times, or again consisting both of $x$ and of $\bar{x}$, in any combination, each occuring any multitude of times — in any case $O(x\bar{x}\gamma)$. Hence principle I is satisfied. On the other hand since $O(x\bar{x})$, while, by 35, 37, $E(xx)$, and $E(\bar{x}\bar{x})$ — the non-equivalence of $x$ and $\bar{x}$ can be readily established, without taking account of any collections except those into which $x$ and $\bar{x}$ either jointly or severally enter. In order that a collection $\beta$ shall consist altogether of complements of $\delta$, while $O(\beta)$ is true, it is necessary, in case $\Sigma$ contains *only* $x$ and $\bar{x}$, that $\beta$ should include both $x$ and $\bar{x}$; while $\delta$ (which, by hypothesis is such that $O(\delta b_n)$ for every element $b_n$, of $\beta$), must then *also* include both $x$ and $\bar{x}$. Hence if $O(\beta)$, $O(\delta)$ follows. Hence principle II would hold true if the system $\Sigma$ consisted only of the pair $(x, \bar{x})$. Principles III and IV would obviously hold true of the same system. And considered with reference to $x$, its obverse, $\bar{x}$, is an element satisfying the requirements of principle VI; while the same holds true of $x$ when it is considered with reference to $\bar{x}$. All the principles except V would therefore be satisfied if the system $\Sigma$ consisted of the single pair $(x, \bar{x})$.

### *Chains of elements, defined by recurrence.*

85. Principle V, however, is not satisfied by the existence of a single pair of elements, such as $(x, \bar{x})$. For, if the pair $(x, \bar{x})$ exists, then (since $x \neq \bar{x}$ by

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38), it follows that principle V demands the existence of $r_1$ such that $E(r_1 x)$ and $E(r_1 \bar{x})$. These two assertions require that $r_1 \neq \bar{x}$ and $r_1 \neq x$ (by 41). Since $r_1 \neq \bar{x}$, there exists an element which we will next symbolize by $\bar{r}_2$, and which, by virtue of principle V, is such that $E(r_1 \bar{r}_2)$, $E(\bar{r}_2 \bar{x})$, and $O(r_1 \bar{r}_2 \bar{x})$. The obverse of $\bar{r}_2$ also exists. Since $E(r_1 \bar{r}_2)$ it follows by 41, that $r_2 \neq r_1$; and for the precisely analogous reason, $r_2 \neq \bar{x}$. At the same time $F(r_2 | r_1 \bar{x})$. Since $r_2 \neq \bar{x}$, there also exists, by principle V, an element $\bar{r}_3$, such that $E(\bar{r}_3 r_2)$ and $E(\bar{r}_3 \bar{x})$ while $O(r_2 \bar{r}_3 \bar{x})$. The obverse $r_3$ of $\bar{r}_3$, therefore, also exists; and is such that $r_3 \neq r_2$, $r_3 \neq \bar{x}$, and $F(r_3 | r_2 \bar{x})$. The procedure whereby the elements $r_1$, $r_2$, and $r_3$ have been defined is obviously a recurrent one. Repeatedly applied it defines a chain of elements: $(r_1, r_2, \cdots, r_n, \cdots)$, whereof any finite number $n$ may be at pleasure defined in a determinate order. These elements, together with their obverses $(\bar{r}_1, \bar{r}_2, \cdots, \bar{r}_n, \cdots)$, all exist in $\Sigma$, and possess the following properties:

$$(1) \qquad x \neq r_1;\ r_1 \neq r_2;\ r_2 \neq r_3;\ \cdots;\ r_{n-1} \neq r_n;\ r_n \neq \bar{x}.$$

$$(2) \qquad F(r_1 | x\bar{x}),\ F(r_2 | r_1 \bar{x}),\ F(r_3 | r_2 \bar{x}),\ \cdots,\ F(r_n | r_{n-1} \bar{x}).$$

$$(3) \qquad F(\bar{r}_1 | x\bar{x}),\ F(\bar{r}_2 | \bar{r}_1 x),\ F(\bar{r}_3 | \bar{r}_2 x),\ \cdots,\ F(\bar{r}_n | \bar{r}_{n-1} x).$$

$$(4) \qquad \bar{x} \neq \bar{r}_1;\ \bar{r}_1 \neq \bar{r}_2;\ \bar{r}_2 \neq \bar{r}_3;\ \cdots;\ \bar{r}_{n-1} \neq \bar{r}_n;\ \bar{r}_n \neq x.$$

The expressions (3) and (4) follow directly from the truth of (1) and of (2), by the definitions of the $F$-collections, and of the relations of equivalent and of obverse elements, and by 30, 36(2), and 52(4).

From these conditions it further follows that, if $m$ and $n$ are (for the moment) viewed, not as symbols for elements of $\Sigma$, but as purely numerical marks or indices, serving to distinguish the ordinal positions of different members of the chain $(r_1, r_2, \cdots, r_m, \cdots, r_n, \cdots)$, and if $m$, in the series of natural numbers, precedes $n$, then:

$$(5) \qquad F(r_n | r_m \bar{x});\ \text{and}\ r_n \neq r_m.$$

$$(6) \qquad F(\bar{r}_n | \bar{r}_m x);\ \text{and}\ \bar{r}_n \neq \bar{r}_m.$$

For, by the laws of the construction of the chain, $F(r_{m+1} | r_m \bar{x})$, and $F(r_{m+2} | r_{m+1} \bar{x})$. Hence $F(r_{m+2} | r_m \bar{x})$, by 71. In the same way, since $F(r_{m+3} | r_{m+2} \bar{x})$, there follows $F(r_{m+3} | r_m \bar{x})$. And the same process of elimination can be repeated any number of times, so that, in fine, $F(r_n | r_m \bar{x})$. Meanwhile, by the principle proved in 72, $r_{m+2} \neq r_m$. In fact, if we suppose $r_{m+2} = r_m$, we have $F(r_{m+1} | r_m \bar{x})$ by construction, as well as $F(r_{m+2} | r_{m+1} \bar{x})$. By the substitution of equivalents we therefore obtain:

$$F(r_{m+1} | r_m \bar{x}) \qquad \text{and} \qquad F(r_m | r_{m+1} \bar{x}).$$



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Whence follows, by 52 (2), $r_{m+1} = r_m$, which contradicts the conditions stated in the expressions (1). Hence $r_{m+2} \neq r_m$. In the same way we can prove that $r_{m+3} \neq r_m$; and in general, $r_n \neq r_m$, where $n$ is any ordinal number that follows $m$. Hence the expressions (5) are proved true; and the expressions (6) follow therefrom by considering the obverses of the elements occurring in (5). All the members of the chain $(r_1, r_2, \cdots, r_n, \cdots)$ are such that $F(r_n | r_1 \bar{x})$.

To sum up: The requirements of principle V include the assertion that, since the pair $(x, \bar{x})$ exists there must also exist the two distinct chains of elements:

$$(r_1, r_2, \cdots, r_n, \cdots) \qquad \text{and} \qquad (\bar{r}_1, \bar{r}_2, \cdots, \bar{r}_n, \cdots).$$

No two of the elements of either one of these chains are mutually equivalent. If any element of the first chain, as $r_m$, were equivalent to a member of the second chain, say $\bar{r}_n$, we should have, since, by the foregoing:

$$F(r_m | r_1 \bar{x}) \qquad \text{and} \qquad F(\bar{r}_n | \bar{r}_1 x),$$

the consequence: $F(r_m | r_1 \bar{x})$ and $F(r_m | \bar{r}_1 x)$. By elimination would follow $F(r_1 \bar{x} | \bar{r}_1 x)$; whence would follow $O(r_1 x)$ and so $r_1 = x$, a result which is rendered impossible by the conditions that define $r_1$. Hence no element of either chain can be equivalent to any member either of the same chain, or of the other chain; and all the elements of both chains are non-equivalent both to $x$ and to $\bar{x}$. Since, if any element $r_n$ exists in the first chain, $r_{n+1}$ also exists (while $\bar{r}_{n+1}$ also belongs to the other chain), the two chains contain each an infinite number of non-equivalent elements. *The system $\Sigma$ consequently contains an infinite number of mutually non-equivalent elements.*

86. Herewith, however, the requirements of principle V are by no means exhausted. For since no two elements of either chain are mutually equivalent, any two successive elements of each chain, as, for instance, $r_m$ and $r_{m+1}$, are such that, by principle V, $\bar{s}_1$ exists such that $E(r_m \bar{s}_1)$ and $E(r_{m+1} \bar{s}_1)$, while $O(r_m r_{m+1} \bar{s}_1)$. In this case $F(s_1 | r_m r_{m+1})$, while $s_1 \neq r_m$ and $s_1 \neq r_{m+1}$. A new recurrent process is thus defined, a process which can be employed to define a chain $(s_1, s_2, \cdots, s_j, \cdots)$ where the subscripts again have the significance of the ordinal numbers.

By 72, no two elements of this chain can be mutually equivalent. Every member $s_j$ of this new chain must be non-equivalent to any member of the chain $(r_1, r_2, \cdots, r_m, \cdots)$, as well as to any member of any chain $(s'_1, s'_2 \cdots, s'_k, \cdots)$, or $(t_1, t_2, \cdots, t_h, \cdots)$, such as can be formed by taking account of those pairs: $(r_1, r_2)$, $(r_3, r_4)$, etc., which are different from the pair $(r_{m+1}, r_m)$, and by applying principle V recurrently to them. In general, if $F(e | r_m r_{m+1})$, and $F(g | r_n r_{n+1})$, where $m$ and $n$ are different subscript numbers, while $m$ precedes $n$ in the ordinal series, we can express the relations already considered by taking, if we choose, $r_1$ as an origin, and by employing the relation of antecedent and consequent. In this case we can write, upon the basis of 85 (5),

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$$r_m \prec_{r_1} r_{m+1},$$

$$F(e \mid r_m r_{m+1});\ e \neq r_m;\ e \neq r_{m+1}.$$

Whence follows $e \prec_{r_1} r_{m+1}$.

Furthermore: $r_{m+1} \prec_{r_1} r_n$ by 85 (5),

$$r_n \prec_{r_1} g,\ r_n \neq g.$$

And so, if $e = g$, we have:

$$g \prec_{r_1} r_{m+1} \prec_{r_1} r_n \prec_{r_1} g.$$

This, however, is impossible. Thus no member of any one of the new chains is equivalent to any member of any other of these new chains, or of the original chains. Since an infinite number of different pairs of non-equivalent members exist in the chain $(r_1, r_2, \cdots, r_n, \cdots)$, it follows that an infinite number of new chains can be constructed upon the basis of these various pairs. The process of forming such chains is itself recurrent.

In addition to the chains thus far defined, other chains of elements exist in $\Sigma$. For since, by 85, $r_1 \neq x$, it is possible to treat the pair $(r_1, x)$ as, in 85, the pair $(r_1, \bar{x})$ was treated. The result would be to define a chain of elements $(r_1, p_1, p_2, \cdots, p_h, \cdots)$, such that $F(p_h \mid r_1 x)$, while, as before, no two elements of the chain are mutually equivalent. If any element of $\Sigma$, say $v$, is such that $F(v \mid r_1 x)$ and $F(v \mid r_1 \bar{x})$, we have $F(v \mid r_1 x)$ and $F(r_1 \mid vx)$, and hence $v = r_1$. See also 72. It follows that no element of the new chain except $r_1$ itself is equivalent to any element of the chain $(r_1, r_2, \cdots, r_n, \cdots)$.

*General properties of resultants.*

87. A very little consideration serves to show that a new application of principle VI, to the chains of elements now defined, will lead to still further results. Before we are prepared to consider these results, we must however survey the properties of a class of elements defined in 17. Of the infinitely numerous non-equivalent elements now known to exist in $\Sigma$, collections can be made comprising any number of elements. These may be either $E$-collections or $O$-collections. If of the former type, the collections so made can be enlarged to $O$-collections by the adjunction of suitable elements. To what laws are such adjunctions subject? This we are next to see.

88. If $\beta$ be any collection, any element $r$ such that $F(r \mid \beta)$, is a resultant of the collection $\beta$, by virtue of the definition stated in 17. Since $r$ is thus any element such that $O(\beta \bar{r})$, and since the obverse of any element of $\beta$ may fill the place of $\bar{r}$ in this $O$-collection, it is obvious that every element of $\beta$ is also a resultant of $\beta$. Furthermore, if any element $x$ forms an $O$-collection with any partial collection of elements of $\beta$, $\bar{x}$ is a resultant of $\beta$. For if $O(\lambda x)$, then $O(\kappa \lambda x)$, so that if $(\kappa, \lambda)$ is the same collection as $\beta$, $F(\beta \mid \bar{x})$.

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89. If $F(r|\beta)$, $F(t|\beta)$ and $F(u|rt)$ are all true there follows, by 63, $F(u|\beta)$. Hence any mediator of a pair of resultants of a given collection is itself a resultant of the collection. By the reasoning used in 85–86, it can therefore be shown that any collection containing at least two non-equivalent elements must possess an infinite number of resultants.

90. If $\rho$ be the collection consisting of all the resultants, $r_1$, $r_2$, etc., of a given collection $\beta$, then any resultant of $\rho$ is also a resultant of $\beta$.

Let $q$ be such that $F(q|\rho)$. Then $O(\rho\bar{q})$, and hence $O(\bar{\rho}q)$. By the definition of a resultant, $\beta$ is a collection which becomes an $O$-collection if any element of $\bar{\rho}$, that is, if the obverse of any one of the resultants of $\beta$, be adjoined to $\beta$. Hence $\beta$ forms an $O$-collection in case any member of the $O$-collection $O(\bar{\rho}q)$, except $q$, is adjoined to $\beta$. By 36(5) it follows that $O(\beta\bar{q})$. Hence $F(\beta|q)$.

The collection $\rho$ is consequently a collection which contains all of its own resultants. It also contains the resultants of all the partial collections which can be found by selecting certain elements from the collection $\beta$.

91. If $y$ is an element of $\Sigma$, selected at pleasure, and if $\beta$ is any collection, and if a determinate element $c$ exists such that, whatever element $b_n$ of $\beta$ is chosen, $c$ is a mediator of the pair $(y, b_n)$, then $c$ is also a mediator of any and every pair that can be formed by combining $y$ with any resultant whatever, say $r_x$ of $\beta$. For if $r_x$ be any resultant of $\beta$, then $O(\beta\bar{r}_x)$. And if $c$ exists such that $F(c|yb_n)$ is true, whatever element $b_n$ of $\beta$ may be chosen, then $O(\bar{c}yb_n)$ is true of every member $b_n$ of $\beta$. Hence, by 36(5), $O(\bar{c}yr_x)$. And hence $F(c|yr_x)$.

The converse of this theorem is obvious. That is, if an element $c$ is a mediator of *every* pair $(y, r_x)$ consisting of $y$ and some resultant of $\beta$, $c$ is a mediator of $(y, b_n)$ where $b_n$ is any element chosen at pleasure from $\beta$. For the elements of $\beta$ are themselves amongst the resultants of $\beta$.

There always exists an element $c$ having the properties here in question, since $y$ itself is such an element. For $F(y|yx)$ is true of every element $x$. But, as will soon appear, there are, in certain important classes of cases, elements possessing this property which are not equivalent to the chosen element $y$.

92. If $\gamma$ be the collection of all those elements, $c_v$, which possess the property discussed in 91, viz., if $\gamma$ be the collection of those elements $c_v$, such that $F(c_v|yr_x)$ is true for a determinate selected element $y$, and for every resultant, $r_x$, of $\beta$, then every resultant of $\gamma$ is itself a member of the collection $\gamma$.

For let $t$ be such that $F(t|\gamma)$, so that $O(\bar{\gamma}t)$. Since $(y, r_x)$ is such that $O(yr_x\bar{c}_v)$ is true of every element $\bar{c}_v$ of $\bar{\gamma}$, while $O(\bar{\gamma}t)$, it follows that $O(yr_x\bar{t})$, by 36(5). Hence $F(t|yr_x)$.

If $\rho$ be the collection of all the resultants of $\beta$, and if $\gamma$ be the collection of all those elements, such as $c_v$, which with reference to a selected element $y$,

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have the property of being, every one of them, such that $F(c_y | yr_x)$ is true of every element $r_x$ of $\rho$, then, as now appears, the collections $\gamma$ and $\rho$ are collections each of which contains all of its own resultants.

*Pairs of conjugate resultants.*

93. We are now prepared to consider more minutely the consequences of principle VI. By that principle, if a collection $\vartheta$ possesses any complement $w$, so that $O(\vartheta w)$, a complement $v$ also exists such that $O(\vartheta v)$ while, whatever element $t_n$ of $\vartheta$ be selected, $O(vwt_n)$. Stating this requirement in terms of the $F$-collections, we have, as the conditions set forth in the hypothesis of the principle, the existence of an element $\bar{w}$, such that $F(\bar{w} | \vartheta)$. The consequence according to the principle is that $\bar{v}$ also exists such that $F(\bar{v} | \vartheta)$, while, since $O(vwt_n)$, $O(\bar{v}\bar{w}\bar{t}_n)$, and therefore $F(t_n | \bar{v}\bar{w})$ for every element $t_n$ of $\vartheta$. Let $\bar{w} = q$, and $\bar{v} = r$. Then principle VI asserts that whatever resultant $q$ of a collection $\vartheta$ be selected, there always exists a resultant of $\vartheta$, namely $r$, such that every element of $\vartheta$ is a mediator (that is a resultant) of the pair $(q, r)$. It readily follows that every resultant of $\vartheta$ is a mediator of $(q, r)$. For if $p_k$ be any resultant of $\vartheta$, then $O(\vartheta\bar{p}_k)$. And since $O(vwt_n)$ is true of every element of $\vartheta$, while $O(\vartheta\bar{p}_k)$, it follows, by 36 (5), that $O(vwp_k)$, that is $O(\bar{q}\bar{r}p_k)$, and hence $F(p_k | qr)$.

94. If $q$ is any determinate resultant of $\vartheta$, and if $r$ is an element related to $q$ in the way set forth in 93, then any element, $r'$, such that $F(qr' | p_k)$ is true for every resultant of $\vartheta$, i. e. any element $r'$ such that $F(qr' | t_n)$ is true of every element of $\vartheta$, is also such that $r' = r$. For, since $r$ and $r'$ are both of them resultants of $\vartheta$, we have $F(qr | r')$, because of the definition of $r$; and also $F(qr' | r)$ because of the definition of $r'$. Hence $r' = r$ by 52 (2). We may consequently let $r$ stand as the unique representative of the class of those resultants of $\vartheta$ which, when $q$ is given, fulfil, with respect to $q$, the requirement of principle VI. With this understanding, we shall henceforth characterize $r$ as the *conjugate resultant of $q$ in, or with respect to* the collection $\vartheta$. If $r$ is given instead of $q$, some equivalent of $q$ is nevertheless predetermined as a conjugate of $r$; and if $q$ be selected as the unique representative of the class of elements which are equivalent to itself, we may regard the relation of $q$ and $r$ as wholly symmetrical; and so we may henceforth speak of the pair $(q, r)$ as a *pair of conjugate resultants of the collection $\vartheta$*, or more briefly as a *conjugate pair in, or with respect to, $\vartheta$*. We shall symbolize the relation in question thus: $J(qr; \vartheta)$. This symbol is to be read as the assertion: "The pair $(q, r)$ is a pair of conjugate resultants of the collection $\vartheta$," or "is a conjugate pair in $\vartheta$," or "with respect to $\vartheta$." Were $\vartheta$ a pair, as for instance $(x, y)$, we could write $J[qr; (x, y)]$.

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95. If the collection $\vartheta$ is made to include all of the elements of $\Sigma$, the pair $(q, r)$ becomes a pair of mutually obverse elements so that $q = \bar{r}$. This appeared already in 27, when the first use of principle VI was made. If $\vartheta$ is an $O$-collection, a conjugate pair $(q, r)$ such that $O(\bar{q}\bar{r}t_n)$ while $O(\vartheta)$ is true, is a pair such, by principle II, that $O(\bar{q}\bar{r})$. Hence, in this case, $O(qr)$, and thus any pair that is a pair of conjugates with respect to an $O$-collection is an $O$-pair. If, in case of any collection $\vartheta$, a pair $(q, r)$, conjugate with respect to $\vartheta$ is such that $q = r$, then, since $F(t_n | qr)$, every element $t_n$ of $\vartheta$ is such that $t_n = q = r$; and therefore, in this case, a single element may be taken as the unique representative both of the whole collection $\vartheta$, and of all of its possible pairs of conjugate resultants (see 52 (1), and 70).

96. In case of any pair of conjugate resultants of a system $\beta$, i. e., in case of $(q, r)$ such that $J(qr; \beta)$, the resultants of the pair $(q, r)$, and the resultants of $\beta$, form precisely identical collections. If $\beta$ is enlarged either to an $O$-collection, or so as to include all of the elements of $\Sigma$, the entire system $\Sigma$ becomes the collection of the resultants of any one of the possible pairs of obverse elements of $\Sigma$, such, for instance as $(x, \bar{x})$; any one of these pairs being, as we now know, such that, if $O(\beta)$, $J(x\bar{x}; \beta)$.

97. If $\beta$ is a given collection, and if $(q, r)$ is a conjugate pair of its resultants, and if $u$ is any third resultant of $\beta$, not equivalent either to $q$ or to $r$, then the conjugate resultant $v$ of $u$ can be found by considering merely the triad $(q, r, u)$. According to principle VI there is, namely, a resultant $v$ of this triad* such that $F(vu | q)$ and $F(vu | r)$. Since $v$ is a resultant of the triad, $F(qru | v)$. But meanwhile, since $J(qr; \beta)$, and $F(\beta | v)$, $F(qr | v)$ by the definition of a conjugate pair, so that from $F(qru | v)$ the element $u$ may be stricken out. Since $v$, then, is such that $F(qr | v)$, while $F(vu | q)$ and $F(vu\ r)$, it is easy to show that any resultant $p_k$ of $\beta$ is such that $F(p_k | vu)$. For $F(p_k | qr)$, since $(q, r)$ is a conjugate pair. But from $F(p_k | qr)$, $F(q | uv)$ and $F(r | uv)$, follows, by 73, $F(p_k | uv)$.

If, then, a single conjugate pair of resultants of a collection $\beta$ is given, viz., $(q, r)$, the conjugate of any third resultant of $\beta$, such as $u$, is equivalent to the conjugate of $u$ in the collection of the resultants of the triad $(q, r, u)$. If $\beta$ is an $O$-collection, or if $\beta$ includes all of the elements of $\Sigma$, the conjugate of $x$ in the triad $(x, z, \bar{z})$ is obviously $\bar{x}$.

98. If $(\beta, x)$ be any collection that includes a given element, $x$, the conjugate resultant of $x$ with respect to the collection $(\beta, x)$ is one of the resultants of $\beta$. For let $q$ be such a resultant. $F(q | qx)$ is in any case true of $q$. But $q$ has, in addition, to be such that $F(q | \beta x)$, while, whatever element $b_n$, of $\beta$,

* Identical with what KEMPE calls the "unsymmetrical resultant" of the triad $(q, r, u)$. KEMPE does not directly define our conjugate resultants in general, but builds his theory upon that of the resultants of triads.

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be selected, $F(qx|b_n)$; i. e., $O(\bar{q}\bar{x}b_n)$. By the adjunction of all elements of $\beta$ besides $b_n$, we obtain, hereupon, $O(\bar{q}\bar{x}\beta)$. But since $F(q|\beta x)$, $O(\bar{q}x\beta)$. Hence $O(\bar{q}\beta)$. Hence $F(\beta|q)$.

99. If $\beta$ and $\alpha$ are such that $F(\beta|\alpha)$, then $\beta$ and $\alpha$ possess at least one common resultant. For let $a$ be an element of $\alpha$, chosen at pleasure. Adjoin $\bar{a}$ to the collection $\beta$, and consider the conjugate resultant of $\bar{a}$ in the collection $(\bar{a}, \beta)$. Let this resultant be the element $b$. By 98, $F(\beta|b)$. By the definition of a conjugate resultant $F(\bar{a}b|b_n)$ is true of every element $b_n$ of $\beta$. By hypothesis, however, $F(\beta|\alpha)$. Hence by 56, $F(\alpha|\bar{a}b)$. Since $a$ is itself an element of $\alpha$, the superfluous obverse $\bar{a}$ may be omitted, so that $F(\alpha|b)$. Hence the element $b$ is a resultant of $\beta$ and also of $\alpha$.

The importance of this theorem for the geometrical application of our theory (since the theorem may be called *the theorem regarding intersections* or *transversals*) justifies a proof directly in terms of $O$-collections.

If, namely, $O(\delta\epsilon)$, there exists $x$ such that $O(\delta\bar{x})$ while $O(\epsilon x)$. For let us select at pleasure any element $d$ of $\delta$. Consider the collections $(\bar{d}, \bar{\epsilon})$, and, with respect to that collection, define the conjugate resultant of $\bar{d}$ in $(\bar{d}, \bar{\epsilon})$. Let $x$ be this resultant. By 98, $O(\bar{\epsilon}\bar{x})$ is true. Hence $O(\epsilon x)$. By the definition of a conjugate resultant $F(\bar{e}_r|x\bar{d})$ is true of every element $\bar{e}_r$ of $\bar{\epsilon}$. Hence $O(e_r x\bar{d})$ is true of every element $e_r$ of $\epsilon$, while $O(\delta\epsilon)$. Hence, by 24 (4), $O(\bar{\delta}x\bar{d})$. $\bar{d}$ is a repetition of some element of $\bar{\delta}$, and may be stricken out (by 37). Hence $O(\bar{\delta}x)$. Hence $O(\delta\bar{x})$, while, as above shown, $O(\epsilon x)$. So the theorem is proved.

Since this process may be repeated for every element of $\delta$ and also of $\epsilon$, the variety of elements of the type $x$, in case the elements of $\delta$ and of $\epsilon$ include non-equivalent pairs, is, in general, by principle V, and 89, infinite.

*The relations of pairs of conjugate resultants in various collections.*

100. It is frequently important to bring the various pairs of conjugate resultants which exist in different collections into relation with one another. The procedure by which this is accomplished will lead us at once to the threshold of the ordinary algebra of logic, which, as originally developed, was based upon observing certain properties of the system $\Sigma$, in cases where this system was interpreted as a collection whose elements are either classes or propositions.

Let the conjugate resultant of $y$ in the collection $(y, \beta)$ be $x$, so that $J(xy; (\beta, y))$. Then, by 98, $F(x|\beta)$ is true. Hereupon, if we select, amongst the resultants of $\beta$, that one, say $z$, which is the conjugate resultant, with respect to $\beta$, of the element $x$ just determined, so that, while $J(xy; (\beta, y))$, it is also true that $J(xz; \beta)$, then it follows that $F(yb_n|z)$ is true for every element $b_n$ of $\beta$. For $z$ is such that $F(xz|b_n)$, and is also such that $F(\beta|z)$. Since $F(b_n|xy)$ is true (by the foregoing) for every element $b_n$ of $\beta$, while

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$F(\beta|z)$, it follows, by 36 (5) and 57, that $F(xy|z)$; and since $F(b_n|xz)$, that is $F(b_n\bar{z}|x)$, while $F(x|z\bar{y})$, we have $F(b_n\bar{z}|z\bar{y})$, whence follows $F(z|b_n y)$.

We have, therefore, the result, that, if $y$ be any element, and $\beta$ any collection, then, in case $x$ is such that $J(xy;\ (\beta, y))$ and $z$ is such that $J(xz;\ \beta)$, the two elements $x$ and $z$ constitute a conjugate pair of the resultants of $\beta$, while just this pair stands in what we may regard as an unique relation to $y$. The pair $(x, z)$ is namely such that, for the first, $z$ is a mediator between $y$ and any element $b_n$ of $\beta$ which may have been chosen for comparison with $y$ and with $z$. By 91, $z$ is consequently also a mediator between $y$ and any resultant, $r_v$, of $\beta$, so that, whatever resultant $r_v$, of $\beta$, we may select $F(r_v y|z)$. Moreover, whatever resultant $z'$ of $\beta$ possesses the property just ascribed to $z$, must be equivalent to $z$. For if $z'$ exists such that $F(z'|\beta)$, while $F(z'|r_v y)$ is true, whatever resultant $r_v$ of $\beta$ we choose to consider, then, since $z$ itself is a resultant of $\beta$, $F(z'|zy)$, while, by the definition of $z$, $F(z|z'y)$, and hence $z = z'$. Therefore $z$ may be taken as the unique representative of its own class of equivalents. Meanwhile, $x$ possesses, with reference to $\beta$ and $y$, the property of being a resultant of $\beta$ such that every element of $\beta$ is a mediator between $x$ and $y$. Consequently, since $F(b_n|xy)$, $F(b_n\bar{y}|x)$, and hence $x$ is a mediator between $\bar{y}$ and whatever element $b_n$ of $\beta$ may have been chosen. By 91, $x$ is accordingly a mediator between $\bar{y}$ and whatever resultant, $r_v$, of $\beta$, may have been chosen. Whatever resultant, $x'$, of $\beta$, possesses the property just ascribed to $x$, is such that $x = x'$.

101. *Conjugate limits of a collection with reference to a base.* We may sum up the result of the foregoing thus: If any element $y$ be chosen, at our pleasure, as what we shall now call a base, and if hereupon any collection $\beta$ be considered with reference to this base, then there exists a pair, and (barring for the moment the consideration of equivalent elements), a *single* pair, of conjugate resultants of $\beta$, which is so related to $y$ and to $\beta$ that (1) one of these two resultants (which we shall now symbolize by $p$) is such that $F(p|b_n y)$ for every element of $\beta$, and $F(p|r_v y)$ for every resultant of $\beta$; while (2) the other of these resultants, which we shall now symbolize by $s$, is such that $F(b_n|ys)$, and $F(r_v|ys)$, i. e., such that $F(b_n\bar{y}|s)$, and $F(r_v\bar{y}|s)$. If $\gamma$ be the collection of the totality of those elements of $\Sigma$ which are mediators between $y$ on the one hand and each and every element and resultant of $\beta$, separately considered, on the other hand (see 92), then $p$ has the property of belonging at once to the collection $\gamma$, and to the collection $\rho$; where $\rho$ is, as before, the collection of all the resultants of $\beta$. Any element possessing the property of belonging at once to $\gamma$ and to $\rho$, is equivalent to $p$, which may therefore be viewed, for present purposes, as the unique representative of its own class of equivalent elements. Barring equivalent elements, then, the collections $\gamma$ and $\rho$ have *only* this element $p$ in common. If $\gamma'$ be the collection of

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those elements of $\Sigma'$ which are mediators between $\bar{y}$ on the one hand, and each and every resultant of $\beta$, separately considered, on the other hand, then $s$ has the property of belonging at once to the collection $\gamma'$, and to the collection $\rho$. Any element possessing the property of belonging at once to $\gamma'$ and to $\rho$, is equivalent to $s$, which may therefore be viewed, for the purposes of forming $O$-collections and $F$-collections, as the unique representative of its own class of equivalent elements. Barring equivalent elements, then, the collections $\gamma'$ and $\rho$ have *only* the element $s$ in common. Of the pair $(p, s)$, each resultant of $\beta$ is a mediator. If, for the collection $\beta$, and for $\rho$, the collection of the totality of the resultants of $\beta$, the pair $(p, s)$ alone is substituted, and if hereupon this pair is treated precisely as, in the foregoing, $\beta$ itself has been treated, that is, if the resultants of $(p, s)$ are first defined, and then their collection, viz., $\rho$, is compared with $y$, the same pair $(p, s)$, is once more found as that pair of conjugate resultants of the collection $(p, s)$ itself, whose relation to $y$ is the relation heretofore characterized.

We shall now call the pair $(p, s)$ a *pair of conjugate limits of $\beta$ with reference to the base $y$*. For a given collection $\beta$, and for a given base $y$, principle VI thus requires us to define one such pair, and (barring equivalent elements), but a single pair, viz., $(p, s)$, which may be viewed as *the* pair of conjugate limits in question. This pair, being a pair of mutually conjugate resultants of $\beta$, is symmetrically disposed with reference to the collection of the resultants of $\beta$. But, as has appeared in the foregoing, the pair $(p, s)$ is not, in general, symmetrically disposed with respect to the enlarged collection $(\beta, y)$. For $p$ is a mediator between $y$ and each resultant of $\beta$ separately considered; while each resultant of $\beta$, separately considered, is a mediator of the pair $(y, s)$. To mark this difference of relative position of $p$ and $s$ we may call: $p$ the inferior limit of $\beta$ with respect to $y$; $s$ the superior limit of $\beta$ with respect to $y$; while $y$ is the base of this pair of conjugate limits of $\beta$. It is at once obvious that if we choose $\bar{y}$ as base instead of $y$, $s$ would become the inferior, and $p$ the superior limit of $\beta$ with respect to $\bar{y}$.

102. We are now in a position to extend our result from the case of a single collection $\beta$, to the case of a set of collections $\beta$, $\gamma$, $\delta$, etc., and in fact to the set of all possible collections of the elements of $\Sigma$. Holding a given base, $y$, chosen at pleasure, constant, we may consider any and all collections of the elements of $\Sigma$ with reference to this one chosen base. If we do so, then, whatever collection, $\alpha$, $\beta$, $\gamma$, $\delta$, etc., we select, we shall find, by a process wholly analogous to the foregoing, that there exists an unique pair of resultants of any one such collection, such that this pair is, for that collection, the pair of conjugate limits of the collection with respect to $y$. Of this pair, one element is the superior, and the one the inferior limit of the collection in question, with respect to this chosen and constant base, while the choice of the base is arbitrary. The

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limits are each time functions of the collection in question, and may be regarded as functions of that alone, *so long as the base is held constant.* *

If, while the base itself remains constant, a collection is altered, by the adjunction, or by the omission of elements, its limits alter or remain invariant, in ways whose laws are now to be defined. If various collections are considered with reference to the same base, their respective superior or inferior limits may be considered as new collections, with results whose laws are also to be determined. But if, instead, while any collection or sets of collections remain constant, the base is altered, so that an element $y'$ or $y''$ takes the place of $y$, then the pairs of conjugate limits of any given collection with reference to the new base remain invariant, or alter, in accordance with still other principles (which we shall consider in chapter VI). We shall discuss these various cases in order. But in the rest of the present chapter, the base shall remain constant.

103. *Adjunction of the base to a collection.* If, to the collection $\beta$, while the base remains constant, the base $y$ is itself adjoined, the inferior limit of $(\beta, y)$ becomes equivalent to $y$ itself; but the superior limit remains invariant. This appears from the reasoning used in 100.

104. *Adjunction of the obverse of the base; adjunction of resultants; other cases.* If the base $y$ is held constant, and if $\bar{y}$ is thereupon adjoined to $\beta$, the inferior limit remains constant, while the superior limit of $(\beta, \bar{y})$ becomes equivalent to $\bar{y}$. The reasoning used in 100 can be employed to prove this also.—If any resultant of $\beta$ is adjoined to $\beta$, or if any collection of the resultants of $\beta$ is adjoined to $\beta$, the superior and inferior limits of the enlarged collection, so long as the base $y$ is held constant, remain invariant; as appears from the reasoning used in 91, 92. Hence, by 89, if to a given collection, any mediator of any pair of its elements or of its resultants is adjoined, the limits remain constant; as they also do if the elements of a collection are repeated any multitude of times. If $y$ is the constant base, and if any element or collection of elements of the collection $\gamma$, of 92 and 101, be adjoined to $\beta$, then, while the inferior limit, in general, alters, the superior limit of the collection remains in so far invariant. If any element or collection of elements chosen from the collection $\gamma'$ of 101 be adjoined to $\beta$, this adjunction leaves the inferior limit of the enlarged collection invariant, while altering, in general, the superior limit.

Elements may be omitted from collections in a manner which is subject to these same laws. Thus, the omission of such repetitions of elements as occur in a collection does not alter either the superior or the inferior limits, etc.

* The conception that the elements usually known as the products and sums of the algebra of logic are relative to a chosen constant base (the zero of the usual algebra of logic), and that the choice of what element of $\Sigma$ shall be treated as the zero-element is essentially arbitrary, is KEMPE's. But KEMPE develops this conception solely on the basis of his theory of the symmetrical and unsymmetrical resultants of *triads*. By the more general concept of the conjugate resultants of collections, I have generalized KEMPE's theory so as to be able to apply it, from the start, to collections of any multitude whatever.

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The operation of determining the superior and inferior limits of a collection is obviously independent of the order in which the elements are arranged or considered, and may consequently be called a *commutative operation.*

105. *The transformation of a collection into its obverse collection.* If a collection $\beta$ be transformed into its obverse collection (the base $y$ remaining constant), if $p$ be the inferior, and $s$ be the superior limit of $\beta$, and if $p'$ be the inferior and $s'$ be the superior limit of $\bar{\beta}$ then the two equivalences hold good:

$$\bar{p} = s' \text{; and } \bar{s} = p'.$$

For, by the definition of an inferior limit, $F(p \mid yr_v)$ for every resultant, $r_v$, of $\beta$, while $F(p \mid \beta)$. Whence there follows (1) $F(\bar{p} \mid \bar{y}\bar{r}_v)$ for every resultant $\bar{r}_v$ of $\bar{\beta}$; while (2) $F(\bar{p} \mid \bar{\beta})$. From (2) it follows that $\bar{p}$ is a determinate resultant of $\bar{\beta}$. From (1) follows, by transfer, $F(\bar{p}y \mid \bar{r}_v)$, whatever resultant $\bar{r}_v$ of $\bar{\beta}$ may be chosen. Thus $\bar{p}$ is such a resultant of $\bar{\beta}$ that whatever element or resultant $\bar{r}_v$ of $\bar{\beta}$ may be chosen, this element or resultant of $\bar{\beta}$ is a mediator between $\bar{p}$ and $y$. Hence $\bar{p}$ is equivalent to the superior limit of $\bar{\beta}$ with respect to the base $y$. Hence $s' = \bar{p}$.

Furthermore, since $s$ is the superior limit of $\beta$ with respect to $y$, we have $F(r_v \mid sy)$, whatever resultant, $r_v$, of $\beta$ we may choose; while, at the same time $F(s \mid \beta)$. It follows that $F(\bar{s} \mid \bar{\beta})$, so that $\bar{s}$ is a determinate resultant of $\bar{\beta}$; while $F(\bar{r}_v \mid \bar{s}\bar{y})$, i. e., $F(\bar{r}_v y \mid \bar{s})$, for every resultant $\bar{r}_v$ of $\bar{\beta}$; so that $\bar{s}$ is a mediator between $y$ on the one hand, and each and every $\bar{r}_v$ of $\bar{\beta}$ on the other hand. Thus $\bar{s}$ is equivalent to the inferior limit of $\bar{\beta}$ for the base $y$. The equivalences in question, viz., $\bar{p} = s'$; and $\bar{s} = p'$, are accordingly proved.

106. *The combination of collections. The associative law in the determination of pairs of conjugate limits with respect to a constant base.* If we consider the inferior limit of a collection $\beta_1$, with respect to a base $y$, and also the inferior limits, with respect to the same base, of collections $\beta_2, \beta_3, \cdots, \beta_n, \cdots$, in any multitude of collections, and if $\pi$ be the collection of all these inferior limits, while $\omega$ is the collection consisting of the totality of collections $(\beta_1, \beta_2, \cdots, \beta_n, \cdots)$, then the inferior limit of $\pi$ is equivalent to the inferior limit of $\omega$ (the subscripts $1, 2, \cdots, n, \cdots$ are now no longer to be viewed as ordinal numbers, but merely as distinguishing marks).

The inferior limit $p_\omega$ of the collection $\omega$ is, in fact, an element such that $F(\omega \mid p_\omega)$, while if $b_n^{(k)}$ is any element of any collection $\beta_n$ in the set $(\beta_1, \beta_2, \cdots, \beta_n, \cdots)$, $F(p_\omega \mid y b_n^{(k)})$. Meanwhile, if $p_n$ is the inferior limit of any collection $\beta_n$, then $F(\beta_n \mid p_n)$, while $F(p_n \mid b_n^{(k)} y)$ for every element $b_n^{(k)}$ of $\beta_n$. If $\pi$ is the collection $(p_1, p_2, \cdots, p_n, \cdots)$, then its inferior limit $p_\pi$ is an element such that $F(\pi \mid p_\pi)$, while, whatever $p_n$ may be selected, $F(p_\pi \mid p_n y)$. Since $F(p_n \mid b_n^{(k)} y)$, and $F(p_\pi \mid p_n y)$, it follows by 71, that $F(p_\pi \mid b_n^{(k)} y)$; and this latter assertion holds true, whatever element $b_n^{(k)}$ of any collection $\beta_n$

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may be selected, that is, whatever element of $\omega$ may be selected. Moreover, whatever element $p_n$ of $\pi$ may be selected, since $p_n$ is a resultant of $\beta_n$, $p_n$ is also a resultant of $\omega$ (88). Thus $F(\omega|p_n)$, whatever element $p_n$ of $\pi$ may be selected, while $F(\pi|p_\pi)$. Hence (by 57), $F(\omega|p_\pi)$. Thus $p_\pi$ is a resultant of $\omega$, while $p_\pi$ is such that $F(p_\pi|b_n^{(k)}y)$ for every element $b_n^{(k)}$ of $\omega$. Hence $p_\pi = p_\omega$.

The operation of determining the inferior limit of a collection $\omega$ which is composed of a set of collections $(\beta_1, \beta_2, \cdots, \beta_n, \cdots)$, is consequently *associative* with respect to the operation of separately determining the inferior limit of each one of these collections $(\beta_1, \beta_2, \cdots, \beta_n, \cdots)$.

By a precisely similar reasoning, one can obviously prove that the operation of determining the superior limit, $s_\omega$, of the collection $\omega$, is associative, in case we separately determine the superior limits of $\beta_1$, $\beta_2$, etc., and use such determination as the basis for determining $s_\omega$.

107. *Cross collections.* If a set of collections $(\beta_1, \beta_2, \cdots, \beta_n, \cdots)$ is given (where the multitude of collections in question is wholly unrestricted), and if a collection $\lambda$ is formed by selecting, at pleasure, one element, and one only (say $b_1^{(i)}$), from $\beta_1$, one element, and one only (say $b_2^{(j)}$), from $\beta_2$, and one element, and only one, from each of the collections of the set (so that, for instance, $b_n^{(r)}$ is selected from $\beta_n$), then the collection $\lambda_z$, that is, the collection $(b_1^{(i)}, b_2^{(j)}, \cdots, b_n^{(r)}, \cdots)$ shall be called a *cross-collection* of the set $(\beta_1, \beta_2, \cdots, \beta_n, \cdots)$. As many distinct cross-collections $(\lambda_t, \cdots, \lambda_x, \cdots, \lambda_z, \cdots)$ exist as there are distinct possible combinations of elements selected one from each of the collections of the set $(\beta_1, \beta_2, \cdots, \beta_n, \cdots)$. Choose now a constant base $y$, to be retained throughout what follows in this and in the next section. Hereupon, let $s_n$ be the superior limit with respect to $y$ of the collection $\beta_n$ of the original set. Let $(s_1, s_2, \cdots, s_n, \cdots)$ be the collection of all such superior limits, with respect to $y$, of $\beta_1$, $\beta_2$, etc. Let the collection $(s_1, s_2, \cdots, s_n, \cdots)$ be here called the collection $\vartheta$. Let the inferior limit of $\vartheta$, with respect to the base $y$, be symbolized by $p_\vartheta$. Next, let $p_z$ be the inferior limit of $\lambda_z$, that is, of the collection $(b_1^{(i)}, b_2^{(j)}, \cdots, b_n^{(r)}, \cdots)$. Let $(p_t, \cdots, p_x, \cdots, p_z, \cdots)$ be the collection comprising *all* such inferior limits of the possible cross-collections $(\lambda_t, \cdots, \lambda_x, \cdots, \lambda_z, \cdots)$. Let the collection $(p_t, \cdots, p_x, \cdots, p_z, \cdots)$ be called the collection $\psi$. Let the superior limit of $\psi$ be symbolized by $s_\psi$. Then we shall next inquire how $p_\vartheta$ is related to $s_\psi$, that is, how the inferior limit of the collection of all the respective superior limits of a given set of collections, is related to the superior limit of the collection of all the respective inferior limits of the corresponding cross-collections. To ask this question is to inquire (in the most general form possible) whether the operation of seeking inferior limits is *distributive* with reference to the operation of seeking the superior limits of given collections.

108. *The Distributive Law.* Using the conventions of the previous section,

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regarding the cross-collections, and retaining the symbols used, we are able to assert that

$$p_\vartheta = s_\psi .$$

For, by the definition of an inferior limit, whatever element $p_z$ we may select from the collection $\psi$, the element $p_z$ is such that $F(p_z | y b_n^{(r)})$ for every element $b_n^{(r)}$ which belongs to the cross-collection $\lambda_z$. And, by the rule according to which this collection $\lambda_z$ has been formed, there exists, in each of the original collections $\beta_n$, an element, $b_n^{(r)}$, for which the foregoing assertion is true. Meanwhile, if we consider the relation of the element $b_n^{(r)}$ to its own collection $\beta_n$, of the original set $(\beta_1, \beta_2, \cdots, \beta_n, \cdots)$, it follows, by the definition of a superior limit, that $F(b_n^{(r)} | s_n y)$. Since, then, the element $b_n^{(r)}$ is such that $F(p_z | y b_n^{(r)})$, that is, such that $F(b_n^{(r)} | p_z \bar{y})$, while $F(b_n^{(r)} | s_n y)$, we have, by the usual rule for the elimination of an element, $F(s_n y | p_z \bar{y})$; whence follows $F(p_z | s_n y)$. Since, whatever $p_z$ may be in question, that element of $\lambda_z$ which $\lambda_z$ has in common with any given collection $\beta_n$ may thus be eliminated, it follows that the relation $F(p_z | s_n y)$ holds of every element $p_z$ of the collection $\psi$, when the relation of $p_z$ to any element, whatever $s_n$, of $\vartheta$, is considered. Since, however, by the definition of a superior limit, $F(s_\psi | \psi)$ is true, while $F(p_z | s_n y)$ holds true of each element, $p_z$, of $\psi$, separately considered, in its relation to each element of $\vartheta$, separately considered, it follows, by 57, that, whatever $s_n$ be chosen, the pair $(s_n, y)$ is such that (1) $F(s_n y | s_\psi)$.

Furthermore, since, by the definition of a superior limit, whatever $p_z$ we may select $F(p_z | s_\psi y)$, while, by the definition of an inferior limit, $F(p_z | \lambda_z)$, it follows that, whatever $\lambda_z$ we may select, $F(\lambda_z | s_\psi y)$ is always true. Since this latter relation is general, and holds for every collection $\lambda_z$, without exception, we have, for every possible cross-collection, $F(b_1^{(i)} b_2^{(j)} \cdots b_n^{(r)} \cdots | s_\psi y)$; an assertion which remains invariant and true if, instead of selecting, from any one of the collections $(\beta_1, \beta_2 \cdots \beta_n \cdots)$ the element which here appears in the collection $\lambda_z$, we select instead any other of the elements of that same collection $\beta_1, \beta_2$, etc. Consider, hereupon, once more, the element $b_n^{(r)}$, here selected from the collection $\beta_n$. Let $\lambda_z^{(n)}$ be what the collection $\lambda_z$ becomes if that element in $\lambda_z$ which is derived from the collection $\beta_n$ is *omitted*. In other words, let $\lambda_z$ be so subjected to partition that $\lambda_z$ is the same collection as $(\lambda_z^{(n)}, b_n^{(r)})$. Then we have, from the foregoing, $F(\lambda_z^{(n)} b_n^{(r)} | s_\psi y)$, that is, by transfer, $F(b_n^{(r)} | s_\psi y \bar{\lambda}_z^{(n)})$; while this relation holds true whatever element $b_n^{(r)}$ is selected from $\beta_n$. Since, however, $s_n$ is such that $F(s_n | \beta_n)$, while for $b_n^{(r)}$ any element whatever of $\beta_n$ may be substituted, we have, by 57, the collection $(s_\psi, y, \bar{\lambda}_z^{(n)})$ such that $F(s_n | s_\psi y \bar{\lambda}_z^{(n)})$; so that, by transfer, $F(\lambda_z^{(n)} s_n | s_\psi y)$. It follows that, while, as before, $F(\lambda_z | s_\psi y)$ is true, this relation remains invariant if *we substitute for any element* $b_n^{(r)}$, *of* $\lambda_z$, *the corresponding superior limit*, $s_n$, *of that collection* $\beta_n$, *from which* $b_n^{(r)}$ *has*

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*been selected.* Such substitution may be accomplished in case of any element $b_1^{(i)}$, $b_2^{(j)}$, etc., independently of whether such a substitution has been made in case of any other element of $\lambda_s$.

For since both the assertions $F(\lambda_s | s_\psi y)$, and $F(\lambda_s^{(m)} s_m | s_\psi y)$, hold good in case an element $b_m^{(q)}$ of $\lambda_s$ is selected from any collection $\beta_m$ which is *not* the collection $\beta_n$, we could, by a repetition of the foregoing process of reasoning, show that the relation expressed in $F(\lambda_s | s_\psi y)$ not only remains invariant whichever one of the two elements, $b_n^{(v)}$ or $b_m^{(q)}$ is selected as that element for which the corresponding $s_n$ or $s_m$ is substituted; but *also* remains invariant when for both of them, simultaneously, the corresponding superior limits of $\beta_n$ and $\beta_m$ are substituted. This result can be extended, at pleasure, therefore, to any number, or to any partial collection, or to *all* of the elements of $\lambda_s$, without regard to their multitude. The relation $F(\lambda_s | s_\psi y)$ therefore remains invariant in case, for *each and every* member of $\lambda_s$, we substitute the corresponding element $s_n$, viz., the superior limit of that collection $\beta_n$ from which the member of $\lambda_s$ which is each time in question was itself selected. Hence carrying out this substitution, we have (2) $F(\vartheta | s_\psi y)$.

But we have seen above, by (1), that $F(s_n y | s_\psi)$ is true. By adjunction it follows from this that $F(\vartheta y | s_\psi)$ is true. Hence we have at once true the *two* assertions (2) $F(y | \vartheta \bar{s}_\psi)$ and $F(y | \bar{\vartheta} s_\psi)$. Hence $F(\vartheta \bar{s}_\psi | \bar{\vartheta} s_\psi)$. Hence (3) $F(\vartheta | s_\psi)$, by 54.

By (1), therefore, $F(s_n y | s_\psi)$ is true, whatever element $s_n$ of $\vartheta$ we may choose; while, by (3), $F(s_\psi | \vartheta)$, so that $s_\psi$ is a resultant of $\vartheta$. It follows that $s_\psi$ is an element such that it is a mediator between $y$ and every element of $\vartheta$ separately considered while $s_\psi$ is also a resultant of $\vartheta$. Hence, by the definition of an inferior limit, $s_\psi = p_\vartheta$; and the theorem is proved.*

109. *The second form of the distributive law.* Still retaining constant the base $y$, let $s_s$ be the superior limit of any cross-collection $\lambda_s$, as such collections were defined in 107. Let the collection of all the superior limits of the cross-

* The demonstration of the distributive law here given may be regarded as a generalization of KEMPE's treatment of the symmetrical resultants of triads; although this generalization involves considerations which are somewhat peculiar to the present form of the theory of conjugate resultants. When the relations of the $F$-collections are regarded as degenerating into the specialized but more familiar relation of antecedent and consequent, the proof of the distributive law becomes subject to those difficulties whose treatment by Mr. C. S. PEIRCE, by SCHROEDER, and by Dr. HUNTINGTON, are summed up by Dr. HUNTINGTON in his *Sets of Postulates for the Algebra of Logic* (these Transactions, July, 1904). The difficulties in question are a test of the sort and of the amount of information which is surrendered when, instead of the $F$-relations viewed, so to speak, in their entirety, we confine ourselves to relations which are defined, for all the collections concerned, merely with reference to a common origin, and when we thereupon define the usual logical "sums" and "products" solely upon the basis of the antecedent-consequent relations. KEMPE's Theory, like the usual one, extends the distributive law by induction from pairs of triads to any number of cases. The present treatment, in this paper, applies the distributive law at once to collections of any multitude whatever.

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collections $(s_t, \cdots, s_x, \cdots, s_z, \cdots)$, be called the collection $\phi$. Let the inferior limit of $\phi$ be $p_\phi$. Let the inferior limit of any collection $\beta_n$ be $p_n$. Let the collection $(p_1, p_2, \cdots, p_n, \cdots)$ of all such inferior limits be the collection $\epsilon$. Let the superior limit of $\epsilon$ be $s_\epsilon$. Then

$$p_\phi = s_\epsilon.$$

For whatever element $b_n^{(r)}$ we may select from the cross-collection $\lambda_z$, $F(s_z y \mid b_n^{(r)})$ is true. And whatever collection $\beta_n$ we may select, and whatever $\lambda_z$ is in question, there exists in $\beta_n$ an element for which the assertion $F(s_z y \mid b_n^{(r)})$ is true. But, in case of the collection $\beta_n$, the assertion $F(y b_n^{(r)} \mid p_n)$ is true. Hence, since $F(b_n^{(r)} \mid p_n \bar{y})$ and $F(b_n^{(r)} \mid s_z y)$, it follows that $F(s_z y \mid p_n)$; an assertion which is true of every $p_n$ in its relation to every $s_z$. From $F(s_z y \mid p_n)$, follows $F(s_z \mid \bar{y} p_n)$, an assertion which is true of every element $s_z$ of the collection $\phi$, while $F(\phi \mid p_\phi)$. Hence the collection $(\bar{y}, p_n)$ is such that $F(\bar{y} p_n \mid p_\phi)$, or (1) $F(p_n \mid p_\phi y)$; an assertion which again holds true whatever $p_n$ of $\epsilon$ may be selected.

Furthermore, since $F(p_\phi \mid y s_z)$, whatever $s_z$ may be selected, while, by the definition of $s_z$, $F(s_z \mid \lambda_z)$, we obtain, by the usual elimination process, $F(p_\phi \bar{y} \mid \lambda_z)$ or $F(p_\phi \bar{y} \mid b_1^{(i)} b_2^{(j)} \cdots b_n^{(r)} \cdots)$. This relation remains invariant whatever element of $\beta_1$ be substituted for $b_1^{(i)}$; whatever element of $\beta_2$ be substituted for $b_2^{(j)}$; whatever element of $\beta_n$ be substituted for $b_n^{(r)}$; and so on. If, as in 108, $\lambda_z^{(n)}$ be what $\lambda_z$ becomes when $b_n^{(r)}$ is omitted, we have, by transfer of the entire collection $\lambda_z^{(n)}$, $F(p_\phi y \bar{\lambda}_z^{(n)} \mid b_n^{(r)})$; and this holds true for every element $b_n^{(r)}$ of $\beta_n$, separately considered. But $F(p_n \mid \beta_n)$. Hence the collection $(p_\phi, \bar{y}, \bar{\lambda}_z^{(n)})$ forms an $F$-collection with every element of $\beta_n$, separately considered, while $F(p_n \mid \beta_n)$; and hence $F(p_\phi \bar{y} \bar{\lambda}_z^{(n)} \mid p_n)$; that is, $F(p_\phi \bar{y} \mid \lambda_z^{(n)} p_n)$. And thus for any element $b_1^{(i)}$, $b_2^{(j)}$, $b_m^{(q)}$, $b_n^{(r)}$, of $\lambda_z$, there may be substituted, either simultaneously with or independently of, any of the other elements, the inferior limit, $p_1, p_2, p_m, p_n$ of the collection $\beta_1, \beta_2, \beta_m, \beta_n$, from which the element in question is selected. If the substitution is effected simultaneously for all the elements of $\lambda_z$, it follows that $F(p_\phi \bar{y} \mid p_1 p_2 \cdots p_n \cdots)$; that is, by transfer (2) $F(p_\phi \bar{\epsilon} \mid y)$.

But by (1) $F(p_n \mid p_\phi y)$. By the adjunction of all the elements of $\epsilon$ besides $p_n$ this becomes $F(\epsilon \mid p_\phi y)$; that is $F(\epsilon \bar{p}_\phi \mid y)$. By the elimination of $y$ it follows, from (1) and (2), that $F(\epsilon \bar{p}_\phi \mid p_\phi \bar{\epsilon})$. Hence $F(\epsilon \mid p_\phi)$. The element $p_\phi$ is so related to $\epsilon$ that $p_\phi$ is a resultant of $\epsilon$, while, by (1), whatever element $p_n$ of $\epsilon$ be selected, $F(p_n \mid p_\phi y)$. Hence $p_\phi$ is equivalent to the superior limit of $\epsilon$, and:

$$p_\phi = s_\epsilon,$$

as was to be proved.

Thus the operation of seeking the superior limits is distributive with reference to the operation of seeking the inferior limits of given collections.

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110. The results of 91–109 may now be summarized in the somewhat less general, but (by virtue of some of our best established mental habits) more easily apprehended form of a series of statements concerning the antecedents and consequents of the members of one or more collections.

Let an element $y$ be chosen, and held constant, *both as the base and as the origin with reference to which antecedents and consequents* are to be determined. What we have shown is:

111. That if a given collection $\beta$ be considered, then, whatever element $c$ is an antecedent, with respect to $y$, of every element of $\beta$, separately considered, is also an antecedent, with respect to $y$, of every resultant of $\beta$ (see 91). Moreover (by 101; see what is there said concerning the collection $\gamma'$), whatever element $c'$ is a consequent of every element of $\beta$, is also a consequent of every resultant of $\beta$. If we consider: (I) the collection $\gamma$, consisting of all elements $c$, each of which is an antecedent of every resultant of $\beta$; (II) the collection $\rho$, consisting of all the resultants and of all the elements of $\beta$; and (III) the collection $\gamma'$, consisting of all the elements $c'$, each of which is a consequent of every element of $\beta$: then each of these three collections is a collection which includes all of its own resultants. Each therefore is an internally complete or "perfect" collection.

112. If $\beta$ includes $y$, or an element equivalent to $y$, then the collection $\gamma$ reduces to the single element $y$ itself. If $\beta$ includes $\bar{y}$, or any element equivalent to $\bar{y}$, then the collection $\gamma'$ reduces to $\bar{y}$. If $\beta$ is an $O$-collection, the same result obtains for both $\gamma$ and $\bar{\gamma}$; that is $\gamma$ reduces to $y$ and $\gamma'$ to $\bar{y}$ (95), all the equivalents of an element being here regarded as represented by that element.

113. The collections $\gamma$ and $\rho$ have an element in common (101). This element is what we have called $p$, the inferior limit of $\beta$ with respect to $y$. Whatever element $\gamma$ and $\rho$ have in common is equivalent to $p$. The collections $\rho$ and $\gamma'$ have an element in common, viz., $s$, the superior limit of $\beta$ with respect to $y$. Whatever element $\gamma'$ and $\rho$ have in common is equivalent to $s$. If all the elements of $\beta$ are mutually equivalent, for instance, if they are all equivalent to $b$, then $b = p = s$. Otherwise, $p \neq s$. The elements $p$ and $s$ remain invariant in case elements of $\beta$ are repeated, any multitude of times, or in case any resultants of $\beta$ are adjoined to $\beta$. Moreover $p$ remains invariant whatever elements of $\gamma'$ are adjoined to $\beta$; and $s$ remains invariant, whatever elements of $\gamma$ are adjoined to $\beta$ (103, 104).

114. The element $p$ is definable as an element which is: (I) An antecedent of every element (and so of every resultant) of $\beta$; and (II) an element such that, whatever element of $\Sigma$ is an antecedent of every element of $\beta$, is also an antecedent of $p$. Or again, $p$ may be defined as that antecedent, of every resultant of $\beta$, which is also itself a resultant of $\beta$. Or finally, $p$ may be defined as that antecedent of every element of $\beta$ which is itself a consequent of every element $c$ which agrees with $p$ in being an antecedent of every element of $\beta$.

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115. The element $s$ is definable as an element which is: (I) A consequent of every element (and so of every resultant) of $\beta$; and (II) an element such that, whatever element of $\Sigma$ is a consequent of every element of $\beta$, is also a consequent of $s$. Or again, $s$ may be defined as that consequent, of every resultant of $\beta$, which is also itself a resultant of $\beta$. Or, finally, $s$ may be defined as that consequent of every element of $\beta$ which is itself an antecedent of every element $c'$ which agrees with $s$ in being a consequent of every element of $\beta$.

116. If $\beta$ is an $E$-collection, no one of whose elements is equivalent to $y$, and no one of whose elements is equivalent to $\bar{y}$, while $\beta$ itself contains at least one pair of non-equivalent elements, then, by 89, and by the reasoning of chapter V, the three collections, $\gamma$, $\rho$ and $\gamma'$, contain, each of them, an infinite number of elements.

117. Of the various collections here in question a principle holds true which is statable in general, on the basis of the foregoing, as a consequence of principle VI, and as holding throughout the system $\Sigma$: If two collections are such that one of them (say $\gamma$) includes all those elements of $\Sigma$ each of which is an antecedent with respect to a given origin $y$, of every element of the other collection (say $\beta$), then there exists in $\Sigma$ an element (the inferior limit, $p$, of $\beta$, with respect to $y$), which is at once a member of the collection $\gamma$, and also a resultant of $\beta$. Or again: If two collections $\alpha$ and $\beta$ exist, such that every element of $\alpha$ is an antecedent, with respect to $y$, of every element of $\beta$, then there exists at least one element of $\Sigma$ which is a consequent of every element of $\alpha$, and which is also an antecedent of every element of $\beta$. The superior limit of $\alpha$ with respect to $y$, and the inferior limit of $\beta$ with respect to $y$, both of them stand in this position. If they are not mutually equivalent, all the elements which are their mediators agree in possessing the character in question. Another and more restricted form of the same principle runs thus: Whatever infinite sequence $\kappa$, consisting of elements of $\Sigma$, is so definable that, with reference to a chosen origin, every element $k_r$ of the sequence possesses a consequent $k_t$ which also belongs to the sequence, there also exists in $\Sigma$ an element which is a consequent of every element of $\kappa$. For $\kappa$ is a collection of elements of $\Sigma$, and consequently possesses, by the foregoing, a superior limit, with respect to $y$, which also belongs to $\Sigma$. The chains of elements, defined in chapter V, consequently all of them possess superior limits belonging to $\Sigma$.*

118. It is now possible, without further difficulty, to point out that the elements of $\Sigma$ possess the properties of a system of logical classes, or of entities to which the ordinary algebra of logic applies. Let the arbitrarily assumed origin

* The definite relation thus brought out between the conceptions of logical products and sums, and the conception of limits, is, so far as I know, a new feature of the present discussion. It is brought to light by defining, from the outset, these conceptions with reference to collections of unrestricted multitude.



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$y$ be taken as the 0 of the ordinary algebra of logic. Let $\bar{y}$ be taken as the 1 of that algebra (otherwise symbolized, by some, as $\infty$). Let the relation $p \prec_y q$ be regarded as the usual relation of logical antecedent and consequent; and let the subscript of the symbol $\prec_y$ be dropped, by virtue of that usual convention which regards the reference to 0, not as reference to an arbitrary origin, but as such that $a \prec b$ has an invariant or absolute sense. So regarded, the system $\Sigma$ possesses an element, 0, such that whatever element $x$ be chosen, $0 \prec x$; and also an element, 1, such that whatever element $z$ be chosen, $z \prec 1$. The relation $\prec$ is transitive. If $a \prec b$ and $b \prec a$, then $a = b$. If $a \prec b$, and also $a \neq b$, the relation $\prec$ is asymmetrical. Elements such as $p$ and $s$ may first be viewed as determined by some given pair of elements, e. g., by the pair $(a, b)$. The element $p$ is then called the *product*, the element $s$ is called the *sum* of this pair; and, in the usual symbols, one may write

$$ab = p; \qquad a + b = s.$$

The definitions of the operations of logical multiplication and of logical addition, may assume the form explained in 114 and 115. Obverses will now appear as elements each of which is what is ordinarily called the negative of the other. Since, in fact, by 95 and 112, the product of an $O$-collection is the origin, and its sum is the obverse of the origin (see also 80, 82), the obverse elements $a$ and $\bar{a}$ are such that $a + \bar{a} = 1$, while $a\bar{a} = 0$. We shall have the known results (easily verifiable on the basis of the foregoing):

$$a + a = a; \qquad aa = a; \qquad a0 = 0; \qquad a + 0 = a;$$

$$a \cdot 1 = a; \qquad a + 1 = 1; \qquad \overline{a + b} = \bar{a}\bar{b}; \qquad \overline{ab} = \bar{a} + \bar{b};$$

$$(ab)c = a(bc); \qquad a(b + c) = ab + ac; \qquad a + bc = (a + b)(a + c).$$

Not only are these results predetermined by the foregoing discussion, but we have in fact given to the principles in question a form much more general than the usual form by so stating the principles from the start that they apply to logical operations upon collections possessing any multitude whatever.

It follows then, that the usual algebra of logic applies without restriction to the system $\Sigma$, which is in so far identical with a totality of logical classes, whereof an infinity are mutually non-equivalent, while all are capable of an unrestricted combination by the operations of logical addition and of logical multiplication.

It is worthy of note that, in terms of the ordinary algebra, the conjugate resultants of a given collection may be defined as follows: Let the logical product of a collection $\beta$ of logical elements be $p$. Let $s$ be the sum of $\beta$. Then any element $q$ such that $p \prec q \prec s$ is a resultant of $\beta$. If $r$ is a resultant of $\beta$

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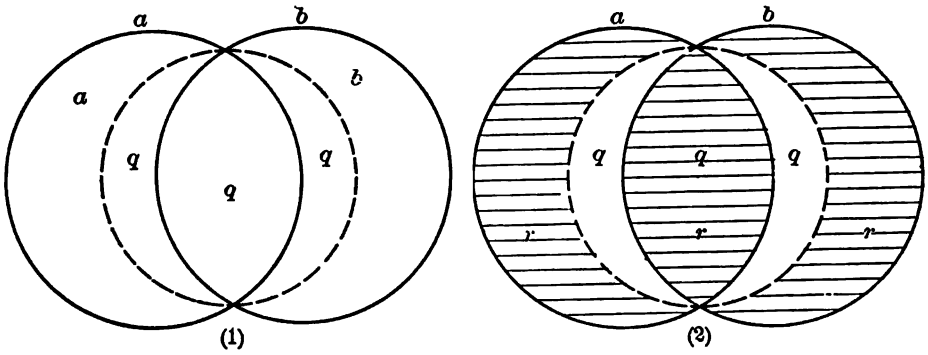
such that $qr = p$, while $q + r = s$, then $q$ and $r$ are conjugate resultants of $\beta$. If $\beta$ is the pair $(a, b)$ represented in the diagram (1), and if $q$ is a resultant of $(a, b)$ so that $q$ includes the common part of $(a, b)$, but does not extend beyond the limits of $a + b$, then, by repeating $(a, b)$ in the diagram (2), we may indicate, by shading, the portion of the repeated diagram where $r$ lies, and




so the extent of $r$, the conjugate resultant of $q$, in the pair $(a, b)$. It will be observed that $q$ and $r$ have the product $ab$ in common, but supplement each other as to the remainder of $a + b$. If $q$ expands so as to coincide with the whole of $a + b$, $r$ shrinks to $ab$, and conversely.

The negatives $\bar{q}$ and $\bar{r}$ are the elements whose existence is directly asserted in principle VI.

## Chapter VI. The system $\Sigma$ as a generalized space-form.

119. The inquiry of the previous chapter was primarily devoted to determining what elements exist in $\Sigma$, and how they are arranged. As an incident to this research, the relations of our system $\Sigma$ to the system defined in the algebra of logic was developed. But the consequences of principles V and VI, in their combination, have still other aspects. In particular, the properties of the system $\Sigma$ to which we have already called attention, make its array analogous to that of the points of space. This we are next to see.

120. By definition, all of the elements of any collection are resultants of that collection. But if a collection $\beta$ contains at least one pair of non-equivalent elements, and if $\beta$ at the same time comprises only a finite collection of the resultants of some one of those pairs of non-equivalent elements which $\beta$ contains, then there exists an infinite collection of resultants of $\beta$ such that these resultants are not themselves elements of $\beta$. This appears from 89, in combination with principle V. If $\beta$ is an $O$-collection, all the elements of $\Sigma$ are resultants of $\beta$, and are also complements of $\beta$. If $\beta$ is not an $O$-collection, $\Sigma$ contains an infinity of elements which are not resultants of $\beta$. For the

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obverses of the resultants of $\beta$ can none of them be resultants of $\beta$; since if $F(\beta|x)$ and $F(\beta|\bar{x})$ are both of them true, $O(\beta)$. The obverses of the resultants of $\beta$ are, by definition, complements of $\beta$. But if $\beta$ is not an $O$-collection, $\Sigma$ also contains an infinity of elements which are *neither* complements nor resultants of $\beta$. For let $(a, b)$ be any pair of conjugate resultants of $\beta$, so that $J(ab; \beta)$. Since $\beta$ is not an $O$-collection, $E(ab)$ is true (by 95). Every resultant of $\beta$ is a mediator of the pair $(a, b)$. Since $E(ab)$, $a \neq \bar{b}$ and $b \neq \bar{a}$. No element $x$ such that $F(x|\bar{a}b)$ is true, and no element $x'$, such that $F(x'|a\bar{b})$ is true, can be a resultant of $\beta$ unless $x = b$ or $x' = a$. Since the complements of $\beta$, being obverses of the resultants of $\beta$, are mediators of the pair $(\bar{a}, \bar{b})$ we can thus define an infinity of elements which are neither complements of $\beta$ nor resultants of $\beta$. All elements, $x$, such that $x \neq \bar{a}$, $x \neq b$ and $F(x|\bar{a}b)$; and all elements $x'$, such that $x' \neq a$, $x' \neq \bar{b}$ and $F(x'|a\bar{b})$, are, namely, neither complements nor resultants of $\beta$. An analogous assertion holds for any other of the pairs of conjugate resultants of the col lection $\beta$. Whatever pair of elements $(a, b)$ we may choose, an infinity of pairs of elements of the form $(a, \bar{b})$, $(b, \bar{a})(c, \bar{d})$, $(d, \bar{c})(e, \bar{f})$, $(f, \bar{e})$, etc., are thereby determined, such that each pair consists of some resultant $c$, $d$, $e$ of the pair $(a, b)$, while the other member of each pair is the obverse of the corresponding conjugate resultant of $(a, b)$, No mediator of any one of the pairs $(a, \bar{b})$, $(c, \bar{d})$, $(\bar{e}, f)$, etc. (except $a$, or $c$, or $f$, as the case may be), is a resultant of the pair $(a, b)$.

If $a, b, c, d, e, f$, are mutually non-equivalent elements, while the pairs $(a, b)$, $(c, d)$, $(e, f)$, are each of them pairs of conjugate resultants of the same pair, or of the same collection $\beta$, then no resultant of $(\bar{a}, b)$, or of $(a, \bar{b})$, or of $(c, \bar{d})$, or of $(\bar{c}, d)$, or of $(e, \bar{f})$, can be equivalent to any resultant of the other pairs thus defined. If $x$, for instance, is a mediator of $(c, \bar{d})$, so that $F(x|c\bar{d})$, $F(dx|c)$, then, with respect to $x$, taken as a base, the conjugate limits of $(a, b)$, or of the collection $\beta$, in question, are, respectively $c$ and $d$; $c$ being the inferior and $d$ the superior limit with respect to the base $x$, of the collection $\beta$. If therefore $x'$ is such that $F(x'|e\bar{f})$, $x'$ can be equivalent to $x$ only in case each of these elements $x$ and $x'$, taken as base, determines, in the manner shown in 100, 101, the same pair of conjugate limits of $\beta$; in which case $c = e$, $d = f$. For if $F(dx|c)$, and $F(xf|e)$, where $(c, d)$, and $(e, f)$ are conjugate pairs, this result follows.

121. If we begin afresh, with a pair $(a, b)$, and then choose a base, $y$, such that $F(y|a\bar{b})$, it is thus plain that, for this base $y$, $a$ is the inferior, and $b$ is the superior limit of the collection of the resultants of $(a, b)$, and that this choice of inferior and superior limits for $(a, b)$, *remains invariant for any base that is a mediator* of $(a, \bar{b})$, while, if the base is changed to some other pair $(c, \bar{d})$, $(d, \bar{c})$, etc.—$(c, d)$ being a pair of conjugate resultants—the inferior

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and superior limits of $(a, b)$ (the products and sums of the ordinary algebra of logic) vary accordingly. The totality of the expressions employed in the ordinary algebra of logic to represent the relations of a system of classes, will remain invariant as to certain values, and undergoes, for other values, perfectly definite transformations, in case the base with reference to which products and sums are reckoned is altered, so that some class $y$ takes the place which has been assigned to the zero of the ordinary algebra. If the base is, for instance, changed from some element $y'$, such that, $F(y' \mid a\bar{b})$, to some element $y$, such that, $F(y \mid \bar{a}b)$, the product of the pair $(a, b)$ is transformed into what was formerly its sum, and the sum into what was formerly the product. If the new base is an element $y$ such that $F(y \mid \bar{a}\bar{b})$, the product and sum of $(a, b)$ are transformed into a new pair of the conjugate resultants of $(a, b)$. These transformations, somewhat analogous, for the algebra of logic, to a transformation of coördinates in a space system, lead to results which are predetermined by the $F$-relations of the elements of the system $\Sigma$. However the base may be changed, the product of any $O$-collection will be equivalent to the new base; the sum of an $O$-collection will be equivalent to the obverse of the base; and so on.

Meanwhile, any pair $(a, b)$, such that $E(ab)$, while $a \neq b$, forms a means of an exhaustive classification of the elements of $\Sigma$. Given, namely, any element of $\Sigma$, say $x$, there is some determinate pair of resultants of $(a, b)$, say the pair $(k, l)$, such $F(x \mid k\bar{l})$. To the resultants of the pair $(k, \bar{l})$ belongs one resultant, $k$ of $(a, b)$, and there are also an infinite number of possible bases, for which $k$ is product and $l$ sum, of $(a, b)$. No element of $\Sigma$ belongs at once to two of the distinct classes thus defined by selecting pairs of conjugate resultants $(i, j)$, $(k, l)$, such that $i \neq k$, $i \neq l$, etc., and by then defining the class of the resultants of $(k, \bar{l})$ and of $(\bar{k}, l)$.

Or, again, one may express our present result by saying that if an element $x$ is *not* a resultant of the pair $(a, b)$, then there exists one and only one pair of conjugate resultants of $(a, b)$, namely the pair $(k, l)$, such that, if the obverse of a determinate one, say $k$, of these two conjugate resultants of $(a, b)$ be chosen, $F(x \mid \bar{k}, l)$, while $x \neq l$.

122. If a pair of elements $(a, b)$ be chosen such that $a \neq b$, and $E(ab)$, it is always possible to find a pair of resultants, $(q, r)$ of $(a, b)$, such that $q$ and $r$ are not mutually conjugate resultants of the pair $(a, b)$, while $q \neq r$, and $E(qr)$, and while $F(a \mid qr)$ and $F(b \mid qr)$ are both of them false. In order to construct such a pair, it is only necessary to choose any resultant $q$, of the pair $(a, b)$, such that $q \neq a$, $q \neq b$, and then to determine $r$ such that $r \neq q$, $r \neq b$, and $F(r \mid qb)$. In this case, since $F(q \mid ab)$, and $F(r \mid qb)$, it is impossible that $F(a \mid qr)$. For if $F(a \mid qr)$ and $F(r \mid qb)$, it follows that $F(a \mid qb)$; while since, at the same time $F(q \mid ab)$, there results $q = a$, contrary to the hypothesis. Moreover, if $F(b \mid qr)$, and $F(r \mid qb)$, it follows that

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$r = b$; and this again is counter to the hypothesis. Hence $q$ and $r$ are both of them resultants of $(a, b)$, but neither $a$ nor $b$ is a resultant of $(q, r)$.

Consider, now, the collection $\rho$ of all the resultants of the pair $(a, b)$, and the collection $\rho_1$ of the resultants of the pair $(q, r)$. Every element of $\rho_1$ belongs to $\rho$. But there exists an infinity of elements of $\rho$ such that no one of them either belongs to $\rho_1$ or is equivalent to any of the elements of $\rho_1$. Since the pair $(q, r)$ is again a pair of non-equivalent elements which is not an $O$-pair, it is possible to determine new pairs $(s, t)$, $(u, v)$, $(w, x)$, etc., without limit, such that the resultants of these pairs form a series, or chain, of collections, $\rho, \rho_1, \rho_2$, whereof each collection is wholly inclusive of all the elements of each later collection, while each collection contains an infinity of elements that are not included in the later collections, and that are equivalent to none of the elements so included.

It is, in the reverse direction, possible to include any collection $\rho$ of the resultants of a given pair (so long as this is not an $O$-pair), in some more inclusive collection, $\rho'$, which then may be enlarged, if necessary, to the collection of the resultants of some new pair, by considering any of the pairs of conjugate resultants of $\rho'$.

123. The structure of $\Sigma$ is, therefore, such as to permit this endless determining of internally complete systems of resultants within systems, every such collection comprising an infinite set of elements. This being the case, the question arises whether there is also any sense in which the system $\Sigma$ may be said to possess a "dimensionality" resembling that of space. The answer is that such a conception, in the system $\Sigma$, is capable of arbitrary definition in an infinite number of ways. And such a way, in fact, is suggested by the relation of any inclusive system $\rho$ of the resultants of a pair $(a, b)$, and any included system such as the collection $\rho_1$ of resultants of the pair $(q, r)$ defined above.

Suppose, namely, that we arbitrarily define the collection of the resultants of the pair $(q, r)$ as a one-dimensional collection, simply because the totality of these resultants is determined by the naming of the single pair of elements $(q, r)$. In precisely the same sense, it would appear that the resultants of $(a, b)$ or any other pair might be regarded as also of one dimension. But if we consider more carefully, it is plain that the following reason appears for a distinction between the systems $\rho$ and $\rho_1$. Let $m$ be any resultant of $(q, r)$, such that $F(qr|m)$ while $m \neq q$, $m \neq r$. In $\rho$, that is, with respect to $(a, b)$, $m$ possesses a conjugate resultant $n$, such that $J[mn; (a, b)]$, i. e., $J(mn; \rho)$. Now it is plain that $F(qr|n)$ is false. For if $F(qr|n)$ and $F(qr|m)$ were both at once true, we should have, by 73, every element of $\rho$ a mediator of $(q, r)$, and so $(q, r)$ would be a pair of conjugate resultants of $(a, b)$, which is contrary to the construction as stated in 122. Consider the triad $(q, r, n)$. Since $F(qr|m)$, any resultant of the pair $(m, n)$ is a resultant of $(q, r, n)$,

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as can readily be shown by the usual elimination-process. Hence any element of $\rho$ is a resultant of the triad $(q, r, n)$. Hence the triad $(q, r, n)$ possesses resultants which are not resultants of any of the pairs $(q, r)$, $(r, n)$, $(q, n)$. The triad $(q, r, n)$ resembles then a triangle, or two-dimensional complex, when viewed with reference to the pairs $(q, r)$, $(r, n)$, $(q, n)$. Thus $\rho$ can be viewed as a two-dimensional complex in relation to $\rho_1$. An analogous result holds whatever pair $(a, q)$, $(b, r)$, etc., we choose from the resultants of $\rho$, so long as the resultants of this selected pair form only a portion of the resultants of $\rho$, while elements equivalent to none of the resultants of the selected pair belong to $\rho$.

But we are not limited in our selection to the whole system $\rho$, in order to be able to define such triads. Consider next the triad $(q, b, n)$. By 122, $q$ and $r$ have been so defined that $F(r|qb)$. Since, by construction, $F(m|qr)$, it follows that $F(m|qb)$. Were $n$ also such that $F(n|qb)$, every resultant of $(a, b)$ would be also a resultant of $(q, b)$, which is false by construction. Hence $F(n|qb)$ is false. Were $F(q|bn)$ true, then since, as just shown, $F(m|qb)$, we should have true $F(m|bn)$, and hence, since $F(b|mn)$, it would follow that $b = m$, which is impossible by construction. For $F(m|qr)$, while $m \neq q$, and $m \neq r$. Finally, if $F(b|qn)$ were true, then, since $F(m|qb)$, it follows that $F(m|qn)$, and hence, since $F(q|mn)$, it would follow that $q = m$, which is again false by construction.

Therefore, no one of the elements of the triad $(q, b, n)$ is a mediator of the other pair. The conjugate resultant, *in this triad*, of the element $n$, is an element which is a mediator of the pair $(q, b)$ (by 98); and hence, since, $m$, a mediator of $(q, r)$, is the conjugate resultant of $n$ in $\rho$, it is impossible that the resultants of the triad $(q, b, n)$ exhaust the collection $\rho$. Meanwhile, the triad $(q, b, n)$ possesses resultants which are not resultants of any one of the pairs $(q, b)$, $(b, n)$, $(q, n)$. And so the triad $(q, b, n)$ may be viewed as a two-dimensional complex.

It thus follows *both* that the resultants of $\rho$, taken as a whole, can be viewed as the resultants of a triad, if we choose, rather than as the resultants of a pair; and that triads such as $(q, b, n)$ can be defined, in $\rho$, in such wise that a triad $(q, b, n)$ possesses resultants which are not resultants of any of its single pairs, and which are still but a part of the resultants of the system $\rho$. Any such triad, however, may be viewed as a two-dimensional structure.

The viewing of $\rho$ as a two-dimensional complex with reference to $\rho_1$ as a one-dimensional complex, is typical of a process which can be repeated any number of times. For, since $\rho_1$ is itself inclusive of $\rho_2$, etc., $\rho_1$ may be viewed, with reference to these included systems, as a complex possessing two, three, or $n$ dimensions, where $n$ is any whole number. According as this is done, $\rho$ comes to be viewed, with reference to a particular series of included collections, as of three, four, or $n + 1$ dimensions.

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The result of the foregoing considerations is that, within any portion of the system $\Sigma$ which contains at least one pair of non-equivalent elements, we can define, pairs, triads, etc., in brief, collections of any finite number of mutually non-equivalent elements, such that, if such a collection, say $\alpha$, possesses $n$ elements, there exist resultants of the whole collection which are not resultants of any partial collection of the elements, containing only $n-1$, or $n-2$, or any less number of these elements themselves.

We may call the complexes of the resultants of such collections $n$-dimensional complexes. But it is observable that any such complex, once given, may also be treated, by the proper choice of conjugate resultants, as a complex of the resultants of a single pair, and so as a one dimensional complex. So that all such dimensionality is entirely relative to processes and structures of the type that we have just been defining.

124. Such structures become, however, of a more positive significance if we take account of the following application.

By a *line* shall be meant a structure of the general type of the chains of 83, only completed by the insertion of certain mediators. A line shall be a collection of elements such that in case of any triad of the elements of the collection, one member of this triad is the resultant of the pair composed of the other two.

And, in particular, the lines that we are here first and mainly to consider are to be subjected to the entirely arbitrary restriction (foreign to the first principles of our system $\Sigma$, but quite capable of being satisfied by a due selection of its elements as their existence has now been established), that if any two non-equivalent entities of a line are given, *no other line*, in the set of lines that we are to consider shall at once contain both of these elements.* In other expression, let the collections which are to be called lines be so selected that, if $(a, b, c)$ is any triad of elements belonging to the same line, $F(abc)$ is true; while, if $F(apq)$ is true and $F(bpq)$ is true, and if at the same time $F(abp)$ is false, then we shall so select that $p = q$; so that if $(a, p, q)$ is a triad of elements belonging to one of the lines now to be selected, while $(b, p, q)$ is a triad belonging to another of these lines, and while $(a, b, p)$ is no linear triad at all, then we shall be required so to select that $p = q$.

125. That selections of this sort are possible the theory of the chains, as developed in 83 sqq., has already shown. Such chains as were there defined might be constructed, as we now may observe, intersecting one another any number of times. For if $(c, d)$ be any pair of elements belonging to a chain, the resultants of $(c, d)$ form no single chain, but lie in sets subject to principle VI, which demands the existence of conjugate resultants, not only in the collection of the resultants of the pair $(c, d)$ itself, but in every one of the countless collections of resultants of the pairs intermediate between $c$ and $d$, as

* The development is here wholly due to KEMPE's initiative.

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these pairs have been characterized, in their mutual relations, in 122, 123. It is possible, within the limits of any pair of non-equivalent elements $(c, d)$, to define any number of segments, that is of intermediate pairs $(p_1, p_2)$, $(p_2, p_3)$, $(p_3, p_4)$, etc., each of which consists of mediators of $(c, d)$, while all the elements concerned form triads such that $F(p_i p_j p_k)$ is true of any one of these triads. A chain, or rather a series of chains, can be run through such a series of intermediate pairs, according to any desired principle of selection from amongst the elements present in the various systems of resultants encountered. By virtue of the results stated in 117, a set of successive chains can be enlarged to a complete line, resembling perfectly, in its structure, a continuous geometrical line, by a mere insertion of intermediates and sets of intermediates. The special principle of selection assigned for the lines now to be considered will not only ensure that two lines have never more than one intersection, but in combination with the definition of a line will also exclude that degree of wealth of elements which forbids the arrangement of all the resultants of any pair of elements in $\Sigma$ in a single linear serial order. For in the system $\Sigma$ as a whole, if $m$ and $n$ are equivalent neither to $a$ nor to $b$ and are conjugate resultants of $(a, b)$, $F(m|ab)$ and $F(n|ab)$ are both true; while $F(m|nb)$ and $F(n|mb)$ are both false. The principle laid down for the selection of line-collections will therefore forbid the inclusion in a given line of more than a single pair of conjugate resultants of any one pair. Thus, if $c$ and $d$ belong to the line, $c$ and $d$ themselves will be conjugate resultants of their own pair. And the intermediate elements of the line will be in $\Sigma$ resultants of that pair. But no conjugate in $(c, d)$ of any such resultant of $(c, d)$ as belongs to the line, will lie in the line, except $c$ and $d$ themselves.

126. Since collections possessing the dimensional structure described in 123 exist in any region of $\Sigma$, it will always be possible to define systems of lines as follows: Let $(d, e, f)$ be any triad such that it possesses resultants not contained amongst the resultants of $(d, e)$, $(e, f)$, $(d, f)$. If $(d, e)$, $(e, f)$, $(d, f)$, belong to lines that are amongst those lines which are here in question, and if $x$ be any resultant of $(d, e, f)$ which is not a resultant of $(d, e)$, nor yet of $(e, f)$, nor yet of $(d, f)$, then it will be possible to regard $(x, d)$ or $(x, e)$, or $x$ united with any element of the lines $(d, e)$, etc., as constituting a new set of segments of lines. The result will be a two dimensional complex of elements. That method of construction of the $n$-dimensional collections or complexes of elements of $\Sigma$ which has been indicated, enables us to regard these complexes, with all the lines, segments, etc., which are involved, as possessing an extent and variety of elements such as to permit us to define new sets of elements beyond any segments or bounded complexes once defined. Therefore, in selecting elements for our present purpose, we may regard these new elements as extensions of the lines and other complexes, while the dimensionality of the

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complexes of lines which we may thus consider is subject altogether to our pleasure, under the conditions now in general laid down.

If, in consequence of the foregoing considerations, we compare the set of relations that we can thus define with the relations known to geometry, a natural method presents itself in the form of a juncture that may now readily be effected between our account and Dr. VEBLEN's *System of Axioms for Geometry*, (Transactions of the American Mathematical Society, July, 1904).

Dr. VEBLEN's expression "in the order $ABC$," becomes, in our terms, the assertion $F(b|ac)$. If we agree, in studying the constitution of our system of lines, to take explicit account only of non-equivalent elements, if we here call our elements *points*, and if we also adopt Dr. VEBLEN's definitions, his axioms appear in our statement as follows:

Dr. VEBLEN's first axiom covers our own principles III and IV, according to which our system contains a pair of elements. Axiom II of Dr. VEBLEN's set, interpreted in our terms, declares that if $F(b|ac)$, $F(b|ca)$. This needs for us, no comment. Axiom III asserts that, if $F(b|ac)$, then $F(c|ba)$ is false. Our own principles require that, in this case, $b = c$; and the axiom may therefore be regarded simply as excluding us from treating certain pairs of equivalent elements as distinct elements. This is merely a principle of selection. Axiom IV asserts that if $F(b|ac)$, then $a \neq c$. Our principles like KEMPE's, require that if $a = c$, $a = b = c$. Axiom IV, therefore, again excludes the regarding of certain equivalent elements as, for the present purpose, distinct. Axiom V of Dr. VEBLEN's set requires that if $a \neq b$, $c$ exists such that $F(b|ac)$. This principle is provided for by our principles, which show that every pair defined in any of our sets of lines may be viewed as included in larger systems possessing linear $F$-relations. Axiom VI defines the important, but for us, quite arbitrary principle that governs the selection of the line-elements: "If points $c$, $d$ $(c \neq d)$ lie on the line $(a, b)$, then $a$ lies on the line $(c, d)$." This agrees with our foregoing statement in 124. Only, with us, this is *merely* a principle of selection. Axioms VII, IX, relate to dimensionality, and demand points existent in triads and tetrads such as we have provided for in the foregoing. For us, such requirements are permitted by the system $\Sigma$ in an infinity of ways.

Axiom VIII, the "triangle transversal" axiom runs, in our terms, thus: If the triad $(a, b, c)$ is of the two-dimensional character described in 123, and if $d$ and $e$ exist such that $F(e|ac)$ and $F(c|bd)$, then $f$ exists such that $F(f|ab)$, and $F(fed)$.

This axiom for us is, if we grant a certain mode of selection of elements, a theorem, resulting from the theorem of 99 — an incidental result, as it may be called, of the theory of conjugate resultants. That is, theorem 99 secures the existence of elements which may be selected so as to verify axiom VIII.

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The theorem of 99 runs that if $F(\beta|\alpha)$, $\beta$ and $\alpha$ have at least one resultant in common.

By the hypothesis of axiom VIII, $F(c|e\bar{a})$ and $F(c|bd)$. Hence $F(e\bar{a}|bd)$. Hence follows $F(e\bar{d}|ba)$. By 99, $(e, \bar{d})$ and $(b, a)$ have in the system $\Sigma$ at least one resultant in common (they have in fact, in $\Sigma$ an infinite number in common). Call this resultant $f$. Then $f$ exists such that $F(f|ab)$, and $F(f|e\bar{d})$; i. e., $F(e|fd)$ so that (at least) $F(fed)$.* That the common resultants here in question should belong as points to the system of lines that we have selected from the system $\Sigma$, is itself a matter of the mode of selection used. The properties of the system $\Sigma$ simply insure the possibility of such a selection.

Axiom XI, Dr. VEBLEN's form of the postulate of continuity, is provided for by our own result, holding for the system $\Sigma$ in general, stated in 117. This result ensures the possibility of the continuity of the line-collections, in case we choose to select suitable sets, precisely as the same result ensures in the system $\Sigma$ as a whole, the existence of "products" and "sums." Axiom X, which limits Dr. VEBLEN's system to three dimensions, is for us a perfectly possible, but again quite arbitrary limitation; and the same can be said of the parallel line axiom XII, which concerns wholly the limitation of the selection of the lines admitted into a given system.

Our own "transversal" theorem, in 99, justifies, in terms of our principles, the remark made by Mr. KEMPE, upon the basis of his postulates, to the effect that any $F$-collection which contains a finite number $n$ of elements that belong to the sets selected as the lines of the foregoing discussion, represents a definite configuration of points in a space of $n - 2$ dimensions.

Thus $F(abc)$ implies that $(a, b, c)$ is a triad of points on one line. $F(ab|cd)$ is to be interpreted as follows: The pairs $(a, b)$ and $(c, d)$, lie by hypothesis, upon some selected pair of lines of our geometrical set. The problem is, how are these two lines to be related? The assertion $F(ab|cd)$ requires, by 99, that, in $\Sigma$, $x$ should exist such that $F(ab|x)$ and $F(cd|x)$. If then $x$ be viewed as one of the selected elements of the geometrical set in question, the assertion $F(ab|cd)$ may be viewed as the assertion that the lines through the segments $(a, b)$ and $(c, d)$ have in common a point of intersection which belongs to the mediators of $(a, b)$ i. e., to the points of the segment $(a, b)$, and also to the segment $(c, d)$. On the other hand, if $F(ab|x)$ and $F(cd|x)$ are given, our principles require that $F(ab|cd)$. So that this form of assertion defines a pair of intersecting lines. The assertion $F(x|abc)$ defines a resultant of the triad $(a, b, c)$. If this triad is to be viewed, in the way heretofore defined, as determining a complex of two dimensions, then $x$ is a point

* KEMPE's theory is explicitly based upon two forms of the transversal theorem, assumed at the outset. For this our statement of the theory substitutes the postulated existence of conjugate pairs. What KEMPE sets at the beginning we thus reach at the end.

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lying within the triangle $(a, b, c)$. If $F(abc|def)$, wherein all the elements are mutually non-equivalent, and wherein each triad is to be viewed as a two-dimensional complex, then $x$ exists common to these two areas, or two-dimensional complexes here in question. Thus all the intersection theorems of geometry may be stated in the form of the assertion of $F$-relations, with a due regard to the limitations of the classes of selected elements.

The principle of continuity is, for such a geometry, merely a principle of the selection of the elements, a principle which the system $\Sigma$ permits, but does not require to be carried out.

Instead of such systems of lines as have here been selected, systems of lines whereof any two have two, three or $n$ intersections, are perfectly permissible, so far as the system $\Sigma$ is concerned. The possibility of a free, but definite variation of space-forms in a infinite number of ways, is thus provided for by the system $\Sigma$; and the outlook for a basis for generalized space-conceptions is all the more attractive, since the structure of the system $\Sigma$, based as it is upon fundamental logical principles, makes a test of the logical possibility of any proposed geometry a perfectly definite task — namely the task of seeing whether $\Sigma$ actually contains complexes which are suitable to embody the desired space-form.

Since $O$-collections at once possess, as their resultants, all of the elements of $\Sigma$ at once, no definite view of their dimensional structure is any longer possible. Hence selections suitable for space-forms must exclude $O$-collections; and so, as KEMPE again points out, no geometrical set contains the obverse of any of the elements of the set. It follows that "spaces," defined in the foregoing way, always occur in $\Sigma$ in pairs, such that to any one space-form $\sigma$ there always corresponds a space-form, or collection $\bar{\sigma}$, constituted of the obverses of the elements of $\sigma$. These two space-forms are related, in KEMPE'S view, somewhat as two hemispheres.

Finally, since metrical relations can be reduced, in the known way, to ordinal relations, KEMPE has briefly pointed out (as mentioned in the introduction to this paper), that sets of the elements of $\Sigma$ can be so selected that operations corresponding to the addition and multiplication of the ordinary algebra of quantity, will enable us to select elements that may be viewed (with reference to certain arbitrarily assumed constant triads of reference-elements, i. e., bases), as the sums or as the products of given pairs of elements. Hence, *without introducing new elements*, the elements of $\Sigma$, if viewed in certain ways, enable us to define, not only the algebra of logic, but the algebra of quantity.

*Note on the independence of the six principles.*

That principle V is independent of the other principles is proved, in 84, by the assumption of a system $\Sigma'$ consisting of a single pair of obverses.

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That principle IV is independent is proved by the reasoning used in 35. For if we assume a system $\Sigma''$, *all* of whose monads and possible collections are to be defined as $O$-collections, while the system itself comprises any arbitrarily chosen number of elements, all the principles except IV and V are satisfied by the possibly existent collections of $\Sigma''$, while principle V is satisfied "vacuously," since no pair of non-equivalent elements exist. But in such a system principle IV is false; since all the elements are, by the definition of equivalence, mutually equivalent.

Principle III, and that principle alone, would be violated by an empty system; and that principle is therefore independent.

If, instead of the $O$-collections, we had used, as the basis of our account of the system $\Sigma$, the indeterminate $F$-collections of $\Sigma$, all the principles I, III, IV, V, VI would remain true if we viewed them as statements regarding indeterminate $F$-collections, and could therefore have been used as principles for the system of $F$-collections. But principle II is false if interpreted as applying to $F$-collections. For $F(xx)$ is always true, since $O(x\bar{x})$. But from $F(\eta x)$ does not follow $F(\eta)$. Hence principle II is independent of the other principles.

If we consider the class of those $O$-collections of the system $\Sigma$ which are either pairs or triads, but which contain no greater number of elements than three, we may call this class the class of the $O_1$-collections. For this class of collections, principle I is false, since the $O_1$-triads cannot be enlarged, by the adjunction of any new members. In order to apply principle II, the hypothesis of that principle must be read as applying to a collection $\beta$ which, in order that it should be an $O_1$-collection at all, must not exceed a triad. If $O_1(\beta)$, where $\beta$ is a pair or a triad (so that $O(\beta)$ is also true), and if $O_1(\delta b_n)$ is true, where $b_n$ is any one of the two (or three) elements of $\beta$, then $\delta$ itself (by the definition of the $O_1$-collections), cannot exceed a pair; otherwise $(\delta, b_n)$ would be a collection of more than three elements. Principle II then becomes equivalent to the assertion that, if $\beta$ is a pair or a triad, and if $\delta$ is a monad or a pair, if $O(\beta)$ is true, and if $O(\delta b_n)$ is true, then $O(\delta)$ is true. Hence principle II is true of the $O_1$-collections. Principles III and IV are obviously true of the elements of $\Sigma$, considered with reference to the $O_1$-collections. The equivalences and non-equivalences are, in fact (because of what is proved in 27–30), unchanged by the limitation of our view to the set of $O_1$-collections, since all equivalences and non-equivalences are already concerned in determining the relations of obverses. And principles V and VI, which require the existence of certain $O$-triads (such as are also $O_1$-collections), remain true, although the hypothesis of principle VI becomes limited, in its application, to the mention of the complements of pairs and of monads. Thus principle I is independent of the other principles.

To prove the independence of principle VI:

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Consider two pairs of mutually obverse collections of elements of $\Sigma$, viz., $\alpha$ and $\bar{\alpha}$, such that each is a line of elements, defined as follows:

(1) Every element $a_n$ of $\alpha$ is a mediator of a pair $(x, y)$ of elements of $\Sigma$, such that neither $x$ nor $y$ belongs to $\alpha$, while $E(xy)$ is true. Thus $F(a_n | xy)$ is true of every element $a_n$ of $\alpha$.

(2) Whatever triad of elements of $\alpha$ be chosen, say the triad $(a_m, a_n, a_r)$, $F(a_m a_n a_r)$ is true of this triad.

(3) Whatever element $a_n$ of $\alpha$ is chosen, $a_q$ and $a_r$ exist, belonging to the collection $\alpha$, and such that $F(a_q | x a_n)$ and $F(a_r | y a_n)$, while $a_q \neq a_n$, and $a_r \neq a_n$.

(4) No two distinct elements of $\alpha$ are mutually equivalent elements of $\Sigma$.

(5) Whatever pair $(a_q, a_r)$ be chosen from amongst the elements of $\alpha$, $a_m$ exists such that $F(a_m | a_q a_r)$ is true.

From this definition of the line $\alpha$, the properties of the obverse collection $\bar{\alpha}$ at once follow. The elements of $\bar{\alpha}$ are mediators of the pair $(\bar{x}, \bar{y})$. No element of $\bar{\alpha}$ can be equivalent to any element of $\alpha$; for if any mediator of $(\bar{x}, y)$ is equivalent to a mediator of $(x, y)$, then $O(xy)$ is true, which opposes condition (1). If then $c$ and $d$ be distinct elements, chosen in any way from the total collection $(\alpha, \bar{\alpha})$, $c \neq d$ by construction.

Whatever pair $(c, d)$ of elements of $(\alpha, \bar{\alpha})$ be chosen, it follows that there exists $g$, also belonging to $(\alpha, \bar{\alpha})$, and such that $g \neq c$, $g \neq d$, and $F(g | cd)$. For if $c$ and $d$ both belong to $\alpha$, the existence of $g$, as an element of $\alpha$, follows directly from condition (5). If $c$ and $d$ be both chosen from $\bar{\alpha}$, a precisely analogous result holds true. But if $c$ be chosen at random from $\alpha$, and $d$ from $\bar{\alpha}$, then let $c = a_n$, and let $d = \bar{a}_q$, where $\bar{a}_q$ is such that its obverse $a_q$ is such that $F(a_q | x a_n)$. Hereupon, choose in $\alpha$ an element $a_t$ such that $F(a_t | a_q x)$. This, by condition (3), is always possible. Since $F(a_q | x a_n)$ and $F(a_t | x a_q)$, it follows that $F(a_t | a_q \bar{a}_n)$ is true; hence $F(\bar{a}_t | \bar{a}_q a_n)$ is true; and thus, if $g = \bar{a}_t$, $F(g | cd)$ is true of an element $g$ which belongs to $(\alpha, \bar{\alpha})$. Similarly, were it true that $c = a_n$, and $d = \bar{a}_r$, where $\bar{a}_r$ is an element such that $F(a_r | y a_n)$, then we might choose $a_w$ such that $F(a_w | y a_r)$. This, by condition (3), is also possible. Eliminating $y$ we have $F(a_w | a_r \bar{a}_n)$; i. e., $F(\bar{a}_w | \bar{a}_r a_n)$; so that, if $g = \bar{a}_w$, we again have $g$ belonging to the total collection $(\alpha, \bar{\alpha})$, and such that $F(g | cd)$ is true. Since all pairs of elements $(c, d)$ thus chosen from $(\alpha, \bar{\alpha})$ are, by construction pairs of mutually non-equivalent elements; since $g$ always exists, belonging to $(\alpha, \bar{\alpha})$ and such that $F(g | cd)$ is true; and since $\bar{g}$ also belongs to $(\alpha, \bar{\alpha})$; it follows that the collection $(\alpha, \bar{\alpha})$ contains sufficient elements to satisfy, with respect to any pair of elements of $(\alpha, \bar{\alpha})$, the demands of principle V, without going beyond the elements of this collection $(\alpha, \bar{\alpha})$ itself. That is, whatever pair of elements of $(\alpha, \bar{\alpha})$ be chosen, an element of $(\bar{\alpha}, \alpha)$ exists which is no obverse of either of the elements of the pair, and which forms an $O$-triad when adjoined to the pair.

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Hereupon let us consider a system $\Sigma_\alpha$, which shall consist solely of the elements of the collection $(\alpha, \bar{\alpha})$. Let there be formed, of the elements of this system $\Sigma_\alpha$, collections which we shall call the $O_\alpha$-collections. These $O_\alpha$-collections shall be identical with those collections of the elements of $(\alpha, \bar{\alpha})$ which are $O$-collections in $\Sigma$. The $O_\alpha$-collections of the system $\Sigma_\alpha$ will now conform, by construction, to principles I–V of the system $\Sigma$.

But, in case of the $O_\alpha$-collections, principle VI will be violated. For (returning to the system $\Sigma$) consider any element $a_n$ of $\alpha$; and consider the totality of those elements $a_q$, belonging to $\alpha$, and such that $F(a_q | x a_n)$ is true in respect of the system $\Sigma$. Call this totality the collection $\alpha'$; so that $\alpha'$ is the collection of those elements of $\alpha$ which, in $\Sigma$, are mediators between $a_n$ and $x$. By condition (1), $x$ itself does not belong to $\alpha$, and so does not belong to $\alpha'$. Now, in the system $\Sigma$, the element $a_n$, since it belongs, by construction, to the collection $\alpha'$, possesses a conjugate resultant with respect to $\alpha'$. Let $r$ be this resultant, so that, in $\Sigma$, $\omega_r(a_n r;\ \alpha')$ is true. It is manifest that $r$ is an element such that $F(r | a_n x)$. Yet $r$ is no element of the collection $\alpha'$ or of the total collection $(\alpha, \bar{\alpha})$. For if $r$ were an element of $\alpha$, then, by conditions (3) and (4), $a_k$ would exist, belonging to $\alpha$, and such that $F(a_k | rx)$, while $a_k \neq r$, and $a_k \neq x$. But $a_k$ would also belong to $\alpha'$, and by the definition of a conjugate resultant $F(a_k | r a_n)$ would be true. Now $F(r | a_n x)$ is true; and from $F(a_k | rx)$ and $F(r | a_n x)$ follows $F(r | a_k a_n)$. If, at the same time, $F(a_k | r a_n)$, $a_k = r$; which is contrary to the hypothesis. Thus $r$ is no element of $\alpha$, and also cannot be any element of $\bar{\alpha}$. For since $F(r | a_n x)$, if there existed $\bar{a}_q$ such that $r = \bar{a}_q$, $F(\bar{a}_q | a_n x)$, and hence $F(\bar{a}_q | xy)$, would be true of some element $a_q$ of which $F(a_q | xy)$ would also be true. In that case $O(xy)$, which is contrary to condition (1).

Since $r$ does not belong to $\alpha$ nor yet to $\bar{\alpha}$, $r$ does not exist in the system $\Sigma_\alpha$. There is then, in $\Sigma_\alpha$, no element capable of meeting the requirements of principle VI as applied, in this system $\Sigma_\alpha$, to the partial collection $\alpha'$. So principle VI fails in $\Sigma_\alpha$; and is therefore independent of the other principles.

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# ON MULTIPLE INTEGRALS

BY

JAMES PIERPONT

## §1. *Introduction.*

In the theory of multiple proper integrals the chief difficulty arises in the consideration of the frontier points of the field of integration $\mathfrak{A}$. The older writers as GAUSS, DIRICHLET, JACOBI and CAUCHY, supposed the field $\mathfrak{A}$ to be bounded by a finite number of surfaces of simple nature. In later years the theory of point aggregates has made it possible to consider more general classes of fields. STOLZ, PRINGSHEIM † and JORDAN ‡ have made valuable contributions in this direction. The work of JORDAN is particularly important as he has not only blocked out the way for multiple integrals in any number of variables but has treated the case of two variables with great generality and completeness.

Two fundamental theorems in this theory relate to the reduction of a multiple integral to an iterated integral and the change of variables in a multiple integral. As far as the writer was aware a year ago when he began to consider the general treatment of these questions, none of the existing methods admitted an extension to $n$ variables, at least not without considerable complication. It occurred to him that by properly generalizing the definition of an integral the grave difficulties which the frontier points create might very readily be obviated.

The theory of integration which is here given, was presented by the author at the April meeting of the American Mathematical Society in 1904. At that time Professor MOORE called his attention to a paper of STOLZ's, § published in 1897, where not only the integrals considered in the present paper but also another class of integrals are introduced. The importance of the integrals first mentioned seems entirely to have escaped STOLZ's observation. At any rate he seems to have made no further use of them, and to have considered them more as curiosities. The only theorem he establishes is theorem 2 of the present paper.

* Presented to the Society, April 30, 1904. Received for publication May 19, 1905.

† Sitzungsberichte der Münchener Akademie, vol. 29 (1899), p. 39.

‡ Journal de Mathematiques, ser. 4, vol. 8 (1892), p. 69, and Cours d'Analyse, vol. 1.

§ *Zwei Grenzwerthe, von welchen das obere Integral ein besonderer Fall ist*, Sitzungsberichte der Wiener Akademie, vol. 106, abt. II, *a*, p. 453.



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## § 2. *Preliminary definitions.*

It is convenient to represent the points $x = (x_1, \cdots, x_m)$ of an $m$-way space $R_m$ by a set of points marked off on $m$ axes. Let us divide each of these axes into intervals of length $\leqq d$; in such a manner, however, that no finite segment of any axis contains more than a finite number of intervals. We shall say that we have effected *a rectangular division of space of norm* $d$. It divides $R_m$ into rectangular *cells* whose sides are of length $\leqq d$. The boundaries of the cells are planes parallel to the coördinate-planes. When the lengths of the intervals are all the same, these cells become cubes. The corresponding division may be called *cubical.*

Let $\mathfrak{A}$ be a limited point aggregate, i. e., let no coördinate of any point of $\mathfrak{A}$ be greater numerically than some fixed number.

Let $f(x_1, \cdots, x_m)$ be a limited one-valued function defined over $\mathfrak{A}$. Let $D$ be a rectangular division of space of norm $d$. Let $d_1, d_2, d_3, \cdots$ denote those cells of $D$ containing at least one point of $\mathfrak{A}$. Without ambiguity we may denote the volume of these cells by the same letters.

Let

$$M_i = \max f, \qquad m_i = \min f, \qquad \text{in } d_i;$$

and

$$|f(x_1, \cdots, x_m)| \leqq F, \qquad \text{in } \mathfrak{A}.$$

We shall set

$$\bar{S}_D = \sum_i M_i d_i, \qquad \underline{S}_D = \sum_i m_i d_i, \tag{1}$$

which may be called the *upper* and *lower sums* for the division $D$. It is sometimes necessary to consider such sums over different aggregates as $\mathfrak{A}, \mathfrak{B}, \cdots$. These may be indicated by

$$\bar{S}_{\mathfrak{A}_D}, \bar{S}_{\mathfrak{B}_D}, \cdots,$$

where the subscripts indicate that the sums (1) are taken over $\mathfrak{A}, \mathfrak{B}, \cdots$ respectively.

The difference $M_i - m_i$ is the oscillation of $f$ in the cell $d_i$. We denote it by

$$\operatorname{osc} f_i.$$

We remark here once for all that all functions and point aggregates considered in this paper are supposed to be limited. The case when either or both are unlimited will be considered in a later paper.

Let $a = (a_1, \cdots, a_m)$, $b = (b_1, \cdots, b_m)$ be two points. The distance between $a$ and $b$ is defined by

$$\operatorname{dist}(a, b) = \sqrt{(a_1 - b_1)^2 + \cdots + (a_m - b_m)^2}.$$



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If $c$ is any other point, we have

$$\text{dist}(a, c) \leqq \text{dist}(a, b) + \text{dist}(b, c).$$

Let $\mathfrak{B}$ be a partial aggregate of $\mathfrak{A}$. If the distance between any frontier point $\mathfrak{A}$ and any frontier point of $\mathfrak{B}$ is $\geqq \rho > 0$, we say $\mathfrak{B}$ is an *inner aggregate* of $\mathfrak{A}$: also $\mathfrak{A}$ is an *outer aggregate* of $\mathfrak{B}$.

If $\mathfrak{A}$ contains all its limiting points, it is *complete*. If it is complete and contains only limiting points, it is *perfect*.

A point aggregate embracing none of its frontier points is called a *region*.

The points $x$ such that dist $(a, x) \leqq \delta$ form the *domain of $a$ of norm $\delta$*. We denote it by $D_\delta(a)$.

## § 3. *Fundamental theorems.*

THEOREM 1. *Let $f(x_1, \cdots, x_m) \geqq 0$ be defined over the field $\mathfrak{A}$. Let*

$$\bar{S} = \min S_D$$

*with respect to all rectangular division $D$ of norm $d$, however small. Then*

$$\lim_{d=0} \bar{S}_D = \bar{S}. \tag{1}$$

The points of $\mathfrak{A}$ lie in a certain cube $\mathfrak{C}$ of edge $C$. The representation of $\mathfrak{C}$ is formed of $m$ segments $\mathfrak{C}_1, \cdots, \mathfrak{C}_m$ on the $x_1, \cdots, x_m$ axes. We may suppose $\mathfrak{C}$ taken so large that no point of $\mathfrak{A}$ comes within a certain distance of any side of $\mathfrak{C}$.

Since $\bar{S}$ is obviously finite, there exists for each $\epsilon > 0$ a division $\Delta$ such that

$$\bar{S} \leqq \bar{S}_\Delta < \bar{S} + \frac{\epsilon}{2}. \tag{2}$$

Let $D$ be an arbitrary division of norm $d$. Let us superimpose $\Delta$ on $D$, forming a division $E$. The division $E$ is formed by interpolating certain points, let us say at most $\mu$ in each of the segments $\mathfrak{C}_1, \cdots, \mathfrak{C}_m$. The interpolation of one of these points may be interpreted as passing a plane parallel to one of the sides of $\mathfrak{C}$. Its effect is to subdivide certain of the cells of $\mathfrak{C}$. The volume of the cells so effected is

$$\leqq d C^{m-1}.$$

Hence the superimposition of $\Delta$ on $D$, being equivalent to passing at most $m\mu$ planes, affects cells of $\mathfrak{C}$ belonging to the original division $D$, whose volume is

$$V < m\mu d C^{m-1}. \tag{3}$$

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Let $\Delta$ subdivide $d_i$, a cell of $D$ containing points of $\mathfrak{A}$, into the cells

$$d_{i1},\ d_{i2},\ \cdots$$

containing points of $\mathfrak{A}$ and into cells

$$\delta_{i1},\ \delta_{i2},\ \cdots$$

containing no points of $\mathfrak{A}$. Then

$$\bar{S}_D = \sum_i M_i d_i + R; \qquad \bar{S}_E = \sum_{ik} M_{ik} d_{ik} + R,$$

where $R$ denotes the sum of those terms common to $\bar{S}_D$ and $\bar{S}_E$ corresponding to cells of $D$ unaffected by the division $\Delta$. But

$$d_i = \sum_{ik} d_{ik} + \sum_{ik} \delta_{ik};$$

hence

$$\bar{S}_D = \sum M_i d_{ik} + \sum M_i \delta_{ik} + R.$$

Thus

$$0 \leqq \bar{S}_D - \bar{S}_E \leqq \sum (M_i - M_{ik}) d_{ik} + \sum M_i \delta_{ik},$$
$$< F \sum d_{ik} + F \sum \delta_{ik} = FV,$$
$$< m\mu d F C^{m-1},$$

by (3). If we take

$$d' < \frac{\epsilon}{2m\mu F C^{m-1}},$$

we have

$$\bar{S}_D < \bar{S}_E + \frac{\epsilon}{2},$$

for any $d \leqq d'$. But regarding $E$ as formed by superimposing $D$ on $\Delta$, we have

$$\bar{S}_E \leqq \bar{S}_\Delta$$

since $f \geqq 0$.

Thus

$$\bar{S} \leqq \bar{S}_D < \bar{S}_\Delta + \frac{\epsilon}{2} < \bar{S} + \epsilon,$$

which proves (1).

In the same way we establish:

*Let $f(x_1 \cdots x_m) \leqq 0$. Let*

$$\underline{S} = \max \underline{S}_D$$

*with respect to all rectangular divisions $D$. Then*

$$\lim_{d=0} \underline{S}_D = \underline{S}.$$

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THEOREM 2. *Let $f(x_1, \cdots, x_m)$ be defined over the field $\mathfrak{A}$. Then the limits*

$$\lim_{d=0} \underline{S}_D, \qquad \lim_{d=0} \bar{S}_D$$

*exist and are finite.* They are called respectively the lower and upper integrals of $f$ over $\mathfrak{A}$ and are denoted by

$$\underline{\int}_{\mathfrak{A}} f d\mathfrak{A} = \underline{\int}_{\mathfrak{A}} f dx_1 \cdots dx_m; \qquad \bar{\int}_{\mathfrak{A}} f d\mathfrak{A} = \bar{\int}_{\mathfrak{A}} f dx_1 \cdots dx_m.$$

This is a corollary of theorem 1. We have only to introduce an auxiliary function

$$g(x_1, \cdots, x_m) = f(x_1, \cdots, x_m) + c$$

and take the constant $c$ so that $g$ is either $\leqq 0$ or $\geqq 0$ in $\mathfrak{A}$ according as we wish to show the existence of the first or second of the above limits.

When

$$\underline{\int}_{\mathfrak{A}} f d\mathfrak{A} = \bar{\int}_{\mathfrak{A}} f d\mathfrak{A},$$

we say $f$ is integrable in $\mathfrak{A}$; the common value of these two limits we denote by

$$\int_{\mathfrak{A}} f d\mathfrak{A} = \int_{\mathfrak{A}} f dx_1 \cdots dx_m$$

and call it the integral of $f$ over $\mathfrak{A}$.

We may now prove at once the following theorem:

THEOREM 3. *Let $D$ be any rectangular division of norm $d$, and $\xi_i$ any point of $\mathfrak{A}$ in the cell $d_i$. Then if $f$ is integrable in $\mathfrak{A}$,*

$$\lim_{d=0} \sum f(\xi_i) d_i = \int_{\mathfrak{A}} f d\mathfrak{A},$$

*however the $D$'s and $\xi$'s are chosen. Conversely if this limit exists, $f$ is integrable in $\mathfrak{A}$.*

It is now easy to show that the usual criteria for integrability hold for the present case.

An important class of integrable functions are those having limited variation. In fact, let $f(x_1, \cdots, x_m)$ be defined over $\mathfrak{A}$. Let $D$ be a cubical division of norm $d \leqq d_0$. If there exists a positive number $\omega$ such that

$$\sum_i d^{m-1} \operatorname{osc} f_i < \omega,$$

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however $D$ be chosen, we say $f$ has *limited variation* in $\mathfrak{A}$; otherwise $f$ has *unlimited variation.*

THEOREM 4. *If $f(x_1, \cdots, x_m)$ has limited variation in $\mathfrak{A}$, it is integrable in $\mathfrak{A}$.*

For,

$$\sum d_i \operatorname{osc} f_i \leqq \sum d^m \operatorname{osc} f_i < \omega d.$$

But

$$\lim_{d=0} \omega d = 0.$$

Hence $f$ is integrable.

## § 4. *Content of point aggregates.*

In all the following work the notion of *content* is fundamental. This has been defined in various ways, e. g., by CANTOR, BOREL, JORDAN. The definition we adopt is equivalent to that of CANTOR and JORDAN, although on the face, it is different. We have chosen it because of its relation to the definition of a multiple integral just given.

Let $\mathfrak{A}$ be any limited point aggregate in $R_m$. Let us effect a division $D$ of space of norm $\delta$. Let

$$d_1, \ d_2, \ d_3, \ \cdots$$

denote those cells of $D$ containing at least one point of $\mathfrak{A}$, while

$$d'_1, \ d'_2, \ d'_3, \ \cdots$$

denote those cells *all* of whose points lie in $\mathfrak{A}$.

*Then the limits*

$$\bar{\mathfrak{A}} = \lim_{\delta=0} \sum d_i; \qquad \underline{\mathfrak{A}} = \lim_{\delta=0} \sum d'_i$$

*exist and are finite.*

For, let

$$g(x_1, \cdots, x_m) = 1 \text{ at points of } \mathfrak{A},$$

$$= 0 \text{ at other points.}$$

Then obviously

$$\bar{\mathfrak{A}} = \overline{\operatorname{cont}}\, \mathfrak{A} = \overline{\int_{\mathfrak{A}}} g d\mathfrak{A}; \qquad \underline{\mathfrak{A}} = \underline{\operatorname{cont}}\, \mathfrak{A} = \underline{\int_{\mathfrak{A}}} g d\mathfrak{A}.$$

The number $\bar{\mathfrak{A}}$, $\underline{\mathfrak{A}}$ are called *the upper and lower content of* $\mathfrak{A}$. When

$$\bar{\mathfrak{A}} = \underline{\mathfrak{A}},$$

their common value is called the *content of* $\mathfrak{A}$; in this case we say $\mathfrak{A}$ is *measurable.*

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We observe that criteria for the integrability of $f(x_1, \cdots, x_m)$ give at once criteria for the measurability of $\mathfrak{A}$. When no ambiguity can arise, we may denote the content of $\mathfrak{A}$ by the same letter, otherwise we denote it by

$$\text{cont}\,\mathfrak{A}.$$

When cont $\mathfrak{A} = 0$, we say that $\mathfrak{A}$ is *discrete*. We show at once:

*For $\mathfrak{A}$ to be measurable, it is necessary that the content of its frontier points $\mathfrak{F}$ be 0; i. e that $\mathfrak{F}$ be discrete.*

With the points of an aggregate $\mathfrak{A}$ let us form partial aggregates $\mathfrak{A}_1, \cdots, \mathfrak{A}_s$, such that the aggregate of the common frontier points of any two of these aggregates is discrete. We shall say that we have divided $\mathfrak{A}$ into the *unmixed aggregates* $\mathfrak{A}_1, \cdots, \mathfrak{A}_s$.

THEOREM 5. *If we divide $\mathfrak{A}$ into the unmixed aggregates $\mathfrak{A}_1, \cdots, \mathfrak{A}_s$, then*

$$\bar{\mathfrak{A}} = \bar{\mathfrak{A}}_1 + \cdots + \bar{\mathfrak{A}}_s; \qquad \underline{\mathfrak{A}} = \underline{\mathfrak{A}}_1 + \cdots + \underline{\mathfrak{A}}_s.$$

For, let $D$ be a rectangular division of norm $\delta$. Let $\bar{\mathfrak{F}}_D$ be the volume of those cells of $D$ which contain points of more than one of the aggregates $\mathfrak{A}_1, \cdots, \mathfrak{A}_s$. Let $\bar{\mathfrak{A}}_{i,D}$ be the volume of those cells containing points of $\mathfrak{A}_i$ $(i = 1, 2, \cdots, s)$. Then

$$\bar{\mathfrak{A}}_D \leqq \bar{\mathfrak{A}}_{1,D} + \cdots + \bar{\mathfrak{A}}_{s,D} \leqq \bar{\mathfrak{A}}_D + s\bar{\mathfrak{F}}_D. \tag{1}$$

By hypothesis

$$\lim_{\delta=0} \bar{\mathfrak{F}}_D = 0.$$

Hence passing to the limit in (1), we get the first half of the theorem. The other half follows similarly.

In showing that certain aggregates have content, the two following theorems are sometimes useful. They rest on the notion of a total difference quotient.

Let $f(x_1, \cdots, x_m)$ be defined over $\mathfrak{A}$. Let $x = (x_1, \cdots, x_m)$ be an arbitrary but fixed point of $\mathfrak{A}$ while $x' = (x_1 + h_1, \cdots, x_m + h_m)$ is a variable point of $\mathfrak{A}$. Let $\Delta f = f(x') - f(x)$, and $\Delta x = \text{dist}\,(x, x')$. The quotient

$$\frac{\Delta f}{\Delta x}$$

may be called the *total difference quotient of $f$ at $x$*. The point $x'$ may, or may not, be restricted to remain near $x$, as required.

Let

$$y_1 = f_1(x_1, \cdots, x_m), \ \cdots, \ y_n = f(x_1, \cdots, x_m)$$

be defined over $\mathfrak{A}$. As $x = (x_1, \cdots, x_m)$ ranges over $\mathfrak{A}$, the point $y = (y_1, \cdots, y_n)$

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will range over an aggregate $\mathfrak{B}$ in an $n$-way space $R_n$. We call $\mathfrak{B}$ the *image of* $\mathfrak{A}$.

THEOREM 6. *If $\mathfrak{A}$ is discrete, and the functions*

$$y_1 = f(x_1, \cdots, x_m), \cdots, y_n = f_n(x_1, \cdots, x_m) \qquad (n = m + p \geqq m)$$

*have limited total difference quotients, the image $\mathfrak{B}$ of $\mathfrak{A}$ is discrete.*

THEOREM 7. *Let the functions*

$$y_1 = f_1(x_1, \cdots, x_m), \cdots, y_n = f_n(x_1, \cdots, x_m) \qquad (n = m + r > m)$$

*have limited total difference quotients in $\mathfrak{A}$ except at points of a discrete aggregate $\Delta$. In the cells of any cubical division of norm $d \leqq d_0$, let at least $m$ of these difference quotients remain limited. Then $\mathfrak{B}$, the image of $\mathfrak{A}$, is discrete.*

We prove only theorem 6, as the demonstration of 7 is similar. Let us effect a cubical division $D$ of the $x$-space of norm $d$. Since the difference quotients are limited,

$$|\Delta f_i| < dG \qquad (i = 1, 2, \cdots, m),$$

when $x$ ranges over any one of the cells $d_i$ of $D$. Hence each coördinate remains in an interval of length $< dG$ as $x$ ranges over the points of $\mathfrak{A}$ in $d_i$. Hence $y = (y_1, \cdots, y_n)$ remains within a cube of volume $d^n G^n$, and thus

$$\bar{\mathfrak{B}} < \sum_{\mathfrak{B}} d^n G^n = d^p G^n \sum_{\mathfrak{A}} d^m = d^p G^n \bar{\mathfrak{A}}_D.$$

As $\lim \bar{\mathfrak{A}}_D = 0$, it follows that $\bar{\mathfrak{B}} = 0$; hence cont $\mathfrak{B} = 0$.

It is convenient to extend the terms cells, division of space into cells, etc., as follows. Let us suppose the points of any aggregate $\mathfrak{A}$, which may be $R_m$, arranged in partial aggregates which we shall call *cells* and which have the following properties:

1°. There are only a finite number of cells in any limited portion of space.

2°. The frontier of each cell is discrete.

3°. Each cell lies in a cube of side $\leqq \delta$.

We shall call this a *division of $\mathfrak{A}$ of norm $\delta$*.

Let $\Delta$ be such a division of space. As in the case of rectangular division, $\bar{\mathfrak{A}}_\Delta$ may denote the contents of all the cells which contain at least one point of $\mathfrak{A}$; while $\underline{\mathfrak{A}}_\Delta$ may denote the content of those cells *all* of whose points lie in $\mathfrak{A}$.

It is now easy to show that

THEOREM 8. $$\lim_{\delta=0} \bar{\mathfrak{A}}_\Delta = \bar{\mathfrak{A}}, \qquad \lim_{\delta=0} \underline{\mathfrak{A}}_\Delta = \underline{\mathfrak{A}}.$$

If $\mathfrak{A}$ does not contain all of its limiting points, these may be adjoined to $\mathfrak{A}$. The resulting aggregate is called the *completed aggregate*.

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THEOREM 9. *An aggregate $\mathfrak{A}$ and its completed aggregate $\mathfrak{B}$ have the same upper content.*

For, let us effect a rectangular division $D$ of norm $d$. The cells containing points of $\mathfrak{B}$ fall into two classes: 1° those cells $d_1, d_2, \cdots$ containing points of $\mathfrak{A}$; 2° those cells $e_1, e_2, \cdots$ containing no point of $\mathfrak{A}$. Each of these latter cells as $e_i$ is contiguous to at least one cell $d_k$. If $e_i, \cdots$ are contiguous to $d_k$ we will join them to $d_k$ to form a new cell $\delta_k$, in such a way that each $e$-cell has been joined to some one $d$-cell. The cells $\delta_1, \delta_2, \cdots$ together with the cells $d_1, d_2, \cdots$ which remain unchanged by this process of consolidation define a division $\Delta$. The norm $\delta$ of this division converges to 0 with $d$.

But for the division $\Delta$,

$$\bar{\mathfrak{A}}_\Delta = \bar{\mathfrak{B}}_\Delta;$$

and by theorem 8,

$$\lim_{\delta=0} \bar{\mathfrak{A}}_\Delta = \bar{\mathfrak{A}}.$$

Hence

$$\bar{\mathfrak{B}} = \bar{\mathfrak{A}}.$$

We may now show at once:

THEOREM 10. *If $\mathfrak{A}$ is measurable, its completed aggregate is also measurable; and both have the same content.*

THEOREM 11. *If $\mathfrak{A}$ is measurable, it has the same content as its derivative $\mathfrak{A}'$.*

Connected with any complete aggregate $\mathfrak{A}$ of upper content $\bar{\mathfrak{A}} > 0$, is an aggregate $\mathfrak{B}$, obtained from $\mathfrak{A}$ by a process of sifting, and therefore it may be called the *sifted aggregate of* $\mathfrak{A}$.

Let $D_1, D_2, \cdots$ be a set of rectangular divisions of space each formed by superimposing a division on the preceding. Let the norms of these divisions converge to zero. The division $D_1$ effects a division of $\mathfrak{A}$ into unmixed partial aggregates. Let $\mathfrak{A}_1$ denote those partial aggregates whose upper content is $> 0$. Then $\bar{\mathfrak{A}}_1 = \mathfrak{A}$. Similarly $D_2$ defines a partial aggregate of $\mathfrak{A}_1$, such that $\bar{\mathfrak{A}}_2 = \mathfrak{A}$. As this process goes on, the cells of $D_n$ containing points of $\mathfrak{A}_n$ diminish in size and, for $n = \infty$, converge to a set of points $\mathfrak{B}$. The upper content of the points of $\mathfrak{A}$ in the domain of any point of $\mathfrak{B}$ is $> 0$. Thus each point of $\mathfrak{B}$ is a limiting point of $\mathfrak{A}$, and hence is a point of $\mathfrak{A}$. It is now easy to show that $\mathfrak{B}$ contains no isolated points, and is hence perfect. Moreover

$$\bar{\mathfrak{B}} = \bar{\mathfrak{A}}.$$

## § 5. *Some properties of multiple integrals.*

THEOREM 12. *Let $f(x_1, \cdots, x_m)$ be defined over the field $\mathfrak{A}$. Let $\Delta$ be any division of norm $\delta$ into cells $\delta_1, \delta_2, \cdots$, not necessarily rectangular. Let $\mathfrak{M}_i, \mathfrak{m}_i$ be respectively the maximum and minimum of $f$ in $\delta$. Then*

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$$\lim_{\delta=0} \bar{S}_\Delta = \lim \sum \mathfrak{M}_i \delta_i = \bar{\int}_{\mathfrak{A}} f d\mathfrak{A}. \tag{1}$$

$$\lim_{\delta=0} \underline{S}_\Delta = \lim \sum \mathfrak{m}_i \delta_i = \underline{\int}_{\mathfrak{A}} f d\mathfrak{A}. \tag{2}$$

For, let $D$ be a cubical division of norm $d$. Let $d_1, d_2, \cdots$ be the cells of $D$ containing points of $\mathfrak{A}$; while $M_i$, $m_i$ denote as usual the maximum and minimum of $f$ in $d_i$. Let $|f| \leqq F$ in $\mathfrak{A}$. For convenience, we also take $F \leqq 1$. Then for each $\varepsilon > 0$ there exists a $d$ such that

$$\left| \bar{S}_D - \bar{\int}_{\mathfrak{A}} \right| < \frac{\epsilon}{2}. \tag{3}$$

We may also take $d$ so small that

$$\bar{\mathfrak{A}}_D - \bar{\mathfrak{A}} < \frac{\epsilon}{8F}. \tag{4}$$

Consider now the division $\Delta$. Those of its cells containing points of $\mathfrak{A}$ fall into two classes: 1° those lying in only one cell of $D$; 2° those lying in more than one cell of $D$. Let $\delta_{i1}, \delta_{i2}, \cdots$ be the cells of the 1° class lying in $d_i$; let $\delta'_1, \delta'_2, \cdots$ denote all the cells of the 2° class.
Then the content of all the cells of $\Delta$ containing points of $\mathfrak{A}$ is

$$\bar{\mathfrak{A}}_\Delta = \sum \delta_{ik} + \sum \delta'_i. \tag{5}$$

But since the frontier of $\bar{\mathfrak{A}}_D$ is discrete, there exists a $\delta_0$ such that

$$\sum \delta'_i < \frac{\epsilon}{16F}, \tag{6}$$

for any $\delta \leqq \delta_0$. As moreover $\bar{\mathfrak{A}}_\Delta$ converges to $\bar{\mathfrak{A}}$, we may also suppose $\delta_0$ small enough that

$$\bar{\mathfrak{A}}_\Delta - \bar{\mathfrak{A}} < \frac{\epsilon}{16F}, \qquad \delta \leqq \delta_0. \tag{7}$$

From (5), (6) and (7) we have

$$|\sum \delta_{ik} - \bar{\mathfrak{A}}| < \frac{\epsilon}{8F},$$

which with (4) gives finally

$$|\sum \delta_{ik} - \bar{\mathfrak{A}}_D| < \frac{\epsilon}{4F}. \tag{8}$$

Let $\mathfrak{M}_{ik}$, $\mathfrak{M}'_i$ be the maxima of $f$ in $\delta_{ik}$, $\delta'_i$ respectively. Then

$$\mathfrak{M}_{ik} \leqq M_i, \qquad \mathfrak{M}_i \leqq F.$$

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Hence

$$\bar{S}_\Delta \leqq \sum M_i \delta_{ik} + F \sum \delta'_i .$$

With (6) this gives

$$(9) \qquad |\bar{S}_\Delta - \sum M_i \delta_{ik}| < \frac{\epsilon}{4} .$$

But

$$|\bar{S}_D - \sum M_i \delta_{ik}| \leqq F(\sum d_i - \sum \delta_{ik}) = F(\mathfrak{A}_D - \sum \delta_{ik})$$

$$(10) \qquad < \frac{\epsilon}{4}, \text{ by (8).}$$

From (3), (9) and (10) we have

$$|\bar{S}_\Delta - \overline{\int_{\mathfrak{A}}}| < \epsilon, \qquad \delta \leqq \delta_0 .$$

This proves (1); in a similar manner we may establish (2).

It is now easy to prove the following theorems:

THEOREM 13. *Let $f(x_1, \cdots, x_m)$ be defined over the measurable field $\mathfrak{A}$. Let $\Delta$ be an unmixed division of $\mathfrak{A}$ of norm $\delta$. Let $\underline{S}$ be the maximum of $\underline{S}_\Delta$ and $\bar{S}$ be the minimum of $\bar{S}_\Delta$ for all divisions $\Delta$ of norm $\delta \leqq \delta_0$. Then*

$$\underline{S} = \underline{\int_{\mathfrak{A}}} f d\mathfrak{A}, \qquad \bar{S} = \overline{\int_{\mathfrak{A}}} f d\mathfrak{A} .$$

THEOREM 14. *Let $f(x_1, \cdots, x_m)$ be integrable in $\mathfrak{A}$. Let $\mathfrak{B}_\eta$ be a partial aggregate depending on $\eta > 0$ such that $\bar{\mathfrak{B}}_\eta \doteq \bar{\mathfrak{A}}$ as $\eta \doteq 0$. Then*

$$\lim_{\eta=0} \int_{\mathfrak{B}} f d\mathfrak{B} = \int_{\mathfrak{A}} f d\mathfrak{A} .$$

THEOREM 15. *Let $f(x_1, \cdots, x_m)$ be defined over $\mathfrak{A}$. Let $\mathfrak{B}$ be a partial aggregate of $\mathfrak{A}$ such that $\bar{\mathfrak{A}} = \bar{\mathfrak{B}}$. Then*

$$\overline{\int_{\mathfrak{A}}} f d\mathfrak{A} = \overline{\int_{\mathfrak{B}}} f d\mathfrak{B}; \qquad \underline{\int_{\mathfrak{A}}} f d\mathfrak{A} = \underline{\int_{\mathfrak{B}}} f d\mathfrak{B} .$$

THEOREM 16. *Let $f(x_1, \cdots, x_m)$ be defined over the measurable field $\mathfrak{A}$, which is an inner field of $\mathfrak{B}$. Let $g(x_1, \cdots, x_m) = 0$ in $\mathfrak{B}$, except at the points of $\mathfrak{A}$ where it $= f(x_1, \cdots, x_m)$. Then*

$$\overline{\int_{\mathfrak{A}}} f d\mathfrak{A} = \overline{\int_{\mathfrak{B}}} g d\mathfrak{B}; \qquad \underline{\int_{\mathfrak{A}}} f d\mathfrak{A} = \underline{\int_{\mathfrak{B}}} g d\mathfrak{B} .$$

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THEOREM 17. *Let $f(x_1, \cdots, x_m)$ be defined over the field $\mathfrak{A}$ which is divided into the unmixed fields $\mathfrak{A}_1, \cdots, \mathfrak{A}_s$. Then*

$$\overline{\int_{\mathfrak{A}}} f d\mathfrak{A} = \overline{\int_{\mathfrak{A}_1}} + \cdots + \overline{\int_{\mathfrak{A}_s}},$$

$$\underline{\int_{\mathfrak{A}}} f d\mathfrak{A} = \underline{\int_{\mathfrak{A}_1}} + \cdots + \underline{\int_{\mathfrak{A}_s}}.$$

THEOREM 18. *Let $f(x_1, \cdots, x_m) \geqq \lambda$ in $\mathfrak{A}$. Then*

$$\overline{\int_{\mathfrak{A}}} f d\mathfrak{A} \geqq \lambda \bar{\mathfrak{A}}.$$

THEOREM 19. *Let $f(x_1, \cdots, x_m)$ be defined over the field $\mathfrak{A}$. Let $\mathfrak{A}_\sigma$ denote the points of $\mathfrak{A}$ at which $|f| \geqq \sigma$. If $\mathfrak{A}_\sigma$ is discrete for any $\sigma > 0$, $f$ is integrable and*

$$\int_{\mathfrak{A}} f d\mathfrak{A} = 0.$$

THEOREM 20. *Let $f(x_1, \cdots, x_m)$ be integrable in the complete field $\mathfrak{A}$. Let $\mathfrak{C}$ be the points of $\mathfrak{A}$ at which $f$ is continuous. Then $\bar{\mathfrak{C}} = \bar{\mathfrak{A}}$.*

For, if $\mathfrak{A}$ is discrete, the theorem is true even if $f$ has no points of continuity in $\mathfrak{A}_i$. Let us therefore suppose that $\bar{\mathfrak{A}} > 0$. Let $\mathfrak{B}$ be the sifted aggregate of $\mathfrak{A}$. Let $D$ be a rectangular division, and $d$ one of its cells containing points of $\mathfrak{B}$. We can choose $D$ so that no cell has points of $\mathfrak{B}$ only on its sides. Let $\mathfrak{a}$ be the points of $\mathfrak{A}$ in $d$. Since $f$ is integrable in $\mathfrak{A}$, it is in $\mathfrak{a}$. We can therefore choose a system of cubical subdivisions of $\mathfrak{a}$ whose norms converge to zero, such that for each subdivision there is a cube lying in one of the preceding division, in which the oscillation is as small as we choose. These cubes converge to a point in $\mathfrak{a}$ at which $f$ is continuous. Returning now to the original division $D$, we see that every cell of $D$ which contains a point of $\mathfrak{B}$ contains a point of $\mathfrak{C}$. Hence $\bar{\mathfrak{C}}_D = \bar{\mathfrak{B}}_D$; hence $\bar{\mathfrak{C}} = \bar{\mathfrak{B}} = \bar{\mathfrak{A}}$.

THEOREM 21. *Let $f(x_1, \cdots, x_m) \geqq 0$ in $\mathfrak{A}$. If*

$$\int_{\mathfrak{A}} f d\mathfrak{A} = 0,$$

*the points $\mathfrak{A}_\sigma$ at which $f$ is equal to or greater than an arbitrarily small positive number $\sigma$, form a discrete aggregate. Let $\mathfrak{Z}$ denote the points at which $f = 0$. If $\mathfrak{A}$ is complete, $\bar{\mathfrak{A}} = \bar{\mathfrak{Z}}$.*

For,

$$0 \geqq \overline{\int_{\mathfrak{A}_\sigma}} \geqq \sigma \bar{\mathfrak{A}}_\sigma.$$

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Hence, $\bar{\mathfrak{A}}_\sigma = 0$. To prove the second part of the theorem, let $b$ be a point of the sifted aggregate $\mathfrak{B}$ of $\mathfrak{A}$, at which $f$ is continuous. Then we can choose $\delta > 0$ so small that $f \geqq \lambda > 0$ in any $D_\delta(b)$. But the upper content of the points $\bar{\mathfrak{a}}$ of $\mathfrak{A}$ in $D(b)$ is $\bar{\mathfrak{a}} > 0$. Hence

$$\int_{\mathfrak{A}} f d\mathfrak{A} \geqq \int_{\mathfrak{a}} f d\mathfrak{a} \geqq \lambda \bar{\mathfrak{a}} > 0.$$

Hence $f = 0$ at every point of continuity of $\mathfrak{A}$ in $\mathfrak{B}$. Let now $D$ be a rectangular division of space. Any cell of $D$ which contains a point of $\mathfrak{B}$ also contains a point of continuity lying in $\mathfrak{B}$. Hence

$$\bar{\mathfrak{Z}}_D \geqq \mathfrak{B}_D;$$

which gives

$$\bar{\mathfrak{Z}} \geqq \bar{\mathfrak{B}}, \qquad \text{or} \qquad \bar{\mathfrak{Z}} = \bar{\mathfrak{B}}.$$

§ 6. *Reduction of multiple integrals to iterated integrals.*

Let $\mathfrak{A}$ be a point aggregate in $R_m$. As $x = (x_1, \cdots, x_m)$ ranges over $\mathfrak{A}$, $x_i$ will range over an aggregate $\mathfrak{x}_i$ on the $x_i$ axis which we call the *projection of* $\mathfrak{A}$ *on this axis.*

The points of $R_m$ for which one of the coördinates as $x_i$ has a fixed value $x_i = \xi_i$ lie in an $(m-1)$-way plane perpendicular to the $x_i$ axis. We denote it by $P_{\xi_i}$ or more shortly by $P_i$. The points of $\mathfrak{A}$ in $P_{\xi_i}$ form a *plane section of* $\mathfrak{A}$ *corresponding to the point* $\xi_i$ in $\mathfrak{x}_i$. We denote it by $\mathfrak{P}_{\xi_i}$ or by $\mathfrak{P}_i$.

As $x = (x_1, \cdots, x_m)$ ranges over $\mathfrak{A}$, the point $(x_1, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_m)$ ranges over an aggregate $\mathfrak{X}_i$, called *the projection of* $\mathfrak{A}$ *on the plane* $x_i = 0$.

Let us fix all the coördinates of $x = (x_1, \cdots, x_m)$ except $x_i$. Then $x$ describes a line; the points of $\mathfrak{A}$ on one of these lines we call the *rectilinear section of* $\mathfrak{A}$ *parallel to the* $x_i$ *axis.* We mention a few properties of these aggregates, whose demonstration is readily given.

THEOREM 22. *If* $\mathfrak{A}$ *is complete, so are the aggregates* $\mathfrak{x}_i$, $\mathfrak{X}_i$, $\mathfrak{a}_i$, $\mathfrak{P}_i$, *above mentioned.*

THEOREM 23. *Let* $\mathfrak{A}$ *be measurable. Let* $\mathfrak{x}_\sigma$ *denote those points of* $\mathfrak{x}_i$ *for which the upper content of the frontier points of* $\mathfrak{P}_i$ *is* $\geqq \sigma$. *Then* $\mathfrak{x}_\sigma$ *is discrete.*

A similar theorem holds for the aggregate $\mathfrak{X}_\sigma$.

THEOREM 24. *Let* $f(x_1, \cdots, x_m)$ *be defined over the measurable field* $\mathfrak{A}$. *Then*

$$(1) \qquad \underline{\int}_{\mathfrak{A}} f d\mathfrak{A} \leqq \underline{\int}_{\mathfrak{X}_i} dx_i \underline{\int}_{\mathfrak{P}_i} f d\mathfrak{P}_i \leqq \overline{\int}_{\mathfrak{X}_i} dx_i \underline{\int}_{\mathfrak{P}_i} f d\mathfrak{P}_i \leqq \overline{\int}_{\mathfrak{P}} f d\mathfrak{A},$$

$$(2) \qquad \underline{\int}_{\mathfrak{A}} f d\mathfrak{A} \leqq \underline{\int}_{\mathfrak{X}_i} dx_i \overline{\int}_{\mathfrak{P}_i} f d\mathfrak{P} \leqq \overline{\int}_{\mathfrak{X}_i} dx_i \overline{\int}_{\mathfrak{P}_i} f d\mathfrak{P}_i \leqq \overline{\int}_{\mathfrak{A}} f d\mathfrak{A}.$$

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Let us prove (1); the demonstration of (2) is similar. Let $\mathfrak{A}$ lie in an outer cube $\mathfrak{B}$, whose projection on the $x_i$ axis is $\mathfrak{b}$, and whose plane sections perpendicular to this axis are $\mathfrak{Q}$. We introduce the auxiliary function

$$g(x_1, \cdots, x_m) = f(x_1, \cdots, x_m) \text{ in } \mathfrak{A},$$
$$= 0 \text{ at other points of } \mathfrak{B}.$$

Let $D$ be a cubical division of $R_m$ of norm $d$. This divides $\mathfrak{B}$ into cells which we denote by $\delta$. It also divides the plane section $\mathfrak{Q}$ into cells $\delta'$, and the segment $\mathfrak{b}$ into intervals which we denote by $\delta''$.

Let $M$, $M'$ be the maxima, and $m$, $m'$ the minima of $g$ in the cells $\delta$, $\delta'$. Let $|f| \leqq F$ in $\mathfrak{A}$. Let $\bar{G}$, $\underline{G}$ denote the upper and lower integrals of $g$ in the field $\mathfrak{B}$. Then for each $\epsilon > 0$ there exists a $d_0$ such that

$$\underline{G} - \epsilon < \sum_{\mathfrak{B}} m\delta \leqq \sum_{\mathfrak{B}} M\delta < \bar{G} + \epsilon \qquad (d \leqq d_0). \tag{3}$$

Also for each $x_i$ of $\mathfrak{b}$

$$\sum_{\mathfrak{Q}} m'\delta' \leqq \underline{\int}_{\mathfrak{Q}} g d\mathfrak{Q} \leqq \sum_{\mathfrak{Q}} M'\delta';$$

or since $m \leqq m'$, and $M' \leqq M$,

$$\sum_{\mathfrak{Q}} m\delta' \leqq \underline{\int}_{\mathfrak{Q}} g d\mathfrak{Q} \leqq \sum_{\mathfrak{Q}} M\delta'.$$

Multiplying by $\delta''$ and summing over $\mathfrak{b}$ we have, since $\delta = \delta'\delta''$,

$$\sum_{\mathfrak{B}} m\delta \leqq \sum_{\mathfrak{b}} \delta' \underline{\int}_{\mathfrak{Q}} \leqq \sum_{\mathfrak{B}} M\delta.$$

Combined with (3), this gives

$$\underline{G} - \epsilon < \sum_{\mathfrak{b}} \delta'' \underline{\int}_{\mathfrak{Q}} < \bar{G} + \epsilon,$$

or

$$\underline{G} \leqq \underline{\int}_{\mathfrak{b}} \underline{\int}_{\mathfrak{Q}} \leqq \int_{\mathfrak{b}} \underline{\int}_{\mathfrak{Q}} \leqq \bar{G}. \tag{4}$$

From (4) and theorem 16 we have

$$\underline{\int}_{\mathfrak{A}} f d\mathfrak{A} \leqq \underline{\int}_{\mathfrak{b}} dx_i \underline{\int}_{\mathfrak{Q}} g d\mathfrak{Q} \leqq \overline{\int}_{\mathfrak{b}} dx_i \underline{\int}_{\mathfrak{Q}} g d\mathfrak{Q} \leqq \overline{\int}_{\mathfrak{A}} f d\mathfrak{A}. \tag{5}$$

Let $\bar{\mathfrak{F}}_{\mathfrak{P}}$ denote the upper content of the frontier points of $\mathfrak{P}_i$. Then

$$\left| \underline{\int}_{\mathfrak{P}_i} f d\mathfrak{P} - \underline{\int}_{\mathfrak{Q}} g d\mathfrak{Q} \right| \leqq F \bar{\mathfrak{F}}_{\mathfrak{P}}. \tag{6}$$

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Let us set

$$h(x_i) = \underline{\int}_{\mathfrak{P}_i} f d\mathfrak{P}_i, \text{ for points of } \mathfrak{x}_i;$$

$$= 0, \text{ for other points of } \mathfrak{b}.$$

Then by (6), since the points of $\mathfrak{x}_i$ for which $\bar{\mathfrak{F}}_{\mathfrak{P}} > \sigma$ form a discrete set,

$$\underline{\int}_{\mathfrak{b}} h dx_i = \underline{\int}_{\mathfrak{b}} dx_i \underline{\int}_{\mathfrak{Q}} g d\mathfrak{Q}. \tag{7}$$

But obviously

$$\underline{\int}_{\mathfrak{b}} h dx_i = \underline{\int}_{\mathfrak{x}_i} h dx_i. \tag{8}$$

Hence from (7), (8),

$$\underline{\int}_{\mathfrak{b}} dx_i \underline{\int}_{\mathfrak{Q}} g d\mathfrak{Q} = \underline{\int}_{\mathfrak{x}_i} dx_i \underline{\int}_{\mathfrak{P}_i} f d\mathfrak{P}_i. \tag{9}$$

In the same way we show

$$\overline{\int}_{\mathfrak{b}} dx_i \underline{\int}_{\mathfrak{Q}} g d\mathfrak{Q} = \overline{\int}_{\mathfrak{x}_i} dx_i \underline{\int}_{\mathfrak{P}} f d\mathfrak{P}_i. \tag{10}$$

By placing (9) and (10) in (5) we get (1).

In a similar manner we may prove:

THEOREM 25. *Let $\mathfrak{A}$ be measurable. Let $\mathfrak{X}_i$ be its projection on the plane $x_i = 0$. Let $\mathfrak{a}_i$ be the rectilinear section of $\mathfrak{A}$ parallel to the $x_i$ axis. Then*

$$\underline{\int}_{\mathfrak{A}} f d\mathfrak{A} \leqq \underline{\int}_{\mathfrak{X}_i} d\mathfrak{X}_i \underline{\int}_{\mathfrak{a}_i} f dx_i \leqq \overline{\int}_{\mathfrak{X}_i} d\mathfrak{X}_i \underline{\int}_{\mathfrak{a}_i} f dx_i \leqq \overline{\int}_{\mathfrak{A}} f d\mathfrak{A}.$$

$$\underline{\int}_{\mathfrak{A}} f d\mathfrak{A} \leqq \underline{\int}_{\mathfrak{X}_i} d\mathfrak{X}_i \overline{\int}_{\mathfrak{y}_i} f dx_i \leqq \overline{\int}_{\mathfrak{X}_i} d\mathfrak{X}_i \overline{\int}_{\mathfrak{a}_i} f dx_i \leqq \overline{\int}_{\mathfrak{A}} f d\mathfrak{A}.$$

Many theorems follow now from 24 and 25 as corollaries. We state only the two following.

THEOREM 26. *Let $f(x_1, \cdots, x_m)$ be integrable in the measurable field $\mathfrak{A}$. Let $\mathfrak{x}_\sigma$ denote those points of $\mathfrak{x}_i$ for which*

$$\overline{\int}_{\mathfrak{P}_i} f d\mathfrak{P} - \underline{\int}_{\mathfrak{P}} f d\mathfrak{P} \geqq \sigma.$$

*Then $\mathfrak{x}_\sigma$ is discrete. If $\mathfrak{A}$ is complete, the upper content of the points at which the above difference vanishes is $\bar{\mathfrak{x}}_i$.*

THEOREM 27. *Let $f(x_1, \cdots, x_m)$ be integrable in the measurable complete field $\mathfrak{A}$. Let $\mathfrak{y}_i$ denote those points of $\mathfrak{x}_i$ for which the integrals over the cor-*

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*responding plane sections* $\mathfrak{P}_i$ *exist. Let* $\mathfrak{Y}_i$ *denote the points of* $\mathfrak{X}_i$ *for which the integrals over the corresponding rectilinear sections* $\mathfrak{a}_i$ *exist. Then*

$$\int_{\mathfrak{A}} f d\mathfrak{A} = \int_{\mathfrak{X}_i} dx_i \int_{\mathfrak{P}_i} f d\mathfrak{P}_i = \int_{\mathfrak{Y}_i} d\mathfrak{Y}_i \int f dx_i .$$

§ 7. *Change of variables.*

Let the transformation $T$ be defined by the equations

$$T; \qquad x_1 = \phi_1(t_1, \cdots, t_m), \cdots, x_m = \phi_m(t_1, \cdots, t_m),$$

where the $\phi$'s have continuous first derivatives in the region $R$. Let the correspondence between $R$ and its image $R_T$ be uniform. Let the determinant of the transformation $J \neq 0$. We say the transformation $T$ is *regular*.

It is easy to show now that to any *inner* aggregate $\mathfrak{A}$ in $R$ corresponds an inner aggregate $\mathfrak{A}_T$ in $R_T$. To inner and frontier points of $\mathfrak{A}$ correspond similar points in $\mathfrak{A}_T$, and conversely.

As a direct consequence of theorem 6 we have

THEOREM 28. *If either* $\mathfrak{A}$ *or* $\mathfrak{A}_T$ *is measurable, so is the other. Also if one is discrete, so is the other.*

We show next how the transformation $T$ can be expressed as the product of two simpler *regular* transformations.

Since $J \neq 0$, not all the first partial derivatives vanish at any point of $R$. Let

$$\frac{\partial \phi_m}{\partial t_m} \neq 0$$

at a point $t$ and hence in a certain domain $D_\delta(t)$ of $t$. Let us define the first of these new transformations by

$$T_1; \qquad u_1 = t_1, \cdots, u_{m-1} = t_{m-1}, \qquad u_m = \phi_m(t_1, \cdots, t_m).$$

Since $T_1$ is regular, it can be inverted, giving

$$t_1 = u_1, \cdots, t_{m-1} = u_{m-1}, \; t_m = \theta(u_1, \cdots, u_m),$$

where $\theta$ is a one-valued function having continuous first partial derivatives in a certain domain $D_\eta(u)$. If $\delta' \leqq \delta$ is taken sufficiently small, the image $U$ of $D_{\delta'}(t)$ lies in $D_\eta(u)$. The second transformation is

$$T_2; \quad x_1 = \phi_1(u_1, \cdots, u_{m-1}, \theta), \cdots, x_{m-1} = \phi_{m-1}(u_1, \cdots, u_{m-1}, \theta), \quad x_m = u_m,$$

and is defined over $U$.

Obviously now

$$T = T_1 T_2$$

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when $t$ ranges over $D_{\delta'}(t)$, and the determinant $J_2$ of $T_2$ is $\neq 0$, since the determinants $J$, $J_1$ of $T$ and $T_1$ are $\neq 0$, by hypothesis.

Let $\mathfrak{A}$ be an inner aggregate of $R$. It is easy now to show that we can effect a cubical division of the $t$-space of norm $d$ such that for the points of $\mathfrak{A}$ in each cell $d_k$, there exist two transformations $T_1^{(k)}$, $T_2^{(k)}$ of the kind just considered such that

$$(1) \qquad T = T_1^{(k)} T_2^{(k)}.$$

For we can take $d$ so small that not all the first partial derivatives vanish in any cell, since otherwise $J = 0$ at some point in this cell. Next, making use of the uniform continuity of our functions, we show that the norms $\eta$ of the domains $D_\eta(u)$, and hence the norms $\delta'$ of $D_{\delta'}(t)$ considered above, do not sink below some positive number, as $t$ ranges over any inner region of $R$. Thus taking $d > 0$ small enough, the relation (1) holds in each cell containing a point of $\mathfrak{A}$.

THEOREM 29. *Let*

$$T; \qquad x_1 = \phi_1(t_1, \cdots, t_m), \cdots, x_m = \phi_m(t_1, \cdots, t_m)$$

*define a regular transformation of determinant $J$ in the region $R$. Let $\mathfrak{T}$ be any inner perfect measurable aggregate of $R$ and let $\mathfrak{X}$ be its image. Let $f(x_1, \cdots, x_m)$ be continuous in $\mathfrak{X}$. Then*

$$(2) \qquad \int_{\mathfrak{X}} f(x_1, \cdots, x_m)\, dx_1 \cdots dx_m = \int_{\mathfrak{T}} |J| f dt_1 \cdots dt_m.$$

For $m = 1$, the theorem is obviously true. Let us therefore assume it correct for $m - 1$, and show it is so for $m$. As just shown, we can effect a cubical division of the $t$-space such that the relation (1) holds in each cell containing points of $\mathfrak{T}$. But if (2) holds for each of these partial aggregates of $\mathfrak{T}$, it obviously holds for $\mathfrak{T}$. We may therefore assume without loss of generality that the relation (1) holds throughout $\mathfrak{T}$. Let $\mathfrak{U}$ be the image of $\mathfrak{X}$ in the $u$-space, while $\mathfrak{y}'_m$, $\mathfrak{P}'_m$ are the images of $\mathfrak{y}_m$, $\mathfrak{P}_m$. Then

$$\begin{aligned}
\int_{\mathfrak{X}} f dx_1 \cdots dx_m &= \int_{\mathfrak{y}_m} dx_m \int_{\mathfrak{P}_m} f dx_1 \cdots dx_{m-1} \\
&= \int_{\mathfrak{y}'_m} du_m \int_{\mathfrak{P}'_m} |J_2| f du_1 \cdots du_{m-1} \\
&= \int_{\mathfrak{U}} |J_2|\, du_1 \cdots du_m.
\end{aligned}$$

The application of the transformation $T_2$ to the integral on the right gives similarly,

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$$\int_{\mathfrak{U}} |J_2| du_1 \cdots du_m = \int_{\mathfrak{T}} |J_1||J_2| f dt_1 \cdots dt_m.$$

This and the preceding equation give (2).

As a corollary of the last theorem we have

THEOREM 30. *Let $d_k$ denote the volume of one of the cells of a rectangular division $D$ of norm $d$ in the t-space. Let $\mathfrak{T}$ be any inner region of $R$ and $\mathfrak{X}$ its image. To the division $D$ correspond a division $\Delta$ of the x-space into cells of volume $\delta_k$. The cells of $\Delta$ following within $\mathfrak{X}$ are unmixed, and*

$$\delta_k = |J| d_k + \epsilon_k, \qquad t \text{ in } d_k,$$

*where $|\epsilon_k| < \epsilon$ uniformly, if $d$ is taken sufficiently small.*

THEOREM 31. *Let $T$ be a regular transformation of determinant $J$ defined over a region $R$. Let $\mathfrak{T}$ be an inner aggregate of $R$ and let $\mathfrak{X}$ be its image. Let $f(x_1, \cdots, x_m)$ be limited in $\mathfrak{T}$. Then*

$$\overline{\int_{\mathfrak{X}}} f(x_1, \cdots, x_m) d\mathfrak{X} = \overline{\int_{\mathfrak{T}}} |J| f d\mathfrak{T}, \tag{3}$$

$$\underline{\int_{\mathfrak{X}}} f(x_1, \cdots, x_m) d\mathfrak{X} = \underline{\int_{\mathfrak{T}}} |J| f d\mathfrak{T}. \tag{4}$$

Let us prove (3); the demonstration of (4) is similar. To the end we effect a cubical division of the $t$-space of norm $d$. To it corresponds a division of $\mathfrak{X}$ into cells of norm $\delta$.

Let us consider the integral on the left of (3). Employing the usual notation, and letting $J_i$ denote the value of $J$ at some point of $d_i$, we have

$$\begin{aligned} \sum M_i \delta_i &= \sum M_i (|J_i| + \epsilon_i) d_i \\ &= \sum M_i |J_i| d_i + \sum \epsilon_i M_i d_i. \end{aligned} \tag{5}$$

Here

$$|\sum \epsilon_i M_i d_i| < \epsilon F \overline{\text{cont}}\, \mathfrak{T} = \eta, \tag{6}$$

which may be made as small as we choose, on taking $d$ sufficiently small.

Let us consider now the integral on the right of (3). In the cell $d_i$,

$$\max J \cdot \min |J| \leqq \max \cdot f|J| \leqq \max f \cdot \max |J|,$$

if $\max f$ is positive, while the signs are reversed if it is negative. Let us set

$$M'_i = \max \cdot f|J|, \text{ in } d_i.$$

Then

$$M'_i = M_i(|J_i| + \epsilon'_i),$$

where $|\epsilon'_i| < \epsilon$ uniformly.

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Thus

$$\sum M'_i d_i = \sum M_i |J| d_i + \sum \epsilon'_i M_i d_i, \tag{7}$$

where, as in (6),

$$|\sum \epsilon'_i M_i d_i| < \eta.$$

Thus (5) and (7) give

$$|\sum M_i \delta_i - \sum M_i d_i| < 2\eta,$$

which proves (3).

OUTER ISLAND, *May*, 1905.

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# SUR L'ÉCART DE DEUX COURBES ET SUR LES COURBES LIMITES*

PAR

MAURICE FRÉCHET

*Définition de l'arc de courbe continue.*

Appelons arc de courbe continue l'ensemble *ordonné* formé par la suite des points que l'on obtient en faisant croître $t$ de $t_0$ à $t_1$ dans les formules :

$$(1) \qquad x = f(t), \qquad y = g(t), \qquad z = h(t),$$

où $f$, $g$, $h$ sont trois fonctions de $t$ uniformément continues de $t_0$ à $t_1$. Il pourra d'ailleurs arriver que $f$, $g$, $h$ soient en même temps constantes dans un intervalle partiel $(t'_0, t'_1)$ de valeurs de $t$; alors toutes les valeurs de $t$ comprises dans cet intervalle correspondent à un même point de la courbe. (Il est utile de ne pas exclure ce cas pour ne pas nuire à la généralité des propositions que nous voulons établir). En particulier, il pourrait se faire que $f$, $g$, $h$ soient constants de $t_0$ à $t_1$; alors la courbe se réduit à un seul point. D'ailleurs, dans tout autre cas, les points de la courbe occupent dans l'espace une infinité non dénombrable de positions.

Lorsque $f$, $g$, $h$ ne sont en même temps constants dans aucun intervalle de valeurs de $t$, nous dirons que les formules (1) fournissent une représentation *normale* de la courbe.

Il peut arriver que les formules (1) donnent à $x$, $y$, $z$ les mêmes valeurs pour deux valeurs différentes de $t$: $t'_0$, $t'_1$ sans que $f$, $g$, $h$ soient constants entre ces deux valeurs de $t$. Alors $t'_0$ et $t'_1$ correspondent à deux points de la courbe *que nous considèrerons comme distincts* quoiqu'ils occupent la même position dans l'espace; nous aurons ainsi un point *multiple* de la courbe.

*Recherche de toutes les représentations paramétriques d'une courbe continue.*

Considérons maintenant, outre les formules (1), les formules :

$$(2) \qquad x = \phi(u), \qquad y = \psi(u), \qquad z = \chi(u)$$

où $\phi$, $\psi$, $\chi$ sont des fonctions de $u$, uniformément continues dans l'intervalle

* Received for publication May 25, 1905. Presented to the Society at the summer meeting, 1905.



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$(u_0, u_1)$. Ces formules (1) et (2) représentent deux ensembles de points ordonnés: $C$ et $\Gamma$. Nous dirons que ces formules représentent la même courbe lorsque les ensembles *ordonnés* $C$ et $\Gamma$ sont superposables. Nous entendons par là qu'on peut établir entre les points de $C$ et de $\Gamma$ une correspondance univoque et réciproque de façon que deux points correspondants occupent la même position dans l'espace et que deux points quelconques de $C$ soient trouvés dans le même ordre relatif que les deux points correspondants de $\Gamma$ quand on parcourt $C$ et $\Gamma$ dans les sens des $t$ et des $u$ croissants. Observons que d'après cette définition, les courbes $C$ et $\Gamma$ peuvent être formées avec les mêmes points de l'espace et être cependant distinctes.*

Cherchons les relations qui doivent exister entre $f, g, h$; $\phi, \psi, \chi$ pour que les formules (1) et (2) représentent la même courbe, au sens que nous venons d'indiquer.

1° Pour cela, supposons d'abord que les représentations (1) et (2) soient normales. Alors à deux valeurs distinctes de $t$ correspondront deux points distincts (quoique pouvant coïncider en position) de la courbe $C$ et réciproquement. De même, tout point de $\Gamma$ correspondra à une seule valeur de $u$ et réciproquement. Par suite la correspondance entre les points de $C$ et de $\Gamma$ se traduira par une correspondance univoque et réciproque entre les valeurs de $t$ et de $u$, dans laquelle $t$ et $u$ croissent en même temps. Autrement dit, on aura, pour deux points correspondants:

$$u = \theta(t),$$

$\theta$ étant une fonction de $t$ qui croît constamment de $u_0$ jusqu'à $u_1$ lorsque $t$ croît de $t_0$ à $t_1$. Cette fonction doit passer par toutes les valeurs de $u$ intermédiaires entre $u_0$ et $u_1$; comme elle est aussi croissante, elle sera donc nécessairement continue. Et l'on aura de $t_0$ à $t_1$:

$$(3) \qquad f(t) \equiv \phi[\theta(t)], \qquad g(t) \equiv \psi[\theta(t)], \qquad h(t) \equiv \chi[\theta(t)].$$

Réciproquement, étant donnée la représentation normale d'une courbe $C$ les formules (1) définiront la même courbe lorsqu'on prendra pour $f, g, h$, les fonctions définies par (3) où $\theta(t)$ est une fonction continue et croissante de $t$; et la représentation paramétrique (1) sera aussi normale.†

2° Supposons maintenant que les courbes $C$ et $\Gamma$ soient données par les représentations (1) et (2) choisies arbitrairement, non nécessairement normales et cherchons encore la condition pour que ces deux courbes coïncident. Tout d'abord, si l'une se réduit à un seul point il en sera de même de l'autre. Dans le cas

* Il suffit de prendre: $f \equiv g \equiv 0$, $h \equiv \sin t$; $\phi \equiv \psi \equiv 0$, $\chi \equiv 2t/\pi \sin^2 \pi^2/2t$, $t_0 = u_0 = -\pi/a$, $t_1 = u_1 = +\pi/a$. Voir *Leçons sur l'intégration*, par HENRI LEBESGUE, Paris, 1904, p. 40.

† En général, on se contente de dire que *par définition*, les formules (1) et (2) définissent la même courbe, si l' on peut trouver une fonction telle que $\theta(t)$ donnant lieu aux identités (3).

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contraire montrons que l'on peut ramener le problème au précédent. Autrement dit : *une courbe continue qui ne se réduit pas à un seul point a toujours au moins une représentation paramétrique normale.* En effet, considérons tous les intervalles $I$ où $f$, $g$, $h$ sont à la fois constants. Ces intervalles sont sans points communs, sans quoi on pourrait les remplacer par de plus grands. Or il est facile de démontrer que si l'on se donne sur l'axe des $t$ entre $t_0$ et $t_1$ un ensemble quelconque d'intervalles $I$ sans points communs, et n'occupant pas tout le segment $(t_0, t_1)$, on peut trouver une fonction continue : $s = a(t)$, allant sans jamais décroître de 0 à 1 quand $t$ croit de $t_0$ à $t_1$ et qui n'est constante que dans chacun des intervalles $I$. Alors à chaque valeur de $s$ correspond une valeur de $t$ ou une infinité de valeurs de $t$ comprises dans un même intervalle où $f$, $g$, $h$ sont constants. Dès lors, à une même valeur de $s$ correspond un seul point de $C$ et on peut écrire :

$$f(t) = f_0(s), \qquad g(t) = g_0(s), \qquad h(t) = h_0(s),$$

$t$ variant de $t_0$ à $t_1$ et $s$ de 0 à 1 ; et à deux valeurs de $s$ distinctes correspondront deux points de $C$ distincts (mais qui coïncident peut-être en position). Alors, si nous considérons les formules :

$$x = f_0(s), \qquad y = g_0(s), \qquad z = h_0(s) \qquad (0 \leq s \leq 1),$$

nous pourrons dire qu'elles constituent une représentation normale de $C$. Car lorsque $s$ croît, on retrouve les points de $C$ chacun une seule fois et dans le même ordre. Du moins cette affirmation sera complètement justifiée dès que nous aurons prouvé que $f_0$, $g_0$, $s_0$ sont uniformément continues de 0 à 1. Or s'il n'en était pas ainsi, on pourrait trouver une infinité de valeurs de $s$ distinctes et comprises entre 0 et 1 : $s_1, s_2, s_3, \cdots, s_n, \cdots$ qui tendent vers un certain nombre $s_0$ et sont telles que par exemple $|f_0(s_n) - f_0(s_0)|$ reste supérieur à un nombre positif fixe $\epsilon$. Or, à tout nombre $s_n$ compris entre 0 et 1, la formule $s = a(t)$ fait correspondre au moins une valeur $t_n$ de $t$ compris entre $t_0$ et $t_1$. Les points $t_1, t_2, t_3, \cdots$ de l'axe des $t$ ont au moins un point limite $t_0$ ; autrement dit, il y a dans la suite $t_1, t_2, t_3, \cdots$ au moins une suite $t_{n_1}, t_{n_2}, \cdots$ qui a une limite déterminée $t_0$. Comme la fonction $a(t)$ est continue, on aura donc :

$$s_0 = \lim s_{n_p} = \lim a(t_{n_p}) = a(t_0)$$

et par suite :

$$|f_0(s_{n_p}) - f_0(s_0)| = |f_0[a(t_{n_p})] - f_0[a(t_0)]| = |f(t_{n_p}) - f(t_0)|.$$

Comme $f(t)$ est une fonction continue, le dernier terme tend vers zéro avec $1/p$ ; il est donc impossible que le premier reste supérieur à $\epsilon$. La proposition est ainsi démontrée.

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Revenons aux courbe $C$ et $\Gamma$. D'après ce qui précède on pourra, en choisissant convenablement deux fonctions $s = a(t)$, $v = \alpha(u)$ continues et allant sans jamais décroître de 0 à 1, écrire les formules (2) sous formes normales:

$$(1)' \qquad x = f_0(s), \qquad y = g_0(s), \qquad z = h_0(s) \qquad (0 \leq s \leq 1),$$

$$(2)' \qquad x = \phi_0(v), \qquad y = \psi_0(v), \qquad z = \chi_0(v) \qquad (0 \leq v \leq 1).$$

Dès lors, si les courbes $C$ et $\Gamma$ coïncident c'est que les deuxièmes membres de ces formules sont égaux deux à deux sous la condition:

$$(3) \qquad v = \theta(s),$$

$\theta$ étant une fonction continue et toujours croissante de 0 à 1. Observons d'ailleurs que la fonction $\theta[a(t)]$ est, comme $a(t)$ une fonction continue allant, sans jamais décroitre, de 0 à 1 et qui n'est constante que dans les intervalles $I$. On aurait donc pu prendre la fonction $\theta[a(t)]$ au lieu de $a(t)$ et alors la relation (3) serait remplacée par $v = s$.

En définitive, *lorsqu'aucune des courbes* (1) *et* (2) *ne se réduisent à un point, la condition nécessaire et suffisante pour qu'elles représentent la même courbe est qu'il existe deux fonctions* $a(t)$ *et* $\alpha(u)$ *continues allant sans jamais décroitre de* 0 *à* 1, *n'étant constantes que dans les intervalles où il en est respectivement de même pour* $f, g, h$ *et pour* $\phi, \psi, \chi$ *et telles que l'égalité:* $a(t) = \alpha(u)$ *entraine:*

$$f(t) = \phi(u), \qquad g(t) = \psi(u), \qquad h(t) = \chi(u).$$

En particulier, on voit que si $\theta(t)$ est une fonction continue de $t$ qui va sans jamais décroitre de $u_0$ à $u_1$ quand $t$ croit de 0 à 1, les formules:

$$x = \phi[\theta(t)],\ y = \psi[\theta(t)],\ z = \chi[\theta(t)] \qquad (0 \leq t \leq 1)$$

représentent la même courbe que les formules (2) quand même la représentation (2) se serait pas normale.

*Ecart de deux courbes.*

*Définition de l'écart.* — Nous généraliserons maintenant une notion introduite par WEIERSTRASS dans le Calcul des Variations sous le nom de voisinage. Si l'on considère deux courbes *infiniment voisines:*

$$y = f(x), \qquad z = g(x), \qquad \text{et} \qquad y = \phi(x), \qquad z = \psi(x)$$

on peut appeler voisinage de ces deux courbes la valeur maxima de la distance de deux points de ces deux courbes ayant même abscisse. Pour parer au cas où les extrémités n'auraient pas mêmes abscisses, on peut remplacer la correspondance entre deux points de même abscisse par une correspondance entre des points

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d'abscisses peu différentes. A chacune de ces correspondances correspondra un nombre particulier, mais ce qui est essentiel c'est que si les deux courbes sont infiniment voisines, tous ces nombres seront infiniment petits.

Nous allons tâcher de généraliser cette notion de façon à l'appliquer à deux arcs de courbes continues quelconques *non nécessairement très voisins*. Pour préciser, nous allons attacher à tout couple $C$, $\Gamma$ d'arcs de courbes continues quelconques un nombre positif ou nul que nous désignerons par la notation $(C, \Gamma)$ ou $(\Gamma, C)$ et que nous appellerons *écart* des arcs $C$, $\Gamma$. Ce nombre sera tel que : 1° l'écart $(C, \Gamma)$ n'est nul que si $C$ et $\Gamma$ coïncident ; 2° $C_1$, $C_2$, $C_3$, étant trois arcs de courbes continues quelconques, si les écarts $(C_1, C_3)$ et $(C_2, C_3)$ sont infiniment petits, il en est de même de $(C_1, C_2)$.*

Pour définir cet écart, considérons deux arcs continus quelconques distincts ou non $C$ et $\Gamma$. On peut comme nous l'avons vu les représenter d'une infinité de façons par des formules telles que :

$$x = f(t), \qquad y = g(t), \qquad z = h(t) \qquad (0 \leq t \leq 1), \tag{4}$$

$$x = \phi(t), \qquad y = \psi(t), \qquad z = \chi(t) \qquad (0 \leq t \leq 1), \tag{5}$$

où $f, g, h$; $\phi, \psi, \chi$ sont uniformément continues de 0 à 1. Désignons maintenant par $\delta(t)$ la distance de deux points correspondant à la même valeur de $t$:

$$\delta(t) = \sqrt{[f(t) - \phi(t)]^2 + [g(t) - \psi(t)]^2 + [h(t) - \chi(t)]^2}.$$

$\delta(t)$ est une fonction continue de $t$ qui atteint un maximum absolu $d$. A chaque système de représentations (normales on non) de $C$ et $\Gamma$ correspond une valeur déterminée positive ou nulle de $d$. *J'appellerai écart de $C$ et de $\Gamma$ la limite inférieure $e = (C, \Gamma)$ de l'ensemble des valeurs de $d$.* Ce nombre sera lui même positif ou nul.

*Simplification du calcul de l'écart.* Avant d'étudier les propriétés de ce nombre, il nous sera utile de montrer qu'on peut simplifier le calcul de $e$ en restreignant le nombre des valeurs de $d$ dont il est la limite inférieure.

Observons tout de suite que si l'une des courbes $C$ ou $\Gamma$, $C$ par exemple se réduit à un point $A$, la quantité $d$ est, dans un système de représentations quelconque de $C$ et de $\Gamma$, le maximum de la distance de $A$ à un point quelconque de $\Gamma$. Donc $e = d$; en particulier, on voit que pour calculer $d$ on peut toujours se restreindre au cas où la représentation de $\Gamma$ est normale si $\Gamma$ n'est pas aussi réduite à un point.

Supposons maintenant qu' aucune des courbes $C$ ou $\Gamma$ ne se réduise à un point ; je dis que $e$ sera égal à la limite inférieure $e'$ des valeurs de $d$ qui correspondent seulement aux représentations *normales* de $C$ et de $\Gamma$ simultanément.

* On pourra donc appliquer les théorèmes énoncés dans les Comptes-Rendus du 20 Mars 1905. *Sur la notion d'écart dans le Calcul fonctionnel.*

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En effet, considérons un système (4), (5) de représentations quelconques de $C$ et de $\Gamma$. On a vu qu'on peut trouver deux fonctions $a(t)$, $\alpha(t)$ continues allant sans jamais décroître de 0 à 1, $a(t)$ et $\alpha(t)$ ne restant respectivement constantes que dans les intervalles où il en est respectivement de même de $f$, $g$, $h$ et $\phi$, $\psi$, $\chi$. On a vu aussi qu'on pouvait alors former des fonctions $f_0$, $g_0$, $h_0$; $\phi_0$, $\psi_0$, $\chi_0$ uniformément continues entre 0 et 1 et telles que

$$(6)\qquad \begin{aligned} &f_0[a(t)] \equiv f(t), \qquad g_0[a(t)] \equiv g(t), \qquad h_0[a(t)] \equiv h(t),\\ &\phi_0[\alpha(t)] \equiv \phi(t), \qquad \psi_0[\alpha(t)] \equiv \psi(t), \qquad \chi_0[\alpha(t)] \equiv \chi(t). \end{aligned}$$

Posons maintenant:

$$a_n(t) \equiv \left(1 - \frac{1}{n}\right) a(t) + \frac{t}{n}, \qquad \alpha_n(t) \equiv \left(1 - \frac{1}{n}\right) \alpha(t) + \frac{t}{n}$$

les fonctions $a_n(t)$ et $\alpha_n(t)$ seront des fonctions de $t$ qui sont continues *toujours croissantes* de 0 à 1 et qui tendent uniformément vers $a(t)$ et $\alpha(t)$. Dès lors puisque les formules

$$\begin{aligned} &x = f_0(u), \qquad y = g_0(u), \qquad z = h_0(u)\\ &x = \phi_0(u), \qquad y = \psi_0(u), \qquad z = \chi_0(u) \end{aligned} \qquad (0 \leq u \leq 1)$$

doivent fournir une représentation *normale* de $C$ et $\Gamma$, il en sera de même des formules:

$$\begin{aligned} &x = f_0[a_n(t)], \qquad y = g_0[a_n(t)], \qquad z = h_0[a_n(t)]\\ &x = \phi_0[\alpha_n(t)], \qquad y = \psi_0[\alpha_n(t)], \qquad z = \chi_0[\alpha_n(t)] \end{aligned} \qquad (0 \leq t \leq 1).$$

Soit $\delta_n(t)$ la distance de deux points correspondants dans cette correspondance particulière; $d_n$ son maximum, $\delta$ et $d$ les quantités analogues dans les représentations (4), (5). Les identités (6) et le fait que $a_n$, $\alpha$ tendent uniformément vers $a$, $\alpha$, tandis que $f_0, \cdots, \chi_0$ sont uniformément continues montrent que $\delta_n(t)$ tend uniformément vers $\delta(t)$. Autrement dit, pour $n > p$, on a:

$$\delta_n(t) - \epsilon < \delta(t) < \delta_n(t) + \epsilon$$

d'où:

$$\delta_n(t) - \epsilon < d$$

quelque soit $t$ et par suite:

$$d_n - \epsilon < d.$$

Mais: $d_n \geq e'$, donc: $d > e' - \epsilon$ quelque soit $\epsilon$, d'où: $d \geq e'$. Mais $d$ est une *quelconque* des quantités dont $e$ est la limite inférieure; donc $e \geq e'$; comme on a évidemment $e' \geq e$, on a $e = e'$.

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Nous pouvons aller plus loin encore. Au lieu de prendre pour calculer $e$ *toutes* les représentations normales simultanées de $C$ et de $\Gamma$, on peut prendre pour $C$ une représentation normale fixe, faire varier seulement la représentation normale de $\Gamma$ et prendre la limite inférieure des quantités $d$ ainsi obtenues. En effet, supposons que les formules (4), (5) donnent une représentation *normale* simultanée de $C$ et de $\Gamma$, choisie une fois pour toutes. Pour avoir toutes les autres, il suffira d'y remplacer $f(t)$, $g(t)$, $h(t)$ par : $f_1(t)=f[\lambda(t)]$, $g_1(t)=g[\lambda(t)]$, $h_1(t)=h[\lambda(t)]$ et $\phi(t)$, $\psi(t)$, $\chi(t)$ par $\phi_1(t)=\phi[\mu(t)]$, $\psi_1(t)=\psi(\mu(t)]$, $\chi_1(t)=\chi[\mu(t)]$ où $\lambda(t)$ et $\mu(t)$ sont deux fonctions de $t$ continues et toujours croissantes de 0 à 1. Or si l'on pose $u=\lambda(t)$, la fonction inverse $t=\lambda_1(u)$ sera aussi continue et croissante de 0 à 1 et il en sera de même de la fonction $\theta(u)=\mu[\lambda_1(u)]$.

Dès lors, on aura :

$$\sqrt{[f_1(t)-\phi_1(t)]^2+[g_1(t)-\psi_1(t)]^2+[h_1(t)-\chi_1(t)]^2}$$
$$=\sqrt{\{f(u)-\phi[\theta(u)]\}^2+\{g(u)-\psi[\theta(u)]\}^2+\{h(u)-\chi[\theta(u)]\}^2}$$

pour $u=\lambda(t)$ et par suite le maximum de ces deux quantités sera le même lorsque $t$ variant de 0 à 1, $u$ varie aussi de 0 à 1. Autrement dit, le maximum $d$ obtenu dans la correspondance normale quelconque est égal à l'un de ceux que l'on obtient en laissant la représentation normale de $C$ fixe et en faisant varier la représentation normale de $\Gamma$ qui pourra être en particulier la suivante :

$$x=\phi[\theta(u)], \qquad y=\psi[\theta(u)], \qquad z=\chi[\theta(u)] \qquad (0\leq u\leq 1).$$

*Propriétés de l'écart.*

THÉORÈME 1. *La condition nécessaire et suffisante pour que deux arcs de courbes continues coïncident est que leur écart soit nul.* En effet, si les deux courbes $C$, $\Gamma$ coïncident, on peut leur donner la même représentation paramétrique ; donc l'une des quantités $d$ dont leur écart $e$ est la limite inférieure sera nulle. Par suite $e$ est aussi nul.

Réciproquement, supposons que l'écart $e$ de deux arcs $C$ et $\Gamma$ soit nul. Si l'un de ces arcs, $C$ par exemple, se réduit à un point $A$, on sait que $e$ est le maximum de la distance de $A$ aux points de $\Gamma$. Donc $\Gamma$ se réduit aussi au point $A$. Supposons maintenant que ni $C$, ni $\Gamma$ ne se réduisent à un point. Alors, nous avons vu que $e$ sera la limite inférieure des quantités $d$ obtenues en prenant pour $C$ une représentation normale fixe (4) et pour $\Gamma$ la représention normale :

$$x=\phi[\theta(t)], \qquad y=\psi[\theta(t)], \qquad z=\chi[\theta(t)] \qquad (0\leq t\leq 1),$$

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où $\theta$ est une fonction de $t$ continue et toujours croissante de 0 à 1 et où pour $\theta(t) \equiv t$, on obtient une représentation normale fixe de $\Gamma$.

Si $e$ est nul, ou bien l'une des quantites $d$ est nulle et alors $C$ est identique à $\Gamma$, ou bien l'on peut trouver une infinité de formes de la fonction $\theta$: $\theta_1, \theta_2, \cdots, \theta_n, \cdots$ telles que les valeurs correspondantes de $d$: $d_1, d_2, \cdots, d_n, \cdots$ tendent vers zéro. On aura donc, quels que soient $t$ et $n$:

$$|f(t) - \phi[\theta_n(t)]| < d_n, \quad |g(t) - \psi[\theta_n(t)]| < d_n, \quad |h(t) - \chi[\theta_n(t)]| < d_n.$$

Par conséquent $\phi[\theta_n(t)]$, $\psi[\theta_n(t)]$, $\chi[\theta_n(t)]$ tendent uniformement vers $f(t), g(t), h(t)$. Cela prouve donc que tout point de $C$ est sur $\Gamma$; en effet pour une valeur arbitraire fixe de $t$: $t_2$, $\theta_n(t_2)$ reste comprise entre 0 et 1, on peut donc prendre dans $\theta_1, \theta_2, \cdots$ une suite $\theta_{n_1}, \theta_{n_2}, \cdots$ telle que $\theta_{n_1}(t_2), \theta_{n_2}(t_2), \cdots$ tendent vers une limite déterminée $\rho$. Alors on a:

$$f(t_2) = \lim_{p=\infty} \phi\,[\theta_{n_p}(t_2)] = \phi(\rho)$$

et de même:

$$g(t_2) = \psi(\rho), \qquad h(t_2) = \chi(\rho)$$

ce qui prouve notre assertion. Inversement puisque $C$ et $\Gamma$ jouent un rôle symmètrique, tout point de $\Gamma$ est sur $C$. Ainsi $C$ et $\Gamma$ sont formées des mêmes points de l'espace, mais il faut montrer que ces points sont rangés dans le même ordre.

Pour cela, rangeons dans une suite déterminée: $t_1, t_2, \cdots, t_n, \cdots$ l'ensemble dénombrable $E$ des nombres rationnels compris entre 0 et 1. Je dis d'abord que l'on peut extraire de la suite $\theta_1, \theta_2, \cdots$ une suite $\theta_{n_1}, \theta_{n_2}, \cdots$ qui soit convergente en tout point de $E$. En effet, les nombres $\theta_1(t_1), \theta_2(t), \cdots$ peuvent être considérées comme les abscisses de points compris entre 0 et 1, points qui ont au moins un point limite $\rho(t_1)$. On peut donc former une suite $\theta^{(1)}_{p_1}, \theta^{(1)}_{p_2}, \cdots$ qui converge vers $\rho(t_1)$ pour $t = t_1$. Mais, *dans cette nouvelle suite*, on pourra de même trouver une suite $\theta^{(2)}_{p_1}, \theta^{(2)}_{p_2}, \cdots$ qui converge vers un certain nombre $\rho(t_2)$ pour $t = t_1$ et qui continuera à converger vers $\rho(t_1)$ pour $t = t_1$. Et ainsi de suite.

Si nous considérons maintenant la suite $S$: $\theta^{(1)}_{p_1}, \theta^{(2)}_{p_2}, \cdots, \theta^{(n)}_{p_n}, \cdots$ elle est formée à partir du rang $n$ de termes qui figurent tous dans la suite $\theta^{(n)}_{p_n}, \theta^{(n)}_{p_{n+1}}, \cdots$. Par conséquent la suite $S$ converge vers un certain nombre $\rho(t_n)$ quelque soit $n$, comme nous voulions le montrer. La fonction $\rho(t)$ qui jusqu'à présent n'est définie que dans $E$, n'est jamais décroissante dans $E$. Car si $t_i > t_k$, on a quel que soit $n$: $\theta^{(n)}_{p_n}(t_i) - \theta^{(n)}_{p_n}(t_k) > 0$ et à la limite $\rho(t_i) - \rho(t_k) \geq 0$. Il en résulte que si $t$ est un nombre quelconque compris entre 0 et 1, les valeurs $\rho(t_i)$ (toutes comprises entre 0 et 1) pourront se ranger en deux classes correspondant aux valeurs de $\rho$ les unes inférieures, les autres supérieures à $t$ et les valeurs de

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la première classe seront inférieures aux valeurs de la seconde. Appelons $\rho(t-0)$ et $\rho(t+0)$ la limite supérieure des premières et la limite inférieure des secondes. Enfin posons pour simplifier : $\lambda_n(t) = \theta_{p_n}^{(n)}(t)$. D'après ce qui précède, on aura évidemment quel que soit $i$ :

$$f(t_i) = \lim_{n=\infty} \phi[\lambda_n(t_i)] = \phi[\rho(t_i)]$$

et par suite quelque soit $t$:

$$f(t) = \phi[\rho(t-0)] = \phi[\rho(t+0)].$$

Si je démontre que quelque soit $t_1$ on a : $\rho(t-0) = \rho(t+0)$ la fonction $\theta(t)$ qui sera égale à cette valeur commune quelque soit $t$ sera évidemment croissante et continue de 0 à 1 et telle que :

$$f(t) \equiv \phi[\theta(t)], \qquad g(t) \equiv \psi[\theta(t)], \qquad h(t) \equiv \chi[\theta(t)] \quad (0 \leq t \leq 1)$$

ce qui prouvera que $C$ et $\Gamma$ sont identiques.

Or supposons qu'il existe une valeur $\zeta$ de $t$ telle que $\rho(\zeta-0) \neq \rho(\zeta+0)$, et soit $u$ un nombre quelconque compris entre les deux précédents. Si $t'_i$ et $t''_k$ sont deux nombres rationnels compris entre 0 et 1, l'un inférieur à $\zeta$, l'autre supérieur à $\zeta$, on aura :

$$\lim_{n=\infty} \lambda_n(t'_i) \leq \rho(\zeta-0), \qquad \lim_{n=\infty} \lambda_n(t''_k) \geq \rho(\zeta+0).$$

Donc pour $n > p$:

$$\lambda_n(t'_i) < \rho(\zeta-0) + \epsilon, \qquad \lambda_n(t''_k) > \rho(\zeta+0) - \epsilon.$$

En prenant $\epsilon$ assez petit le nombre $u$ sera compris entre $\rho(\zeta-0)+\epsilon$ et $\rho(\zeta+0)-\epsilon$. Par suite, il y aura une valeur $\xi_{i,k}^{(n)}$ de $t$ comprise entre $t'_i$, $t''_k$ telle que la fonction continue croissante $\lambda_n$ prenne la valeur $\lambda_n(\xi_{i,k}^{(n)}) = u$, d'où :

$$f(\xi_{i,k}^{(n)}) = \phi(u), \qquad g(\xi_{i,k}^{(n)}) = \psi(u), \qquad h(\xi_{i,k}^{(n)}) = \chi(u).$$

Lorsque $n$ croit indéfiniment les points $\xi_{i,k}^{(n)}$ ont au moins un point limite $\xi_{i,k}$ compris entre $t'_i$ et $t''_k$; donc :

$$f(\xi_{i,k}) = \phi(u), \qquad g(\xi_{i,k}) = \psi(u), \qquad h(\xi_{i,k}) = \chi(u).$$

Faisons maintenant tendre $t'_i$ et $t''_k$ vers $\zeta$, on aura :

$$\phi(u) = \lim f(\xi_{i,k}) = f(\zeta) = \phi[\rho(\zeta-0)] = \phi[\rho(\zeta+0).$$

et de même :

$$\psi(u) = \psi[\rho(\zeta-0)] = \psi[\rho(\zeta+0)], \qquad \psi(u) = \chi[\rho(\zeta-0)] = \chi[\rho(\zeta+0)].$$

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Il en résulte que quelque soit le nombre $u$ compris entre $\rho(\zeta-0)$ et $\rho(\zeta+0)$, les fonctions $\phi(u)$, $\psi(u)$, $\chi(u)$ conservent chacune une valeur constante. Donc la représentation (5) ne serait pas normale ce qui est contraire à l'hypothèse. Ainsi $C$ et $\Gamma$ sont bien identiques.

THÉORÈME 2. *Etant donnés trois arcs de courbes continues quelconques $C_1$, $C_2$, $C_3$; on a toujours la relation suivante entre leurs écarts*

$$(C_1, C_2) \leq (C_1, C_3) + (C_2, C_3).$$

En effet, considérons trois représentations quelconques de $C_1$, $C_2$, $C_3$:

$$x = f_i(t), \qquad y = g_i(t), \qquad z = h_i(t), \qquad (0 \leq t \leq 1) \qquad (i = 1, 2, 3).$$

On a:

$$\sqrt{(f_1-f_2)^2+(g_1-g_2)^2+(h_1-h_2)^2} \leq \sqrt{(f_1-f_3)^2+(g_1-g_3)^2+(h_1-h_3)^2} + \sqrt{(f_2-f_3)^2+(g_2-g_3)^2+(h_2-h_3)^2}.$$

Appelons $d_3$, $d_2$, $d_1$ les maxima respectifs des trois radicaux quand $t$ varie. On aura quel que soit $t$:

$$\sqrt{(f_1-f_2)^2+(g_1-g_2)^2+(h_1-h_2)^2} \leq d_2 + d_1,$$

par suite le maximum $d_3$ du premier membre sera au plus égal à $d_1 + d_2$. D'ailleurs quel les que soient les représentations adoptées, on aura: $d_3 \geq (C_1, C_2)$ et par suite:

$$(C_1, C_2) \leq d_2 + d_1.$$

Le premier membre est fixe, le second a pour limite inférieure $(C_1, C_3)+(C_2, C_3)$, la proposition est donc établie.

*Limite de courbes continues.*

Il est à peine utile de faire observer l'analogie entre la notion de distance de deux points et celle d'écart de deux courbes, qui est mise en évidence par les théorèmes 1 et 2. Nous profiterons de cette analogie pour définir ainsi la limite d'une suite de courbes: *Nous dirons qu'une suite d'arcs continus*

$$C_1, C_2, C_3, \cdots, C_n, \cdots,$$

*a pour limite un arc continu $C$ lorsque l'écart $(C, C_n)$ tend vers zéro avec $1/n$.* Les théorèmes 1 et 2 montrent que cette définition satisfait aux deux conditions suivantes: 1° si $C_1, C_2, \cdots, C_n, \cdots$ sont des courbes continues identiques à une même courbe $C$, elles ont pour limite $C$; 2° si $C_1, C_2, C_3, \cdots, C_n, \cdots$ ont

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une limite $C$, toute suite infinie extraite de la première en conservant l'ordre tendra aussi vers $C$.*

On peut ramener cette définition à la définition ordinaire au moyen de la proposition suivante :

THÉORÈME 3. *La condition nécessaire et suffisante pour qu'un arc continu $C_n$ tende vers un arc continu déterminé $C$ lorsque $n$ croit indéfiniment, est que, quelle que soit la représentation paramétrique de $C$:*

$$x = f(t), \qquad y = g(t), \qquad z = h(t) \qquad (0 \leq t \leq 1), \tag{6}$$

*on puisse choisir celle de $C_n$:*

$$x = f_n(t), \qquad y = g_n(t), \qquad z = h_n(t) \qquad (0 \leq t \leq 1), \tag{7}$$

*de façon que $f_n$, $g_n$, $h_n$ tendent uniformément vers $f$, $g$, $h$ respectivement.*

En effet, on a en appelant $d_n$ le maximum de la distance $\delta_n(t)$ de deux points correspondants :

$$(C, C_n) \leq d_n$$

et comme $\delta_n(t)$ tend uniformément vers zéro, $d_n$ et par consequent $(C, C_n)$ tendent aussi vers zéro.

Réciproquement, la représentation de $C$ étant fixée, en faisant varier celle de $C_n$, on obtient différentes valeurs de $d_n$, dont $(C, C_n)$ est la limite inférieure. Par conséquent, on pourra choisir la représentation de $C_n$ de façon que l'on ait :

$$d_n < (C, C_n) + \frac{1}{n}.$$

Et alors pour une telle représentation, on aura,

$$|f - f_n| < (C, C_n) + \frac{1}{n}, \quad |g - g_n| < (C, C_n) + \frac{1}{n}, \quad |h - h_n| < (C, C_n) + \frac{1}{n}.$$

Donc si $(C, C_n)$ tend vers zéro avec $1/n$, les fonctions $f_n$, $g_n$, $h_n$ tendront uniformément vers $f$, $g$, $h$.

*Remarque.* On voit que si $C_n$ tend vers $C$ (non réduit à un point), on peut établir entre les points de $C_n$ et de $C$ *une* correspondance univoque et réciproque dans laquelle l'ordre soit conservé, de façon que si $M_n$ de $C_n$ correspond à $M$ de $C$, $M_n$ tend vers $M$ et cela uniformément. On peut se demander, ce que devient le point $M_n$ qui correspond à $M$ dans une correspondance *quelconque* analogue à la première, en supposant toujours que $C_n$ tend vers $C$. Dans le cas où $C$ se

* On peut donc lui appliquer les théorèmes énoncés dans les Comptes-Rendus du 21 Nov. 1904. *Généralisation d'un théorème de Weierstrass* et du 2 Janvier 1905 : *Sur les fonctions limites et les opérations fonctionnelles.*

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réduit à un point, $M_n$ tendra encore vers ce point. Si $C$ ne se réduit pas à un point, il en sera de même de $C_n$ pour $n$ assez grand; il y aura donc parmi les représentations de $C$ et $C_n$, une représentation *normale* simultanée (6), (7) de $C$ et $C_n$, telle que $f_n$, $g_n$, $h_n$ tendent uniformément vers $f$, $g$, $h$. On obtiendra une correspondance quelconque en remplaçant dans $f_n(t)$, $g_n(t)$ et $h_n(t)$: $t$ par $\theta_n(t)$ fonction continue et croissante de 0 à 1. Le point $M_n$ qui correspondra ainsi au point $M$ de $C$ donné par une valeur $t$ du paramètre, aura pour coordonnées:

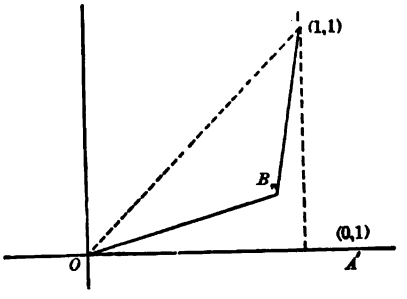
$$x = f_n[(\theta_n(t)], \qquad y = g_n[\theta_n(t)], \qquad z = h_n[\theta_n(t)].$$

Lorsque $n$ croit indéfiniment ce point peut avoir un ou plusieurs points limites en général distincts de $M$, mais: *tous ces points limites sont des points de $C$.* En effet, soit $N$ une point limite de $M_n$: on peut trouver $n_1, n_2, \cdots$ tels que la suite $M_{n_1}, M_{n_2}, \cdots$ tend vers $N$. Mais, pour la valeur de $t$ considérée, les nombres $\theta_{n_1}(t), \theta_{n_2}(t), \cdots$ sont les abscisses de points compris entre 0 et 1 qui ont par suite un point limite $\theta(t)$. Il y a donc dans $\theta_{n_1}(t), \theta_{n_2}(t), \cdots$ une suite: $\theta_{p_1}(t), \theta_{p_2}(t), \cdots$ qui tend vers un nombre déterminé $\theta(t)$. Or on a:

$$|f_{p_i}[\theta_{p_i}(t)] - f[\theta(t)]| < |f_{p_i}[\theta_{p_i}(t)] - f[\theta_{p_i}(t)]| + |f[\theta_{p_i}(t)] - f[\theta(t)]|.$$

Puisque $f_n$ tend vers $f$ *uniformément* le premier terme du second membre tend vers zéro. Il en est de même du second puisque $f(t)$ est continue. Donc $f_{p_i}[\theta_{p_i}(t)]$ tend vers $f[\theta(t)]$; de même pour les deux autres coordonnées; par conséquent les points $M_{p_1}, M_{p_2}, \cdots$ qui tendent vers $N$, tendent aussi vers le point de $C$ de coordonnées: $f[\theta(t)], g[\theta(t)], h[\theta(t)]$. Donc $N$ est sur $C$.

La réciproque de cette proposition n'est pas vraie: *tout point de $C$ n'est pas nécessairement point limite de points $M_n$.* Ainsi prenons en particulier




pour $\theta_n(t)$ une fonction continue croissante de 0 à 1 mais qui tend vers 0 pour $t \neq 1$ et vers 1 pour $t = 1$. Alors on voit que l'on établit entre $C$ et $C_n$ une correspondance telle que tout point $M_n$ de $C_n$ correspondant à un point $M$ fixe

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quelconque de $C$ tend vers l'origine de l'arc $C$, sauf si $M$ est l'extrémité de $C$, cas où $M_n$ tend vers cette extrémité. [Il suffit pour construire les fonctions $\theta_n$, de considérer l'équation $y = \theta_n(x)$ comme l'équation dans le plan $0xy$ de la ligne brisée $0B_nA$ où $A$ est un point fixe de coordonnées $(1, 1)$ et où $B_n$ est un point du triangle rectangle $0A'A$ tendant vers le point $A'$, $(1, 0)$.]

THÉORÈME 4. *La condition nécessaire et suffisante pour qu' un arc continu $C_n$ tende vers un arc continu déterminé $C$ lorsque $n$ croit indéfiniment est que, quel que soit $\epsilon$, on puisse trouver un nombre entier $n$ tel que quel que soit l'entier $p$ l'écart $(C_n, C_{n+p})$ soit inférieur à $\epsilon$.*

Cette condition est évidemment nécessaire, car si $C_n$ tend vers un arc continu $C$, on pourra prendre un entier $g$ tel que l'on ait $(C, C_n) < \epsilon/2$ pour $n > g$. Alors on aura :

$$(C_n, C_{n+p}) \leq (C, C_n) + (C, C_{n+p}) < \frac{\epsilon}{2}$$

quel que soit $p$ pour $n > g$.

Réciproquement, supposons qu'une telle condition soit vérifiée. Alors quel que soit l'entier $n$, on pourra trouver un entier croissant $r_n$, tel que l'on ait quel que soit $p$ :

$$(C_{r_n}, C_{r_n+p}) < \frac{1}{n^2}.$$

Appelons $\Gamma_n$ la courbe $C_{r_n}$; on voit que l'on aura quels que soient $n$ et $p$ :

$$(\Gamma_n, \Gamma_{n+p}) < \frac{1}{n^2}.$$

Ceci étant, choisissons les représentations paramétriques de $\Gamma_1, \Gamma_2, \cdots$; on sait qu'étant donnée la représentation d'une arc continu $\Gamma_n$, on peut toujours choisir celle de $\Gamma_{n+1}$ de façon qu'en appelant $d_{n,n+1}$ le maximum de la distance de deux points correspondants, on ait :

$$d_{n,n+1} < (\Gamma_n, \Gamma_{n+1}) + \frac{1}{n^2}.$$

Nous pourrons ainsi choisir successivement les représentations de $\Gamma_1, \Gamma_2, \cdots$ de façon à satisfaire à cette condition et par conséquent à :

$$d_{n,n+1} < \frac{2}{n^2}.$$

D'ailleurs, on aura évidemment par un raisonnement analogue à celui employé pour le théorème 2 :

$$d_{n,n+p} \leq d_{n,n+1} + d_{n+1,n+2} + \cdots + d_{n+p-1,n+p}$$

quels que soient $n$ et $p$ et par suite :

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$$d_{n,n+p} < \frac{2}{n^2} + \frac{2}{(n+1)^2} + \cdots + \frac{2}{(n+p-1)^2} < \rho_n$$

$\rho_n$ étant le reste de la série dont le terme général est $2/n^2$. Si donc la représentation choisie pour $\Gamma_n$ est:

$$x = f_n(t), \qquad y = g_n(t), \qquad z = h_n(t) \qquad (0 \leq t \leq 1)$$

on voit que l'on aura quels que soient $n$ et $p$:

$$|f_n - f_{n+p}| < \rho_n, \qquad |g_n - g_{n+p}| < \rho_n, \qquad |h_n - h_{n+p}| < \rho_n \quad (0 \leq t \leq 1).$$

$\rho_n$ étant un nombre indépendant de $t$ qui tend vers zéro avec $1/n$. Par conséquent $f_n$, $g_n$, $h_n$ convergent uniformément vers trois fonction continues $f$, $g$, $h$ (d'après un théorème connu de CAUCHY). Dès lors $\Gamma_n$ tend vers un arc continu $\Gamma$. Alors il en aussi de même des courbes $C_1, C_2, \cdots$. En effet, on a:

$$(\Gamma, C_p) \leq (\Gamma, C_{r_n}) + (C_{r_n}, C_p).$$

Donc en prenant $p > r_n$ et $r_n$ assez grand, on pourra rendre chacun des deux derniers termes plus petit qu'une quantité arbitraire donnée. Par suite $(\Gamma, C_p)$ tend vers zéro: il est bien démontré que $C_p$ tend vers un arc continu quand $p$ croit indéfiniment. D'ailleurs il pourra fort bien arriver que cet arc soit réduit à un point.

*Ensembles compacts.*

Nous dirons qu'un ensemble $E$ d'arcs continus est compact s'il ne comprend qu'un nombre fini d'éléments ou bien, dans le cas contraire, si tout ensemble $H$ formé d'une infinité de courbes de $E$ admet au moins une courbe limite. * Il est entendu qu'un ensemble $H$ admet une élément limite $\Gamma$, s'il existe dans $H$ une infinité d'éléments *distincts*: $C_1, C_2, \cdots, C_n, \cdots$ qui tendent vers $\Gamma$.

Nous allons maintenant chercher la condition pour qu'un ensemble de courbes soit compact. Pour y arriver nous introduirons *l'indice de compacité* d'un ensemble de courbes: Nous appellerons ainsi un nombre $\nu$ égal à zéro si l'ensemble $E$ considéré ne comprend qu'un nombre fini de courbes distinctes et défini ainsi dans le cas contraire: Soit $S$ une suite infinie de courbes distinctes $C_1, C_2, \cdots, C_n, \cdots$ de l'ensemble $E$; appelons $\epsilon_n$ la limite supérieure des écarts $(C_n, C_{n+1}), (C_n, C_{n+2}), \cdots$ et $\lambda_S$ la plus grande des limites de $\epsilon_n$. Si $H$ est un ensemble formé d'une infinité de courbes de $E$, à toute suite infinie $S$ contenue dans $H$ correspondra un nombre déterminé $\lambda_S$ et nous pourrons prendre la limite inférieure $\mu_H$ des nombres $\lambda_S$. Par définition, l'indice de compacité de $E$ sera la limite supérieure $\nu$ de $\mu_H$ quand $H$ est quelconque dans $E$. Les quantités $\epsilon_n$, $\lambda_S$, $\mu_H$, $\nu$ sont d'ailleurs positives on nulles, finies on infinies, mais bien déterminées.

* Voir la note déja citée des Comptes-Rendus du 21 Nov., 1904.

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THÉORÈME 5. *La condition nécessaire et suffisante pour qu'un ensemble $E$ d'arcs continus soit compact est que son indice de compacité soit nul.*

La condition est nécessaire. En effet, si $E$ n'a qu'un nombre fini d'éléments, $\nu$ est nul par définition. Dans le cas contraire, quel que soit $H$ dans $E$, il y a une suite convergente $S$ de courbes de $H$ et d'après le théorème V, on aura évidemment $\lambda_S = 0$ d'où $\mu_H = 0$. Alors $\mu_H$ étant constamment nul, sa limite supérieure $\nu$ sera aussi nulle.

Réciproquement supposons nul l'indice de compacité de $E$. On bien $E$ n'a qu'un nombre fini d'éléments et alors c'est un ensemble compact, on bien $\mu_H$ sera nul pour tout ensemble $H$ formé d'une infinité de courbes de $E$. Considérons l'un quelconque d'entre eux; je dis qu'il admet une courbe limite (ce qui suffit pour démontrer le théorème). En effet, dans le cas contraire, il n'y aurait aucune suite infinie $S$ de courbes de $H$ telles que $\lambda_S$ soit nul. Puisque la limite inférieure $\nu$ des $\lambda_S$ est nulle, il faut donc que quel que soit le nombre positif $\epsilon_1$, on puisse trouver dans $H$, une suite $S_1$ telle que $0 < \lambda_{S_1} < \epsilon_1$. Mais $S_1$ étant formée d'éléments de $E_1$, on a $\mu_{S_1} = 0$ et le même raisonnement s'applique à $S_1$ comme à $H$. $S_1$ n'admet aucune limite sans quoi il en serait de même de $H$. On peut donc trouver dans la suite $S_1$: $C_1^{(1)}, C_2^{(1)}, C_3^{(1)}, \cdots, C_n^{(1)}, \cdots$ une suite infinie $S_2$ de courbes: $C_1^{(2)}, C_2^{(2)}, C_3^{(2)}, \cdots, C_n^{(2)}, \cdots$ telle que $0 < \lambda_{S_2} < \epsilon_2$. On peut même supposer que les éléments de la suite $S_2$ se trouvent disposés dans le même ordre que dans $S_1$. On pourra répéter le même raisonnement sur $S_2$ et ainsi de suite. On formera ainsi des suites $S_1, S_2, \cdots, S_n, \cdots$ telles que $\lambda_{S_n} < \epsilon_n$ quel que soit $n$. Prenons par exemple $\epsilon_n = 1/n$ et considérons la suite $S$: $C_1^{(1)}, C_2^{(2)}, C_3^{(3)}, \cdots, C_n^{(n)}, \cdots$. A partir du rang $n$, elle est formée d'éléments qui figurent tous dans $S_n$ et dans le même ordre. On en déduit facilement l'inégalité $\lambda_S \leq \lambda_{S_n}$ quelque soit $n$ et par suite $\lambda_S = 0$. Donc il y aurait dans $H$ une suite $S$ telle que $\lambda_S = 0$; nous arrivons ainsi à la contradiction annoncée.

*Remarque.* C'est à M. ASCOLI que revient le mérite d'avoir trouvé le premier une condition nécessaire et suffisante pour qu'un ensemble de courbes continues soit compact.* Mais la condition que nous avons donnée au théorème 5 nous parait préferable au point de vue théorique. En effet, il est nécessaire de connaître une représentation paramétrique déterminée des courbes de l'ensemble pour s'assurer si la condition de M. ASCOLI est satisfaite. Au contraire, il suffit pour calculer notre indice de compacité de connaître l'ensemble des valeurs des écarts des courbes de $E$ deux à deux. Toute autre indication sur la forme ou la position des courbes de $E$ est superflue pour ce calcul. Nous n'utilisons donc que des éléments géométriques et seulement ceux qui sont essentiels.

PARIS, *Mai*, 1905.

* G. ASCOLI, *Sulle curve limite di una varietà data di curve*, Accademia dei Lincei, 1884. Voir aussi : C. ARZELÀ : *Funzioni di linee*, Accademia dei Lincei, 1889.

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# ON A CERTAIN SYSTEM OF CONJUGATE LINES ON A SURFACE CONNECTED WITH EULER'S TRANSFORMATION*

BY

JOHN EIESLAND

In a paper published in vol. 26 of the American Journal of Mathematics I have proved a number of theorems concerning curves and two-dimensional surfaces in five-dimensional space which belong to a so-called asymptotic complex whose lines satisfy the differential equations

$$(1)\qquad \begin{aligned} dx_5 + x_2\,dx_1 - x_1\,dx_2 + x_4\,dx_3 - x_3\,dx_4 &= 0, \\ dx_1\,dx_2 + dx_3\,dx_4 &= 0. \end{aligned}$$

If $(u)$ and $(v)$ are the coördinate lines on a surface belonging to such a complex,† and if we make use of the transformation ‡

$$(2)\qquad x_1 = \frac{P_1}{2},\ x_2 = X_1,\ x_3 = \frac{P_2}{2},\ x_4 = X_2,\ x_5 + x_1x_2 + x_3x_4 = X_3,$$

where $X_1, X_2, X_3, P_1, P_2, -1$ are the coördinates of a surface-element in ordinary space, we obtain, as I have shown, a surface in three dimensional space on which the lines $(u)$ and $(v)$ are asymptotic lines.

The geometry of asymptotic complexes is thus seen to be closely connected with the general theory of surfaces; in fact, in five dimensions, to any geometric property of a two-dimensional point-manifoldness belonging to an asymptotic complex corresponds a property of surfaces in ordinary space.

In the first part of this paper it is shown that a certain single projective transformation of the complex (1) will lead to EULER's classical transformation

* Presented to the Society under a slightly different title at the Boston summer meeting, September 1, 1903. Received for publication October 24, 1903, and April 10, 1905.

† A surface is said to belong to the complex (1) whenever the linear tangents along the $u$ and $v$ lines are lines of the complex.

‡ This transformation, which was used by LIE, when generalized for $2r+1$ variables establishes a correspondence between two spaces $M_{2r+1}$ and $M_r$ of such a nature that the projective group of the linear complex $dx_{2r+1} + \Sigma(x_\nu\,dx_\mu - x_\mu\,dx_\nu) = 0$ becomes a group of irreducible contact-transformations in the space $M_r$. See LIE, *Theorie der Transformationsgruppen*, Abschnitt II, p. 522.



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in three-dimensional space. This transformation, moreover, transforms the asymptotic lines on a surface into a definite system of conjugate lines having a well-defined geometric property which characterizes the system. I have called these lines *Euler's lines*, since they are inseparably bound up with the transformation that bears his name.

The problem to find all surfaces such that EULER's lines are lines of curvature is then considered, and it is found that the determination of such surfaces leads to the integration of a partial differential equation with equal invariants and quadratures. It follows as an immediate corollary that on these surfaces the lines of curvature correspond to asymptotic lines on the transformed surface by EULER's transformation.

While in LIE's sphere-geometry asymptotic lines correspond to lines of curvature by virtue of the well-known contact-transformation that transforms all the $\infty^4$ lines in space into $\infty^4$ spheres, in the case of EULER's transformation such correspondence is established only in the case of certain surfaces. It is also worthy of notice that while in LIE's sphere-geometry to a real surface corresponds in general an imaginary surface (the sphere being imaginary), in EULER's transformation corresponding elements are either both real or both imaginary.*

The second part has been devoted to the geometrical definition of EULER's lines and the derivation of their differential equation from this definition.

In the third part it is shown that EULER's transformation is only one among $\infty^{10}$ which change asymptotic lines on a surface into EULER's lines on the corresponding surface. A group of contact-transformations leaving EULER's lines invariant is also considered, and it is shown that it contains $\infty^{10}$ such transformations.

## PART 1.

Let there be given in the space $M_5$ a two-dimensional surface belonging to the complex

$$dx_5 + x_2 dx_1 - x_1 dx_2 + x_4 dx_3 - x_3 dx_4 = 0, \tag{3}$$

and let it be written

$$x_i = \phi_i(u, v) \qquad (i = 1, 2, \cdots, 5). \tag{4}$$

The following conditions must evidently be fulfilled:

$$\begin{aligned} &\frac{\partial \phi_5}{\partial u} + \phi_2 \frac{\partial \phi_1}{\partial u} - \phi_1 \frac{\partial \phi_2}{\partial u} + \phi_4 \frac{\partial \phi_3}{\partial u} - \phi_3 \frac{\partial \phi_4}{\partial u} = 0, \\ &\frac{\partial \phi_5}{\partial v} + \phi_2 \frac{\partial \phi_1}{\partial v} - \phi_1 \frac{\partial \phi_2}{\partial v} + \phi_4 \frac{\partial \phi_3}{\partial v} - \phi_3 \frac{\partial \phi_4}{\partial v} = 0. \end{aligned} \tag{5}$$

* See LIE, *Geometrie der Berührungstransformationen*, vol. 1, pp. 411–480.



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Using the transformation (2) we obtain in the space $M_3$ a surface

$$X_1 = \phi_2, \qquad X_2 = \phi_4, \qquad X_3 = \phi_1\phi_2 + \phi_3\phi_4 + \phi_5,$$

which is the image of the surface (4) considered as a point-locus in $M_5$. If the complex (3) be asymptotic, that is, if the additional Monge equation

$$dx_1 dx_2 + dx_3 dx_4 = 0$$

be satisfied, and if the surface (4) belongs to this complex, we obtain, as I have proved in the paper mentioned above, a surface in $M_3$ on which $(u)$ and $(v)$ are asymptotic lines. The analytical conditions which must be satisfied by the functions $\phi_i$ are, besides (5), the following

$$(6) \qquad \begin{gathered} \frac{\partial\phi_1}{\partial u}\frac{\partial\phi_2}{\partial u} + \frac{\partial\phi_3}{\partial u}\frac{\partial\phi_4}{\partial u} = 0, \qquad \frac{\partial\phi_1}{\partial v}\frac{\partial\phi_2}{\partial v} + \frac{\partial\phi_3}{\partial v}\frac{\partial\phi_4}{\partial v} = 0, \\ \frac{\partial\phi_2}{\partial u}\frac{\partial\phi_3}{\partial v} + \frac{\partial\phi_2}{\partial v}\frac{\partial\phi_3}{\partial u} = 0. \end{gathered}$$

From this it may be proved that the coordinates $\phi_2$ and $\phi_4$ must satisfy the differential equation with equal invariants *

$$(7) \qquad \frac{\partial^2\theta}{\partial u\,\partial v} + \tfrac{1}{2}\frac{\partial}{\partial v}\log R\cdot\frac{\partial\theta}{\partial u} + \tfrac{1}{2}\frac{\partial}{\partial u}\log R\cdot\frac{\partial\theta}{\partial v} = 0,\dagger$$

where

$$R = \frac{\dfrac{\partial\phi_3}{\partial u}}{\dfrac{\partial\phi_2}{\partial u}} = -\frac{\dfrac{\partial\phi_3}{\partial v}}{\dfrac{\partial\phi_2}{\partial v}}.$$

Conversely, whenever particular solutions $\phi_2$ and $\phi_4$ of this equation can be found, the other functions $\phi_1$, $\phi_3$ and $\phi_5$ can be obtained by quadratures and the corresponding surfaces thus be determined‡. It may also be proved that to

* A partial differential equation of the second order

$$\frac{\partial^2\phi}{\partial u\,\partial v} + a\frac{\partial\phi}{\partial u} + b\frac{\partial\phi}{\partial v} + c\phi = 0$$

is said to be one of equal invariants whenever

$$\frac{\partial a}{\partial u} + ab - c = \frac{\partial b}{\partial v} + ab - c.$$

See DARBOUX, *Théorie des Surfaces*, vol. II, chapter 2.

† It should also be noticed that that the function $X_3 = \phi_1\phi_2 + \phi_3\phi_4 + \phi_5$ satisfies (7).

‡ American Journal of Mathematics, vol. 26, pp. 130–134.

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any surface in $M_3$ referred to its asymptotic lines corresponds in $M_5$ a two-dimensional surface belonging to an asymptotic complex.

Closely associated with an asymptotic complex is a complex defined by the equations

$$(8)\qquad \begin{gathered} dx_5 + x_2 dx_1 - x_1 dx_2 + x_4 dx_3 - x_3 dx_4 = 0, \\ dx_1 dx_2 - dx_3 dx_4 = 0; \end{gathered}$$

in fact, if we employ the transformation

$$(9)\qquad x_1 = \frac{\bar{x}_2}{2},\ x_2 = -2\bar{x}_1,\ x_3 = \bar{x}_3,\ x_4 = \bar{x}_4,\ x_5 = \bar{x}_5,$$

we obtain an asymptotic complex. In the space $M_3$ this transformation is equivalent to the well-known EULER's transformation *

$$(9')\qquad P_1 = \bar{X}_1,\ X_1 = -\bar{P}_1,\ P_2 = \bar{P}_2,\ X_2 = \bar{X}_2,\ X_3 = \bar{X}_3 - \bar{P}_1\bar{X}_1$$

which transforms the surface-elements of $M_3$ into the surface-elements of the corresponding space $\bar{M}_3$; moreover, it is a contact-transformation, since

$$(10)\qquad dX_3 - P_1 dX_1 - P_2 dX_2 = d\bar{X}_3 - \bar{P}_1 d\bar{X}_1 - \bar{P}_2 d\bar{X}_2.$$

Let there now be given a surface in $M_5$ belonging to the complex (8). We have

$$(11)\qquad \begin{gathered} \frac{\partial\phi_5}{\partial u} + \phi_2\frac{\partial\phi_1}{\partial u} - \phi_1\frac{\partial\phi_2}{\partial u} + \phi_4\frac{\partial\phi_3}{\partial u} - \phi_3\frac{\partial\phi_4}{\partial u} = 0, \\ \frac{\partial\phi_5}{\partial v} + \phi_2\frac{\partial\phi_1}{\partial v} - \phi_1\frac{\partial\phi_2}{\partial v} + \phi_4\frac{\partial\phi_3}{\partial v} - \phi_3\frac{\partial\phi_4}{\partial v} = 0. \end{gathered}$$

$$(12)\qquad \frac{\partial\phi_1}{\partial u}\frac{\partial\phi_2}{\partial u} - \frac{\partial\phi_3}{\partial u}\frac{\partial\phi_4}{\partial u} = 0, \qquad \frac{\partial\phi_1}{\partial v}\frac{\partial\phi_2}{\partial v} - \frac{\partial\phi_3}{\partial v}\frac{\partial\phi_4}{\partial v} = 0.$$

In order that $d\phi_5$ shall be an exact differential, we must also have

$$\frac{\partial\phi_1}{\partial u}\frac{\partial\phi_2}{\partial v} - \frac{\partial\phi_1}{\partial v}\frac{\partial\phi_2}{\partial u} + \frac{\partial\phi_3}{\partial u}\frac{\partial\phi_4}{\partial v} - \frac{\partial\phi_3}{\partial v}\frac{\partial\phi_4}{\partial u} = 0,$$

which, by the aid of the two equations (12), reduces to the form

$$\left(\frac{\partial\phi_3}{\partial u}\frac{\partial\phi_1}{\partial v} + \frac{\partial\phi_3}{\partial v}\frac{\partial\phi_1}{\partial u}\right)\left(\frac{\partial\phi_4}{\partial u}\frac{\partial\phi_1}{\partial v} - \frac{\partial\phi_4}{\partial v}\frac{\partial\phi_1}{\partial u}\right) = 0.$$

* See, for example, LIE, *Berührungstransformationen*, p. 645.

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We shall assume that the second factor is different from zero,* so that we may put

$$\frac{\partial \phi_3}{\partial u}\frac{\partial \phi_1}{\partial v}+\frac{\partial \phi_3}{\partial v}\frac{\partial \phi_1}{\partial u}=0, \tag{13}$$

which may be replaced by the equivalent one

$$\frac{\partial \phi_1}{\partial u}\frac{\partial \phi_2}{\partial v}+\frac{\partial \phi_3}{\partial u}\frac{\partial \phi_4}{\partial v}=0. \tag{13'}$$

obtained by eliminating $\partial\phi_1/\partial v$ and $\partial\phi_3/\partial v$ from (13) and the second of (12).

If now we transform the surface $x_i=\phi_i\,(u, v)$ by the transformation (2) *we obtain in $M_3$ a surface on which $(u)$ and $(v)$ are conjugate lines.* In fact, introducing the coördinates of $M_3$ in (13′) we get

$$\frac{\partial X_1}{\partial v}\frac{\partial P_1}{\partial u}+\frac{\partial X_2}{\partial v}\frac{\partial P_2}{\partial u}=0,$$

which is the condition that $(u)$ and $(v)$ shall be conjugate lines. Hence the

THEOREM. *To a surface in $M_5$ belonging to the complex*

$$dx_5+x_2dx_1-x_1dx_2+x_4dx_3-x_3dx_4=0,$$

$$dx_1dx_2-dx_3dx_4=0$$

*corresponds by virtue of the transformation*

$$x_1=\frac{P_1}{2},\ x_2=X_1,\ x_3=\frac{P_2}{2},\ x_4=X_2,\ x_1x_2+x_3x_4+x_5=X_3$$

*a surface in $M_3$ on which $(u)$ and $(v)$ are conjugate lines.*

Suppose now that $\overline{M}_5$ be a space with coördinates $\bar{x}_i$ and let a one-to-one correspondence be established between it and the space $M_5$ by means of the transformation (9). Since the complex (8) is transformed into an asymptotic complex in $\overline{M}_5$, any surface belonging to the former is transformed into a surface belonging to the latter, and conversely; hence, if we obtain the images in $M_3$ and $\overline{M}_3$ of the respective surfaces $S_3$ and $\overline{S}_3$, using the transformation (2), these will be of such a nature that EULER's transformation transforms $S_3$ into $\overline{S}_3$ and, moreover, to the conjugate lines on $S_3$ correspond asymptotic lines on $\overline{S}_3$ and conversely, so that we *may* say:

*By means of Euler's transformation a one-to-one correspondence is established between two spaces $M_3$ and $\overline{M}_3$ such that all the surfaces $S_3$ in $M_3$ whose images in $M_5$ are surfaces belonging to the complex*

*It may easily be proved that if this factor vanishes the surface $x_i=\phi_i$ $(i=1,\cdots,5)$ will degenerate into a curve; hence the assumption.

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$$dx_5 + x_2 dx_1 - x_1 dx_2 + x_4 dx_3 - x_3 dx_4 = 0, \qquad dx_1 dx_2 - dx_3 dx_4 = 0,$$

*are transformed into surfaces in* $\overline{M}_3$ *whose images in* $\overline{M}_5$ *are surfaces belonging to an asymptotic complex. To the conjugate curves* $(u)$ *and* $(v)$ *on* $S_3$ *correspond the asymptotic lines* $(u)$ *and* $(v)$ *on* $\overline{S}_3$, *and conversely, to the asymptotic lines* $(u)$ *and* $(v)$ *on* $\overline{S}_3$ *correspond a set of conjugate lines* $(u)$ *and* $(v)$ *on* $S_3$.

If now we put

$$R = \frac{\dfrac{\partial \phi_2}{\partial u}}{\dfrac{\partial \phi_4}{\partial u}} = -\frac{\dfrac{\partial \phi_2}{\phi v}}{\dfrac{\partial \phi_4}{\partial v}}, \tag{14}$$

the conditions (12) and (13′) may be written

$$\frac{\partial \phi_3}{\partial u} = R\frac{\partial \phi_1}{\partial u}, \qquad \frac{\partial \phi_3}{\partial v} = -R\frac{\partial \phi_1}{\partial v}, \tag{15}$$

$$\frac{\partial \phi_2}{\partial u} = R\frac{\partial \phi_4}{\partial u}, \qquad \frac{\partial \phi_2}{\partial v} = -R\frac{\partial \phi_4}{\partial v}. \tag{16}$$

Eliminating $\partial\phi_3/\partial u, \partial\phi_3/\partial v, \partial\phi_2/\partial u, \partial\phi_2/\partial v$ we find that $\phi_4$ and $\phi_1$ must satisfy the differential equation

$$\frac{\partial^2 \phi}{\partial u \partial v} + \tfrac{1}{2}\frac{\partial}{\partial v}\log R \cdot \frac{\partial \phi}{\partial u} + \tfrac{1}{2}\frac{\partial}{\partial u}\log R \cdot \frac{\partial \phi}{\partial v} = 0. \tag{17}$$

Conversely, if we know two particular solutions, $\phi_4$ and $\phi_1$, of this equation, $\phi_3$ and $\phi_2$ may be obtained by quadratures from (15) and (16) and $\phi_5$ likewise from (11).* The surface

$$X_1 = \phi_2, \qquad X_2 = \phi_4, \qquad X_3 = \phi_5 + \phi_1\phi_2 + \phi_3\phi_4, \tag{18}$$

has $(u)$ and $(v)$ for conjugate lines and is transformed by EULER's transformation into a surface on which $(u)$ and $(v)$ are asymptotic lines.

Since the surface (18) is referred to a set of conjugate lines the differential equation of its asymptotic lines must take the form †

$$A du^2 + B dv^2 = 0;$$

in fact, we find by an easy calculation that this equation is

$$\frac{\partial \phi_1}{\partial u}\frac{\partial \phi_2}{\partial u} du^2 + \frac{\partial \phi_1}{\partial v}\frac{\partial \phi_2}{\partial v} dv^2 = 0. \tag{19}$$

* It may also be shown that the function $\phi_5 + \phi_3\phi_4 - \phi_1\phi_2$ satisfies the above equation.

† See DARBOUX, *Leçons*, vol. 1, p. 139, § 110.

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*Example* I. Let $R = \text{const.}$, then $\phi_1 = \rho_1(u) + \sigma_1(v) = c_1(u - v)$, and $\phi_4 = \rho_4(u) + \sigma_4(v)$. From (15) we find that $\phi_3$ must have the form $\phi_3 = cc_1(u + v)$; likewise from (16) we get $\phi_2 = c[\rho_4(u) - \sigma_4(v)]$ and from (11)

$$\phi_5 + \phi_1\phi_2 + \phi_3\phi_4 = 4cc_1 \int u\rho_4' du + 4cc_1 \int v\sigma_4' dv.$$

If now we put

$$\int \rho_4 du = -F(u), \qquad \int \sigma_4 dv = -F_1(v), \qquad c = k, \qquad c_1 = \frac{1}{k},$$

we obtain in $M_3$ the surface

$$(20) \quad X_1 = -\frac{k}{4}(F' - F_1'), \; X_2 = -F' - F_1', \; X_3 = -uF' - vF_1' + F + F_1.$$

By means of EULER's transformation we obtain the surface

$$(21) \quad \begin{aligned} \overline{X}_1 &= \frac{u - v}{k}, \qquad X_2 = -F' - F_1', \\ \overline{X}_3 &= -\frac{u + v}{2}F' - \frac{u + v}{2}F_1' + F + F_1', \end{aligned}$$

which may also be obtained directly by considering a translation surface in $M_3$ belonging to an asymptotic complex.* DARBOUX † has derived the same surface by a different method.

The equation of asymptotic lines reduces in the case of the surfaces (20) to the form

$$F''(u)du^2 - F_1''(v)dv^2 = 0$$

which may be integrated by quadratures; it follows, therefore, that the corresponding set of conjugate lines on (21) may be so obtained. ‡

If we take note of the fact that the surface (20) is the most general form of a translation surface whose generating curves are in planes forming a constant angle $\theta = \tan^{-1} 8k/(k^2 - 16)$ with each other, we have the

THEOREM. *The asymptotic lines of all translation surfaces whose generating curves lie in two intersecting planes may be found by quadratures.* If $k = 4$, the planes are perpendicular to each other for which special case this theorem has been proved by BIANCHI. §

* American Journal of Mathematics, vol. 26, p. 131.

† *Leçons*, vol. 1, pp. 141–142.

‡ As is shown by DARBOUX's method the lines $(u - v)/k = \alpha$, $(u + v)/2 = \beta$ are also a set of conjugate lines which may be constructed by KOENIG's method. On all surfaces (21) we know therefore two different sets of conjugate lines.

§ *Lezioni di geometria differenziale*, p. 111; German edition, p. 113.

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*Example* II. Let $R = 1$; $\phi_2 = u - v$, $\phi_4 = u + v$, $\phi_1 = -\phi_1(v)$, $\phi_3 = \phi_1(v)$ and

$$\phi_5 + \phi_1\phi_2 + \phi_3\phi_4 = -4\int \phi_1(v)\,dv.$$

In $M_3$ we get the cylinder

$$X_1 = u - v, \qquad X_2 = u + v, \qquad X_3 = \xi(v) \tag{22}$$

whose elements are $v = \text{const.}$ In $\overline{M}_3$ we obtain the ruled surface

$$\overline{X}_1 = 2\phi_1(v),\ \overline{X}_2 = u + v,\ \overline{X}_3 = \xi(v) - 2(u - v)\phi_1(v) \tag{22'}$$

on which $(u)$ and $(v)$ are asymptotic lines. In this case then we observe that to the lines of curvature $(u)$ and $(v)$ on the cylinder correspond asymptotic lines on the ruled surface (22′).

This particular example raises the question whether it is possible to determine all the surfaces on which there is a one-to-one correspondence between lines of curvature and asymptotic lines by virtue of EULER's transformation. To do this we must introduce the condition that the conjugate lines shall be at right angles. We find

$$F = (1 + 4\phi_1^2)\frac{\partial\phi_2}{\partial u}\frac{\partial\phi_2}{\partial v} + (1 + 4\phi_3^2)\frac{\partial\phi_4}{\partial u}\frac{\partial\phi_4}{\partial v} = 0, \tag{23}$$

which by virtue of (14) may be written

$$(1 + 4\phi_1^2)R^2 - (1 + 4\phi_3^2) = 0. \tag{24}$$

This equation may be satisfied if we assume $\phi_2$ and $\phi_4$ functions of only one variable, but we shall exclude this case, since the surface then degenerates into a curve. Introducing the value of $R$ obtained from (24) in (15), we obtain the two equations

$$\begin{aligned} \tfrac{1}{2}\frac{\partial}{\partial u}\log\left(2\phi_1 + \sqrt{1 + 4\phi_1^2}\right) &= \frac{\dfrac{\partial\phi_3}{\partial u}}{\sqrt{1 + 4\phi_3^2}}, \\ \tfrac{1}{2}\frac{\partial}{\partial v}\log\left(2\phi_1 + \sqrt{1 + 4\phi_1^2}\right) &= \frac{\dfrac{\partial\phi_3}{\partial v}}{\sqrt{1 + 4\phi_3^2}}, \end{aligned} \tag{25}$$

from which we obtain the following differential equation for $\phi_3$,

$$\frac{\partial^2\phi_3}{\partial u\partial v} + \tfrac{1}{2}\frac{\partial}{\partial v}\log\frac{1}{\sqrt{1 + 4\phi_3^2}}\cdot\frac{\partial\phi_3}{\partial u} + \tfrac{1}{2}\frac{\partial}{\partial u}\log\frac{1}{\sqrt{1 + 4\phi_3^2}}\cdot\frac{\partial\phi_3}{\partial v} = 0.$$

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When simplified, this becomes

$$\frac{\partial^2 \phi_3}{\partial u \partial v} - \frac{4\phi_3}{1+4\phi_3^2}\frac{\partial \phi_3}{\partial v}\frac{\partial \phi_3}{\partial u} = 0,$$

which integrated gives

$$\phi_3 = \frac{\rho^2(u)\sigma^2(v)-1}{4\rho(u)\sigma(v)}.$$

Substituting this value in (25) and integrating, we also find

$$\phi_1 = \frac{\rho^2-\sigma^2}{4\rho\sigma},$$

$\rho$ and $\sigma$ being arbitrary functions of $u$ and $v$ respectively. The value of $R$ is now found to be

$$R = \frac{1+\rho^2\sigma^2}{\rho^2+\sigma^2}.$$

In order to find $\phi_2$ and $\phi_4$ we substitute this value of $R$ in (16) and eliminate $\partial\phi_4/\partial u$ and $\partial\phi_4/\partial v$. We obtain a differential equation of the form

$$\frac{\partial^2 \phi_2}{\partial u \partial v} + \frac{\sigma\sigma'(1-\rho^4)}{(\rho^2+\sigma^2)(1+\sigma^2\rho^2)}\frac{\partial \phi_2}{\partial u} + \frac{\rho\rho'(1-\sigma^4)}{(\rho^2+\sigma^2)(1+\sigma^2\rho^2)}\frac{\partial \phi_2}{\partial v} = 0,$$

which has equal invariants and of which we know one particular solution, $\phi_3$, as may be easily verified. Having obtained a particular solution $\phi_2$ (different from $\phi_3$) we may obtain $\phi_4$ from (16) by quadratures and $\phi_5$ may then be found from (11) as before. *We have thus found a surface on which the lines of curvature by* EULER'S *transformation correspond to asymptotic curves on the transformed surface.*

If we put $\rho(u) = u$ and $\sigma(v) = v$, remembering that $\phi_2 = X_2 = \overline{X}_2$ and $2\phi_3 = P_1 = \overline{X}_1$, we may state the preceding result thus:

*If a surface, referred to its lines of curvature,* $(u)$, $(v)$, *is transformed by* EULER'S *transformation into a new surface on which* $(u)$ *and* $(v)$ *are asymptotic lines, the cartesian coördinates* $\overline{X}_1$ *and* $\overline{X}_2$ *of the surface must satisfy the differential equation*

$$(26)\qquad \frac{\partial^2 \phi}{\partial u \partial v} + \frac{v(1-u^4)}{(u^2+v^2)(1+u^2v^2)}\frac{\partial \phi}{\partial u} + \frac{u(1-v^4)}{(u^2+v^2)(1+u^2v^2)}\frac{\partial \phi}{\partial v} = 0.$$

*Conversely, whenever two particular solutions,* $\overline{X}_1$ *and* $\overline{X}_2$, *of* (26) *can be obtained, a surface can be found such that Euler's transformation transforms the surface into a new surface on which the asymptotic lines* $(u)$ *and* $(v)$ *correspond to the lines of curvature on the original surface.*

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There is one particular case not included in the above theorem, viz., when either $\rho$ or $\sigma$ is a constant. We shall consider this case later. The case where $R =$ const. was considered on p. 457, where for $k = 4$ lines of curvature on a cylinder were obtained.

Surfaces of this class are, as a rule, transcendental. It is not at all difficult to obtain a particular solution of (26) differing from $\phi_3$ and depending on an arbitrary constant. In fact, putting

$$\phi_2 = \phi_3 + k\left[\xi(u) + \eta(v)\right] \qquad (k = \text{const.}),$$

and substituting in (26) we obtain

$$v(1-u^4)\frac{d\xi}{du} + u(1-v^4)\frac{d\eta}{dv} = 0,$$

which may be satisfied by putting

$$\frac{d\xi}{du} = \frac{-u}{1-u^4}, \qquad \frac{d\eta}{dv} = \frac{v}{1-v^4},$$

so that $\phi_2$ will have the form

$$\phi_2 = \phi_3 - k\int\frac{udu}{1-u^4} + k\int\frac{vdv}{1-v^4}.$$

We may now determine $\phi_4$ from (16) by quadratures. We find

$$\phi_4 = \frac{u^2-v^2}{4uv} - k\int\left[\frac{u^2+v^2}{1+u^2v^2}\cdot\frac{u}{1-u^4}du + \frac{u^2+v^2}{1+u^2v^2}\frac{v}{1-v^4}dv\right];$$

so that we have the following functions

$$\phi_1 = \frac{u^2-v^2}{4uv}, \qquad \phi_2 = \frac{u^2v^2-1}{4uv} + \frac{k}{2}\log\frac{1-u^2}{1+u^2}\cdot\frac{1+v^2}{1-v^2},$$

$$\phi_3 = \frac{u^2v^2-1}{4uv}, \qquad \phi_4 = \frac{u^2-v^2}{4uv} - \frac{k}{4}\log\frac{(1+u^2v^2)^2}{(1-u^4)(1-v^4)}.$$

$\phi_5$ may then be calculated without difficulty from (11). In $M_3$ we obtain the surface

$$X_1 = \frac{u^2v^2-1}{4uv} + \frac{k}{2}\log\frac{1-u^2}{1+u^2}\cdot\frac{1+v^2}{1-v^2},$$

$$(27) \qquad X_2 = \frac{u^2-v^2}{4uv} - \frac{k}{4}\log\frac{(1+u^2v^2)^2}{(1-u^4)(1-v^4)},$$

$$X_3 = \frac{(u^2v^2-1)(u^2-v^2)}{8u^2v^2} + k\tan^{-1}uv,$$

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on which $(u)$ and $(v)$ are lines of curvature. For $k=0$ we obtain the quadric surface $2X_1X_2=X_3$ whose rectilinear generators are $(u^2v^2-1)/4uv=u'$, $(u^2-v^2)/4uv=v'$. Transforming (27) into $\overline{M}_3$ we get the surface

$$(28)\qquad \begin{aligned} \bar{X}_1 &= \frac{u^2-v^2}{2uv}, \qquad \bar{X}_2 = \frac{u^2-v^2}{4uv} - \frac{k}{4}\log\frac{1-u^2}{1+u^2}\cdot\frac{1+v^2}{1-v^2}, \\ \bar{X}_3 &= -\frac{k}{4}\,\frac{u^2-v^2}{uv}\log\frac{(1-u^2)(1+v^2)}{(1+u^2)(1-v^2)} + k\tan^{-1}uv, \end{aligned}$$

on which $(u)$ and $(v)$ are asymptotic lines. For $k=0$ we obtain the straight line $\bar{X}_3=0$, $\bar{X}_1=\bar{X}_2$ which is the transform of the quadric $2\bar{X}_1\bar{X}_2-\bar{X}_3=0$.

We shall now consider the case where $R$ is a function of one variable only, say $v$. For this purpose it will be sufficient to put $\rho=1/\sqrt{c}$ and $\sigma^2=v^2/c$, so that $R$ will take the form

$$R=\frac{c^2+v^2}{c(1+v^2)}.$$

Introducing this value in (16) and eliminating $\partial\phi_4/\partial u$ and $\partial\phi_4/\partial v$, we obtain the differential equation

$$\frac{\partial^2\phi}{\partial u\partial v}+\tfrac{1}{2}\frac{\partial}{\partial v}\log\frac{c(1+v^2)}{c^2+v^2}\cdot\frac{\partial\phi}{\partial u}=0.$$

By integrating this we find, putting the arbitrary function of $u$ which occurs in the integral equal to $u$,

$$\phi_2=u\sqrt{R}+\int\sigma_2'R\,dv,$$

and from (25) and (16),

$$\phi_3=\tfrac{1}{4}\left(\frac{c}{v}-\frac{v}{c}\right),\qquad \phi_1=\tfrac{1}{4}\left(v-\frac{1}{v}\right),\qquad \phi_4=\frac{u}{\sqrt{R}}-\sigma_2,$$

where $\sigma_2$ is an arbitrary function of $v$. The surface in $M_3$ is

$$X_1=u\sqrt{R}+\int\sigma_2'R\,dv,\qquad X_2=\frac{u}{\sqrt{R}}-\sigma_2,$$

$$X_3=\frac{c^2-1}{\sqrt{c}}\,\frac{uv}{\sqrt{(1+v^2)(c^2+v^2)}}+\frac{1}{c}\int\frac{(v^4-c^2)\sigma_2'}{v(1+v^2)}\,dv;$$

a ruled surface on which $(u)$ and $(v)$ are lines of curvature. It is, moreover, developable as may easily be proved by forming the differential equation of the asymptotic curve.* In $\overline{M}_3$ we get the ruled surface,

* Since $\partial\phi_1/\partial u=0$, the differential equation (19) becomes

$$\frac{\partial\phi_1}{\partial v}\frac{\partial\phi_2}{\partial v}dv^2=0,$$

which means that the surface is developable.

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$$\overline{X}_1 = \tfrac{1}{2}\left(v - \frac{1}{v}\right), \quad \overline{X}_2 = \frac{u}{\sqrt{R}} - \sigma_2,$$

$$\overline{X}_3 = \frac{c^2-1}{\sqrt{c}} \frac{uv}{\sqrt{(1+v^2)(c^2+v^2)}} - \frac{(v^2-1)u\sqrt{R}}{2v}$$
$$+ \frac{1}{c}\int \frac{(v^4-c^2)\sigma_2' dv}{v(1+v^2)} - \tfrac{1}{2}\frac{v^2-1}{v}\int \sigma_2' R dv,$$

on which $v =$ const. are the rectilinear generators and $u =$ const. the family of asymptotic lines corresponding to the generators. It will be noticed that all the rectilinear generators lie in a plane parallel to the $X_2X_3$-plane. For the special value $c = 1$, $R$ becomes equal to unity and we obtain in $M_3$ a cylinder

$$X_1 = u + \sigma_2, \qquad X_2 = u - \sigma_2, \qquad X_3 = \xi(v)$$

whose elements are parallel to the plane $X_3 = 0$. If we put $\sigma_2(v) = -v'$ this surface takes the form

$$X_1 = u - v, \qquad X_2 = u + v, \qquad X_3 = \xi(v)$$

which is the cylinder obtained on p. 457. The transform of this surface is the surface (22′).

## Part 2.

Since through any point on a surface there pass two asymptotic lines, and since Euler's transformation establishes a one-to-one correspondence between the asymptotic lines on a surface in $\overline{M}_3$ and a set of conjugate lines on the transform in $M_3$, it is clear that this system must be a definite one of all the $\infty^1$ pairs of conjugate lines that can be made to pass through a given point. The question is, therefore, what geometric property distinguishes this system from all the others. To answer this question we proceed as follows:

Suppose given in $M_3$ any surface $S_3$ and let it be referred to a family of conjugate lines $(u)$ and $(v)$. This surface, considered as an ensemble of $\infty^2$ surface-elements, will when subjected to the transformation (2) become a two-dimensional point-locus in $M_5$ whose coördinates $(u)$, $(v)$ belong to the null-system

$$dx_5 + x_2 dx_1 - x_1 dx_2 + x_4 dx_3 - x_3 dx_4 = 0.$$

If the coördinates of $S_3$ be written

$$(30) \qquad X_1 = \phi_2, \qquad X_2 = \phi_4, \qquad X_3 = \phi,$$

the image in $M_5$ will be

$$x_1 = \frac{P_1}{2} = \phi_1, \qquad x_2 = X_1 = \phi_2, \qquad x_3 = \frac{P_2}{2} = \phi_3,$$

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$$x_4 = X_2 = \phi_4, \qquad x_5 = X_3 - \frac{P_1}{2} X_1 - \frac{P_2}{2} X_2 = \phi_5,$$

where

$$P_1 = \frac{\partial X_3}{\partial X_1}, \qquad \text{and} \qquad P_2 = \frac{\partial X_3}{\partial X_2}.$$

Since $(u)$ and $(v)$ are conjugate lines, we have

$$(31) \qquad \begin{aligned} \frac{\partial X_3}{\partial u} &= P_1 \frac{\partial X_1}{\partial u} + P_2 \frac{\partial X_2}{\partial u}, \\ \frac{\partial X_3}{\partial v} &= P_1 \frac{\partial X_1}{\partial v} + P_2 \frac{\partial X_2}{\partial v}, \end{aligned}$$

$$(32) \qquad \frac{\partial P_1}{\partial u} \frac{\partial X_1}{\partial v} + \frac{\partial P_2}{\partial u} \frac{\partial X_2}{\partial v} = 0,$$

from which it also follows, since $dX_3$ must be an exact differential,

$$(33) \qquad \frac{\partial P_1}{\partial v} \frac{\partial X_1}{\partial u} + \frac{\partial P_2}{\partial v} \frac{\partial X_2}{\partial u} = 0.$$

We now put

$$(34) \qquad \frac{\dfrac{\partial X_1}{\partial u}}{\dfrac{\partial X_2}{\partial u}} = R_1(u, v), \qquad \frac{\dfrac{\partial X_1}{\partial v}}{\dfrac{\partial X_2}{\partial v}} = R_2(u, v),$$

so that the equations (32) and (33) take the form

$$(35) \qquad R_2 \frac{\partial P_1}{\partial u} + \frac{\partial P_2}{\partial u} = 0, \qquad R_1 \frac{\partial P_1}{\partial v} + \frac{\partial P_2}{\partial v} = 0.$$

Eliminating $\partial X_1/\partial u$, $\partial X_1/\partial v$ from (34) and $\partial P_2/\partial u$, $\partial P_2/\partial v$ from (35), we find that $X_2$ and $P_1$ must satisfy the differential equation

$$(36) \qquad \frac{\partial^2 \theta}{\partial u \partial v} + \frac{\dfrac{\partial R_1}{\partial v}}{R_1 - R_2} \cdot \frac{\partial \theta}{\partial u} - \frac{\dfrac{\partial R_2}{\partial u}}{R_1 - R_2} \cdot \frac{\partial \theta}{\partial v} = 0.$$

Conversely, if two particular solutions of this equation can be found, we obtain from (34), (35) and (31) by means of quadratures a surface on which $(u)$ and $(v)$ are conjugate lines. Now suppose, in particular, that $-R_2 = R_1 = R$. Since $R_1$ and $R_2$ are the respective slopes of the tangents drawn to a point of intersection of the projections of any pair of conjugate lines on the $X_1 X_2$-plane,

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this relation expresses the geometrical property that *the triangle formed by any pair of such tangents and either axis is isosceles.**

The equation (36) may now be written

$$\frac{\partial^2 \theta}{\partial u \partial v} + \tfrac{1}{2}\frac{\partial}{\partial v}\log R \cdot \frac{\partial \theta}{\partial u} + \tfrac{1}{2}\frac{\partial}{\partial u}\log R \cdot \frac{\partial \theta}{\partial v} = 0.$$

But, since $X_2 = \phi_4(u, v)$, and $P_1 = 2\phi_1$, this is the equation (17) on page 455. We have therefore the

THEOREM. *If a surface possesses a set of conjugate lines* $(u)$ *and* $(v)$ *such that the pair of tangents drawn to the point of intersection of the projection of the curves on the* $X_1 X_2$*-plane form with either axis an isosceles triangle, then will* EULER'S *transformation transform it into a surface on which the asymptotic lines correspond to this family of conjugate lines.* We shall call such a set of conjugate lines *Euler's lines.*

The relation $R_2 = -R_1$ may be written

$$\frac{\partial X_2}{\partial u}\frac{\partial X_1}{\partial v} + \frac{\partial X_2}{\partial v}\frac{\partial X_1}{\partial u} = 0$$

from which, by the use of (32) and (33) we deduce the equations

$$\frac{\partial X_1}{\partial u}\frac{\partial P_1}{\partial u} - \frac{\partial X_2}{\partial u}\frac{\partial P_2}{\partial u} = 0, \qquad \frac{\partial X_1}{\partial v}\frac{\partial P_1}{\partial v} - \frac{\partial X_2}{\partial v}\frac{\partial P_2}{\partial v} = 0.$$

That is to say, the image of the surface in $M_5$ belongs to the complex (7) which we shall call a *conjugate complex.* Conversely, the relation $R_2 = -R_1$ is satisfied for all surfaces in $M_3$ whose images in $M_5$ by the transformation (2) belong to a conjugate complex, so that we may say:

*The necessary and sufficient condition that a surface in* $M_3$ *referred to a set of conjugate lines shall have as image in* $M_5$ *a surface belonging to a conjugate complex is that the conjugate lines shall be a family of* EULER'S *lines.*

Let there be given on any surface in $M_3$ a system of lines satisfying the differential equation

$$dX_1 dP_1 - dX_2 dP_2 = 0, \tag{37}$$

which expanded may be written

* This may also be expressed thus: The conjugate lines, when projected on the $X_1 X_2$-plane form infinitesimal parallelograms whose areas equal

$$2\frac{\partial X_1}{\partial u}\frac{\partial X_2}{\partial v}\,du\,dv;$$

this may easily be proved geometrically.

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$$(38)\quad \left(\frac{\partial X_1}{\partial u}\frac{\partial P_1}{\partial u}-\frac{\partial X_2}{\partial u}\frac{\partial P_2}{\partial u}\right)du^2$$
$$+\left(\frac{\partial X_1}{\partial v}\frac{\partial P_1}{\partial u}+\frac{\partial X_1}{\partial u}\frac{\partial P_1}{\partial v}-\frac{\partial X_2}{\partial v}\frac{\partial P_2}{\partial u}-\frac{\partial X_2}{\partial u}\frac{\partial P_2}{\partial v}\right)dudv$$
$$+\left(\frac{\partial X_1}{\partial v}\frac{\partial P_1}{\partial v}-\frac{\partial X_2}{\partial v}\frac{\partial P_2}{\partial v}\right)dv^2=0.$$

We shall investigate some of the properties of these lines. Since the differential equation is of the second degree it is satisfied by an infinite system of curves $\phi_1=c_1$, $\phi_2=c_2$. Let us suppose that the surface be referred to this system; since now $u=$ const., $v=$ const. must be integrals of the equation we have

$$(39)\quad \frac{\partial X_1}{\partial u}\frac{\partial P_1}{\partial u}-\frac{\partial X_2}{\partial u}\frac{\partial P_2}{\partial u}=0,\qquad \frac{\partial X_1}{\partial v}\frac{\partial P_1}{\partial v}-\frac{\partial X_2}{\partial v}\frac{\partial P_2}{\partial v}=0.$$

But we also have

$$\frac{\partial X_3}{\partial u}=P_1\frac{\partial X_1}{\partial u}+P_2\frac{\partial X_2}{\partial u},\qquad \frac{\partial X_3}{\partial v}=P_1\frac{\partial X_1}{\partial v}+P_2\frac{\partial X_1}{\partial v}.$$

Eliminating $\partial X_3/\partial u$ and $\partial X_3/\partial v$ we get

$$(40)\quad \frac{\partial X_1}{\partial u}\frac{\partial P_1}{\partial v}+\frac{\partial X_2}{\partial u}\frac{\partial P_2}{\partial v}-\frac{\partial P_1}{\partial u}\frac{\partial X_1}{\partial v}-\frac{\partial P_2}{\partial u}\frac{\partial X_2}{\partial v}=0,$$

and eliminating $\partial P_1/\partial u$, $\partial P_1/\partial v$ from equations (39) and (40) we obtain the equation

$$\left(\frac{\partial X_2}{\partial u}\frac{\partial X_1}{\partial v}+\frac{\partial X_2}{\partial v}\frac{\partial X_1}{\partial u}\right)\left(\frac{\partial X_1}{\partial u}\frac{\partial P_2}{\partial v}-\frac{\partial X_1}{\partial v}\frac{\partial P_2}{\partial u}\right)=0.$$

If the second factor vanishes we find that combined with (39) it will cause the coefficient of $dudv$ in (38) to vanish, in which case every line on the surface will satisfy the equation; this can happen only for special surfaces. Excluding this case we have

$$(41)\quad \frac{\partial X_2}{\partial u}\frac{\partial X_1}{\partial v}+\frac{\partial X_2}{\partial v}\frac{\partial X_1}{\partial u}=0;$$

combining this again with the second of equations (39) we obtain

$$\frac{\partial P_1}{\partial u}\frac{\partial X_1}{\partial v}+\frac{\partial P_2}{\partial u}\frac{\partial X_2}{\partial v}=0.$$

This is the condition that $(u)$ and $(v)$ shall be conjugate lines. Moreover,

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these lines constitute a family of EULER's lines, since the condition (41) is just the relation $R_2 = -R_1$. Hence the

THEOREM. *If an infinite family of curves $\phi_1 = c_1$, $\phi_2 = c_2$ on a surface satisfy the differential equation*

$$\left(\frac{\partial P_1}{\partial u}\frac{\partial X_1}{\partial u} - \frac{\partial P_2}{\partial u}\frac{\partial X_2}{\partial u}\right)du^2$$

$$(38) \qquad + \left(\frac{\partial X_1}{\partial v}\frac{\partial P_1}{\partial u} + \frac{\partial X_1}{\partial u}\frac{\partial P_1}{\partial v} - \frac{\partial X_2}{\partial v}\frac{\partial P_2}{\partial u} - \frac{\partial X_2}{\partial u}\frac{\partial P_2}{\partial v}\right)dudv$$

$$+ \left(\frac{\partial P_1}{\partial v}\frac{\partial X_1}{\partial v} - \frac{\partial P_2}{\partial v}\frac{\partial X_2}{\partial v}\right)dv^2 = 0,$$

*then will these lines be a family of* EULER's *lines.**

Conversely, *a family of* EULER's *lines on a surface satisfies the differential equation* (38).

PROOF. Let the curves be $\xi_1(u, v) = c_1$, $\xi_2(u, v) = c_2$, which by hypothesis are EULER's lines. Introducing the curvilinear coördinates $\xi_1 = u'$, $\xi_2 = v'$ in the rectangular coördinates of the surface, we have by hypothesis

$$(42) \qquad \begin{aligned} \frac{\partial P_1}{\partial u'}\frac{\partial X_1}{\partial v'} + \frac{\partial P_2}{\partial u'}\frac{\partial X_2}{\partial v'} &= 0, \\ \frac{\partial X_2}{\partial u'}\frac{\partial X_1}{\partial v'} + \frac{\partial X_2}{\partial v'}\frac{\partial X_1}{\partial u'} &= 0, \end{aligned}$$

to which we may also add the condition

$$(43) \qquad \frac{\partial P_1}{\partial v'}\frac{\partial X_1}{\partial u'} + \frac{\partial P_2}{\partial v'}\frac{\partial X_2}{\partial u'} = 0,$$

since $dX_3$ must be an exact differential also after the introduction of the new coördinates. The conditions (42) and (43) reduce to the following

$$(43') \qquad \frac{\partial P_1}{\partial u'}\frac{\partial X_1}{\partial u'} - \frac{\partial P_2}{\partial u'}\frac{\partial X_2}{\partial u'} = 0, \qquad \frac{\partial P_1}{\partial v'}\frac{\partial X_1}{\partial v'} - \frac{\partial P_2}{\partial v'}\frac{\partial X_2}{\partial v'} = 0.$$

Expressing $\partial P_i/\partial u'$, $\partial P_i/\partial v'$, $\partial X_i/\partial u$, $\partial X_i/\partial v$ $(i = 1, 2)$ in terms of $\partial P_i/\partial u$, $\partial P_i/\partial v$, $\partial X_i/\partial u$, $\partial X_i/\partial v$, $\xi_1$ and $\xi_2$ and substituting in (43′) we get

* If $u$ and $v$ are a set of asymptotic lines on the surface this equation is simplified; in fact, an easy calculation will show that it reduces to

$$\frac{\partial P_2}{\partial u}\frac{\partial X_2}{\partial u}du^2 + \frac{\partial P_2}{\partial v}\frac{\partial X_2}{\partial v}dv^2 = 0.$$

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$$\left(\frac{\partial P_1}{\partial v}\frac{\partial X_1}{\partial v}-\frac{\partial X_2}{\partial v}\frac{\partial P_2}{\partial v}\right)\left(\frac{\partial \xi_1}{\partial v}\right)^2$$
$$-\left(\frac{\partial X_2}{\partial u}\frac{\partial P_2}{\partial v}+\frac{\partial X_2}{\partial v}\frac{\partial P_2}{\partial u}-\frac{\partial X_1}{\partial u}\frac{\partial P_1}{\partial v}-\frac{\partial X_1}{\partial v}\frac{\partial P_1}{\partial u}\right)\frac{\partial \xi_1}{\partial u}\frac{\partial \xi_1}{\partial v}$$
$$+\left(\frac{\partial P_1}{\partial u}\frac{\partial X_1}{\partial u}-\frac{\partial P_2}{\partial u}\frac{\partial X_2}{\partial u}\right)\left(\frac{\partial \xi_1}{\partial u}\right)^2=0,$$

and an equation of identically the same form in $\partial\xi_2/\partial u$, $\partial\xi_2/\partial v$. But, since $\xi_1 = c_1$ and $\xi_2 = c_2$, we have

$$\frac{du}{dv}=-\frac{\dfrac{\partial \xi_1}{\partial u}}{\dfrac{\partial \xi_1}{\partial v}}=-\frac{\dfrac{\partial \xi_2}{\partial u}}{\dfrac{\partial \xi_2}{\partial v}},$$

and substituting these ratios in the above, we obtain the differential equation (38). Q. E. D.

The theorem on p. 458 may now be stated thus:

*The problem of finding all surfaces on which Euler's lines are lines of curvature depends on the integration of the equation* (26) *and quadratures. These surfaces have the following geometric property. The projection of the lines of curvature on the $X_1X_2$-plane form a system such that the two tangents drawn at a point of intersection of any pair of curves form with either axis an isosceles triangle.*

If the system $S_1$ be referred to a set of asymptotic lines, the equation (38) of EULER's lines takes the simple form

$$\frac{\partial P_2}{\partial u}\frac{\partial X_2}{\partial u}du^2+\frac{\partial P_2}{\partial v}\frac{\partial X_2}{\partial v}dv^2=0. \tag{38'}$$

If $S_1$ be transformed into $S_2$ the asymptotic curves $u = c$, $v = c$ become EULER's lines on $S_2$, and the equation of asymptotic lines on $S_2$ will be identically of the same form as (38′), since by EULER's transformation $\bar{X}_2 = X_2$ and $\bar{P}_2 = P_2$. It follows therefore that if on any surface $S_1$ we know the asymptotic lines, the EULER lines are known on the corresponding surface $S_2$.

## PART 3.

The following question now presents itself: Are there contact-transformations other than EULER's that will change asymptotic lines into EULER's lines? We shall find that there exist $\infty^{10}$ such transformations. To establish this it will be convenient to go back to the space $M_5$ from which we started.

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A null-system in $M_5$, defined by the equations

$$(44)\qquad \begin{gathered} x_i = \rho_i x_5 + \sigma_i \qquad (i=1,2,3,4), \\ \rho_2\sigma_1 - \rho_1\sigma_2 + \rho_4\sigma_3 - \rho_3\sigma_4 + 1 = 0, \end{gathered}$$

will when subjected to the transformation (2) become an ensemble of surface-elements consisting of element-bands formed by $\infty^5$ parabolæ

$$(45)\qquad \begin{gathered} \rho_4 X_1 - \rho_2 X_2 = \sigma_2\rho_4 - \sigma_4\rho_2, \\ X_3 = \frac{\rho_1\rho_2 + \rho_3\rho_4}{\rho_2^2}(X_1 - \sigma_2)^2 + \frac{2(\rho_2\sigma_1 + \rho_4\sigma_3)}{\rho_2}(X_1 - \sigma_2) + \sigma_1\sigma_2 + \sigma_3\sigma_4, \end{gathered}$$

lying in planes parallel to the $X_3$-axis.* Moreover, the coördinates of the plane of each surface-element are subject to the conditions

$$(46)\qquad \begin{gathered} \rho_2 \frac{P_1}{2} = \rho_1 X_1 + \sigma_1\rho_2 - \sigma_2\rho_1, \\ \rho_2 \frac{P_2}{2} = \rho_3 X_1 + \sigma_3\rho_2 - \sigma_2\rho_3. \end{gathered}$$

To each line of the nullsystem corresponds one of the parabolæ considered as an element-band, or, in other words, to the $\infty^1$ points of the line correspond the $\infty^1$ surface-elements of the parabolæ. We shall now consider the effect of EULER's transformation on these element-bands when the nullsystem becomes a conjugate complex, that is, when the additional MONGE equation

$$dx_1\,dx_2 - dx_3\,dx_4 = 0$$

is satisfied. Now in order that the lines (44) shall be lines of such a complex we must have

$$\rho_1\rho_2 - \rho_3\rho_4 = 0.$$

Introducing this condition in (45) we obtain the parabolæ

$$(47)\qquad \begin{gathered} \rho_4 X_1 - \rho_2 X_2 = \sigma_2\rho_4 - \sigma_4\rho_2, \\ X_3 = \frac{2\rho_1}{\rho_2}(X_1 - \sigma_2)^2 + \frac{2(\rho_2\sigma_1 + \rho_4\sigma_3)}{\rho_2}(X_1 - \sigma_2) + \sigma_1\sigma_2 + \sigma_3\sigma_4, \end{gathered}$$

while the conditions (46) become

$$(46')\qquad \begin{gathered} \rho_2 \frac{P_1}{2} = \rho_1 X_1 + \sigma_1\rho_2 - \sigma_2\rho_1, \\ \rho_4 \frac{P_2}{2} = \rho_1 X_1 + \sigma_3\rho_4 - \sigma_2\rho_1. \end{gathered}$$

* American Journal of Mathematics, vol. 26.



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Next I shall show that *Euler's transformation transforms these* $\infty^5$ *parabolæ into* $\infty^4$ *straight lines.* In fact, transforming (46′) and (47) and reducing, we have

$$(48)\qquad \begin{aligned} &\rho_4\bar{X}_1 - 2\rho_1\bar{X}_2 = 2(\sigma_1\rho_4 - \rho_1\sigma_4),\\ &\bar{X}_3 = \frac{\rho_4\sigma_3 - \rho_1\sigma_2}{\rho_1}\bar{X}_1 - \frac{2\sigma_1\rho_4\sigma_3}{\rho_1} + \sigma_1\sigma_2 + \sigma_3\sigma_4, \end{aligned}$$

which represent $\infty^4$ straight lines in $M_3$.

In the same way we may prove without difficulty that if the null-system (44) becomes asymptotic so that $\rho_1\rho_2 + \rho_3\rho_4 = 0$, the $\infty^4$ straight lines

$$(49)\qquad \begin{aligned} &\rho_4 X_1 - \rho_2 X_2 = \sigma_2\rho_4 - \sigma_4\rho_2,\\ &X_3 = \frac{2(\rho_2\sigma_1 + \rho_4\sigma_3)}{\rho_2}(X_1 - \sigma_2) + \sigma_1\sigma_2 + \sigma_3\sigma_4. \end{aligned}$$

are by EULER's transformation transformed into $\infty^5$ parabolæ of a form similar to (47).

Since all the contact-transformations that transform (47) into (48) also transform EULER's lines into asymptotic lines and vice versa our problem is now simplified. Let us consider an asymptotic complex in $M_5$. There exist $\infty^{10}$ projective transformations which leave it invariant.* In $M_3$ we get a group of $\infty^{10}$ contact transformations which leave invariant the differential equations

$$(50)\qquad \begin{aligned} dX_3 - P_1 dX_1 - P_2 dX_2 &= 0,\\ dX_1 dP_1 + dX_2 dP_2 &= 0. \end{aligned}$$

These transformations have the form

$$(51)\qquad \begin{aligned} \bar{X}_1 &= a_1 X_1 + b_1 X_2 + d_1,\\ \bar{X}_2 &= a_2 X_1 + b_2 X_2 + d_2,\\ \bar{X}_3 &= a_3 X_1 + b_3 X_2 + c_3 X_3 + d_3,\\ \bar{P}_1 &= \frac{b_2 c_3}{\Delta} P_1 - \frac{a_2 c_3}{\Delta} P_2 + \frac{a_3 b_2 - a_2 b_3}{\Delta},\\ \bar{P}_2 &= \frac{-b_1 c_3}{\Delta} P_1 + \frac{a_1 c_3}{\Delta} P_2 + \frac{a_1 b_3 - a_3 b_1}{\Delta}, \end{aligned}$$

where $\Delta = a_1 b_2 - a_2 b_1$. We notice that they are projective point-transformations since the coördinates $\bar{X}_1, \bar{X}_2, \bar{X}_3$ do not involve $P_1$ and $P_2$; all these

*American Journal of Mathematics, vol. 26, p. 146.

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transformations form a subgroup of the general linear group. We shall denote any one of them by the symbol $T_a$.

If now we superpose EULER's transformation on (51) we obtain $\infty^{10}$ contact-transformations

$$
\begin{aligned}
&\bar{X}_1 = a_1 P_1 + b_1 X_2 + d_1, \\
&\bar{X}_2 = a_2 P_1 + b_2 X_2 + d_2, \\
(52) \qquad &\bar{X}_3 = a_3 P_1 + b_3 X_2 + b_3 (X_3 - P_1 X_1) + d_3, \\
&\bar{P}_1 = -\frac{b_2 c_3}{\Delta} X_1 - \frac{a_2 c_3}{\Delta} P_2 + \frac{a_3 b_2 - a_2 b_3}{\Delta}, \\
&\bar{P}_2 = \frac{b_1 c_3}{\Delta} X_1 + \frac{a_1 c_3}{\Delta} P_2 + \frac{a_1 b_3 - a_3 b_1}{\Delta},
\end{aligned}
$$

which change the differential equation

$$(53) \qquad d\bar{X}_1 d\bar{P}_1 + d\bar{X}_2 d\bar{P}_2 = 0,$$

into

$$(54) \qquad dX_1 dP_1 - dX_2 dP_2 = 0,$$

*and, therefore, also asymptotic lines into* EULER's *lines.* They also change $\infty^5$ parabolæ (47) into the $\infty^4$ straight lines (48); we shall denote them by $T_a E$. The transformations (52) do not form a group, for, if so we should have $T_a E T_b E = T_k E$, where $T_a$, $T_b$, $T_k$ are transformations of the group (51), that is, $T_a E T_b = T_k$. But the succession of $T_a$, $E$ and $T_b$ cannot be equivalent to a projective transformation.

All the transformations leaving the equation (53) invariant are included in the group (51) and LEGENDRE's transformation.* The transformation $EL$ will, therefore, have the same property as any one of (52). But the succession of two EULER's transformations is a LEGENDRE's transformation so that the combination of a Legendre transformation with any one of the transformations (52), or, what is the same thing, *the succession of $T_a$ and an odd number of* EULER's *transformations will transform* EULER's *lines $u = c$, $v = c$ on $S_1$ into the asymptotic lines $u = c$, $v = c$ on $S_2$.*

If we transform the coördinates on the left hand side in (52) by EULER's transformation we obtain $\infty^{10}$ contact-transformations which leave invariant the equation (54). *These transformations form a group;* in fact calling any one of them $T_c$ we have $T_c = E^{-1} T_a E$ and $E^{-1} T_a E\, E^{-1} T_b E = E^{-1} T_c E$. This

*See p. 148 of my article, American Journal of Mathematics, vol. 26. By a curious mistake the name EULER's transformation is there given to the well-known one of LEGENDRE:

$$\bar{X}_1 = P_2, \quad \bar{X}_2 = -P_1, \quad \bar{X}_3 = X_3 - P_1 X_1 - P_2 X_2, \quad \bar{P}_1 = -X_2, \quad \bar{P}_2 = X_1.$$

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group is a subgroup of a group of contact-transformations connected with the projective group of the non-special nullsystem in $M_5$

$$dx_5 + x_2 dx_1 - x_1 dx_2 + x_4 dx_3 - x_3 dx_4 = 0,$$

which has been discussed by LIE in the second volume of his *Theorie der Transformationsgruppen.* The transformations are

$$\begin{aligned}
P_1 &= a_1 P_1 + b_1 X_2 + d_1, \\
P_2 &= \frac{c_3 b_1 X_1}{\Delta} + \frac{a_1 c_3}{\Delta} P_2 + \frac{a_1 b_3 - a_3 b_1}{\Delta}, \\
X_1 &= \frac{b_2 c_3 X_1}{\Delta} + \frac{a_2 c_3}{\Delta} P_2 - \frac{a_3 b_2 - a_2 b_3}{\Delta}, \\
X_2 &= a_2 P_1 + b_2 X_2 + d_2, \\
X_3 - P_1 X_1 &= a_3 P_1 + b_3 X_2 + c_3 (X_3 - P_1 X_1) + d_3.
\end{aligned} \tag{55}$$

Now since LEGENDRE's transformation also leaves (54) invariant, all the transformations that leave this equation invariant are obtained by superposing LEGENDRE's on any one of the $\infty^{10}$ transformations (55). We have thus found all the transformations which change the $\infty^5$ parabolæ (47) into themselves. It should be noticed that the group (55) is similar to the projective group (51).*

In conclusion we shall give the following resumé putting in evidence the relations of the spaces $M_5$ and $M_3$:

| $M_5$. | $M_3$. |
|---|---|
| I. $\begin{cases}(a)\ dx_5 + x_2 dx_1 - \cdots = 0, \\ (b)\ dx_1 dx_2 + dx_3 dx_4 = 0.\end{cases}$ | I. $\begin{cases}(a)\ dX_3 - P_1 dX_1 - P_2 dX_2 = 0, \\ (b)\ dX_1 dP_1 + dX_2 dP_2 = 0.\end{cases}$ |
| II. Projective group of nullsystem $(a)$ (21 parameters). | II. Irreducible group, $C_3$, of contact transformations. |
| III. Sub-group $G_a$ of $\infty^{10}$ transformations leaving the asymptotic complex I invariant. | III. Subgroup $C_a$ of $C_3$ leaving I invariant, i. e., transforming asymptotic lines $(u)$, $(v)$ on $S_1$ into asymptotic lines $(u)$, $(v)$ on $S_2$. |
| IV. Special projective transformation not included in II leaving I invariant ($\bar{x}_2 = 2x_3$, $\bar{x}_4 = -2x_1$, $2\bar{x}_1 = -x_4$, $2\bar{x}_3 = x_2$, $\bar{x}_5 = x_5$). | IV. LEGENDRE's transformation leaving I invariant. |

* See LIE-SCHEFFER's *Continuierliche Gruppen*, p. 427.

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| $M_5$. | $M_3$. |
|---|---|
| V. $\begin{cases}(a) & dx_5 + x_2 dx_1 - \cdots = 0, \\ (b) & dx_1 dx_2 - dx_3 dx_4 = 0.\end{cases}$ | V. $dX_3 - P_1 dX_1 - P_2 dX_2 = 0,$ $dX_1 dP_1 - dX_2 dP_2 = 0.$ |
| VI. Linear projective group of $\infty^{10}$ transformations leaving V invariant (denoted by $G_c$). | VI. Group of contact transformations similar to $C_3$, leaving V invariant, i. e., changing EULER's lines $(u)$, $(v)$ on $S_1$ into EULER's lines $(u)$, $(v)$ on $S_2$. |
| VII. Special projective transformation not included in VI, changing I into V. | VII. EULER's transformation changing EULER's lines into asymptotic lines. |
| VIII. $\infty^{10}$ projective transformations transforming an asymptotic complex I into a conjugate complex V. | VIII. $\infty^{10}$ contact transformations changing I into V, or changing EULER's lines $(u)$, $(v)$ on $S_1$ into asymptotic lines $(u)$, $(v)$ on $S_2$. |

ANNAPOLIS, *April 7, 1905*

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# SURFACES OF CONSTANT CURVATURE AND THEIR TRANSFORMATIONS*

BY

LUTHER PFAHLER EISENHART

BIANCHI † has called attention to the fact that upon a surface of positive total curvature there is an infinity of conjugate systems of lines which are characterized by the property that when such a system is parametric the parameters can be chosen so that the second quadratic form becomes

$$\lambda(du^2 + dv^2),$$

where $\lambda$ is generally a function of both $u$ and $v$; BIANCHI calls such a system *isothermal-conjugate.* In a similar manner it can be shown ‡ that an infinity of conjugate systems exist on a surface of negative curvature such that the second quadratic form is reducible to

$$\lambda(du^2 - dv^2);$$

we call these also *isothermal-conjugate.* When the total curvature is constant and positive, the surface is called *spherical.* We find that when a spherical surface is referred to an isothermal-conjugate system the fundamental quantities for the surface and the equations which they satisfy take very simple forms—forms which suggest transformations carrying the given surface into new spherical surfaces. This paper is given over to a study of these transformations.

In § 1 we derive the fundamental equations for a spherical surface referred to isothermal-conjugate lines, show that its lines of curvature and characteristic lines § are isothermal-conjugate, derive the expressions for the spherical representation of the given isothermal system and for the fundamental functions when this given system is composed of the lines of curvature or the characteristic lines.

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† Lezioni, vol. I, p. 167; German translation by LUKAT, p. 135.

‡ *Isothermal-conjugate systems of lines on surfaces*, American Journal, vol. 25 (1903), p. 220.

§ *Three particular systems of lines on a surface*, Transactions, vol. 5 (1904), p. 429.



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In § 2 we discover a transformation which changes a spherical surface into a surface of the same curvature with the same second quadratic form as the given surface and the lines of curvature are in correspondence on the two surfaces. When the given surface is referred to its lines of curvature it is found that this is the transformation of HAZZIDAKIS.* The new transformation is the transformation of HAZZIDAKIS in a general form, for it can be shown that the latter is the only transformation of spherical surfaces leaving the total curvature and second quadratic forms unaltered and preserving lines of curvature.†

An interesting example of the effect of the HAZZIDAKIS transformation is afforded by applying it to the spherical surfaces of revolution. In addition to this study § 2 is given over to the consideration of surfaces of constant mean curvature, which are always associated with spherical surfaces, as BONNET has shown.

In § 4 another transformation is found which changes a spherical surface into another, leaving the total curvature and the second quadratic form unaltered, and changing lines of curvature into characteristic lines and vice versa. After a study of the effects of combinations of these and HAZZIDAKIS transformations, it is shown that these transformations or transformations of the two kinds are the only ones leaving the total curvature and second quadratic forms altered and interchanging the lines of curvature and characteristic lines.

In § 5 the pseudospherical surfaces are considered and their equations of condition are given forms analogous to those in § 1 for spherical surfaces, but one cannot find transformations similar to the above.

## § 1. *Spherical surfaces referred to isothermal-conjugate lines.*

Consider a surface, $S$, of constant positive total curvature, $1/a^2$. Let the parametric curves form an isothermal-conjugate system ‡ upon the surface and choose the parameters of these lines in such a way that

$$D = D'', \qquad D' = 0, \tag{1}$$

where these $D$'s are the coefficients of the second quadratic form of the surface. The GAUSS § and CODAZZI ‖ equations can then be written in the respective forms

$$D^2 = H\left[\frac{\partial}{\partial v}\left(\frac{H}{E}\left\{\begin{matrix}11\\2\end{matrix}\right\}\right) - \frac{\partial}{\partial u}\left(\frac{H}{E}\left\{\begin{matrix}12\\2\end{matrix}\right\}\right)\right], \tag{2}$$

* BIANCHI, Lezioni, II, p. 436; German translation, p. 472.

† In a subsequent paper we shall show the part which the HAZZIDAKIS transformation plays in the theory of the deformation of surfaces.

‡ BIANCHI, Lezioni, I, p. 167; German translation, p. 135.

§ Ibid., I, p. 93; p. 68.

‖ Ibid., I, p. 119; p. 91.

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and

$$(3)\qquad \frac{\partial \log D}{\partial v} = \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} - \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}, \qquad \frac{\partial \log D}{\partial v} = \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} - \begin{Bmatrix} 22 \\ 1 \end{Bmatrix},$$

where the CHRISTOFFEL symbols $\{ {r \atop i} \}$ are formed with respect to the linear element of $S$ written

$$(4)\qquad ds^2 = Edu^2 + 2Fdudv + Gdv^2,$$

and we have put for the sake of brevity

$$(5)\qquad H = \sqrt{EG - F^2}.$$

Since now

$$(6)\qquad K = \frac{D^2}{H^2} = \frac{1}{a^2},$$

we have

$$\frac{\partial \log D}{\partial v} = \frac{\partial \log H}{\partial v} = \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 22 \\ 2 \end{Bmatrix},$$

$$\frac{\partial \log D}{\partial u} = \frac{\partial \log H}{\partial u} = \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} + \begin{Bmatrix} 11 \\ 1 \end{Bmatrix}.$$

When these equations are combined with (3), the latter may be replaced by

$$(7)\qquad 2\frac{\partial F}{\partial u} = \frac{\partial E}{\partial v} - \frac{\partial G}{\partial v}, \qquad 2\frac{\partial F}{\partial v} = \frac{\partial G}{\partial u} - \frac{\partial E}{\partial u}.$$

From this it follows that the functions $F$ and $E - G$ are solutions of the equation

$$(8)\qquad \frac{\partial^2 \theta}{\partial u^2} + \frac{\partial^2 \theta}{\partial v^2} = 0.$$

The general integral of this equation is

$$\theta = \Phi(u + iv) + \Psi(u - iv),$$

where the functions $\Phi$ and $\Psi$ are arbitrary in form; moreover, in order that the surface be real $\Psi$ must be conjugate to $\Phi$. Consider this case and put

$$F = \frac{\Phi(u + iv) + \Phi_0(u - iv)}{2},$$

when $\Phi_0$ denotes the conjugate function of $\Phi$; from (7) it follows that

$$E - G = i(\Phi - \Phi_0).$$

Since equations (7) have replaced the CODAZZI equations, the only other equation of condition to be satisfied is the GAUSS equation (2). Hence $\Phi$ can be chosen

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arbitrarily and there remains a partial differential equation of the second order to be satisfied by $G$; this equation is found by substituting the values for $D^2$, $F'$, and $E$ in (2).

As equation (8) is satisfied by $\theta = 0$, it follows that a solution is

$$(9) \qquad F = 0, \qquad E - G = \text{const.}$$

Since the latter constant is zero only in case $S$ is a sphere, we have the theorem:

*The lines of curvature upon a spherical surface form an isothermal-conjugate system; these lines are isothermal-orthogonal only in case of the sphere.*

Another solution of equations (7), suggested by the solution $\theta = 0$ of equation (8), is

$$(9') \qquad E = G, \qquad F = \text{const.},$$

where the constant is equal to zero only for the sphere. Now

$$\frac{D}{E} = \frac{D''}{G}, \qquad D' = 0.$$

But the latter are the conditions that the conjugate parametric system is composed of the lines cutting one another under the minimum angle between conjugate lines;* REINA† has called the latter the *characteristic lines*. Hence the theorem:

*The characteristic lines upon a spherical surface are isothermal-conjugate.*

From the general expressions for the coefficients of the linear element of the spherical representation of a surface referred to a conjugate system, namely,‡

$$(10) \qquad \mathcal{E} = \frac{GD^2}{EG - F^2}, \qquad \mathcal{F} = -\frac{FDD'}{EG - F^2}, \qquad \mathcal{G} = \frac{ED''^2}{EG - F^2},$$

it is seen that for the present case

$$(11) \qquad \mathcal{E} = \frac{G}{a^2}, \qquad \mathcal{F} = -\frac{F}{a^2}, \qquad \mathcal{G} = \frac{E}{a^2}.$$

In consequence of (7) one has

$$(12) \qquad 2\frac{\partial \mathcal{F}}{\partial u} = \frac{\partial \mathcal{E}}{\partial v} - \frac{\partial \mathcal{G}}{\partial v}, \qquad 2\frac{\partial \mathcal{F}}{\partial v} = \frac{\partial \mathcal{G}}{\partial u} - \frac{\partial \mathcal{E}}{\partial u},$$

that is, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ satisfy equations (7). Conversely, when $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ are three functions satisfying (12), one has

* *Three particular systems of lines on a surface*, Transactions, vol. 5 (1904), pp. 9, 443.

† *Di alcune proprietà delle linee caratteriche*, Roma Accademia Lincei Rendiconti (4) V (1899), pp. 881-5.

‡ BIANCHI, Lezioni, vol. I, p. 150; German translation, p. 120.

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$$\left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\}' + \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}' = 0, \qquad \left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\}' + \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}' = 0$$

where the CHRISTOFFEL symbols $\left\{ \begin{matrix} rs \\ t \end{matrix} \right\}'$ are formed with respect to

$$d\sigma^2 = \mathcal{E}du^2 + 2\mathcal{F}dudv + \mathcal{G}dv^2.$$

Hence if $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ satisfy also the GAUSS equation for the unit sphere, the parametric lines upon the sphere represent an isothermal-conjugate system of lines on a unique spherical surface. *

## § 2. *The transformation of Hazzidakis.*

The results of the last section suggest a transformation of spherical surfaces. Let $S$ be such a surface with the linear element (4), whose coefficients satisfy equations (7). We remark that the latter equations are satisfied also by $E_1$, $F_1$, $G_1$, where

$$E_1 = G, \qquad F_1 = -F, \qquad G_1 = E. \tag{13}$$

If these values be subtituted in an equation similar to (2), namely,

$$D_1^2 = H_1 \left[ \frac{\partial}{\partial v} \left( \frac{H_1}{E_1} \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\} \right) - \frac{\partial}{\partial u} \left( \frac{H_1}{E_1} \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\} \right) \right], \tag{13'}$$

when the CHRISTOFFEL symbols are formed with respect to

$$ds_1^2 = E_1 du^2 + 2F_1 dudv + G_1 dv^2,$$

and one remarks that

$$H_1 = H,$$

it follows in consequence of (7) that this becomes

$$D_1^2 = H \left[ \frac{\partial}{\partial u} \left( \frac{H}{G} \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\} \right) + \frac{\partial}{\partial v} \left( \frac{H}{G} \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} \right) \right],$$

where now the symbols are formed with respect to (4). But the right-hand member of this equation is equivalent to the similar member of (2).† Hence, the GAUSS equation is satisfied by taking $D_1 = D_1' = D$. This gives the theorem:

*When a spherical surface is referred to an isothermal-conjugate system and the parameters are chosen so that* (1) *is satisfied, there exists another spherical surface of the same total curvature and with the same second quadratic form as the given surfaces; the linear element is*

$$ds_1^2 = Gdu^2 - 2Fdudv + Edv^2. \tag{14}$$

* Ibid., p. 168; p. 136.

† Ibid., p. 77; p. 53.

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It will be convenient to refer to the new surface $S_1$ as the *conjugate* of the given one. One remarks that the relation between the two surfaces is entirely reciprocal.

If the linear element of the spherical representation of $S_1$ is written

$$d\sigma_1^2 = \mathcal{E}_1 du^2 + 2\mathcal{F}_1 du\, dv + \mathcal{G}_1 dv_1^2$$

it follows from (10), (11) and (13) that

$$\mathcal{E}_1 = \frac{E}{a^2} = \mathcal{G}, \qquad \mathcal{F}_1 = -\mathcal{F}, \qquad \mathcal{G}_1 = \mathcal{E}. \tag{15}$$

Hence the coefficients of the spherical representation of $S$ and $S_1$ bear the same relation to one another as the first fundamental quantities of these surfaces.

A comparison of the formulas (11), (13) and (15) leads to the theorem:

*A spherical surface of unit total curvature is applicable to the unit sphere in such a way that an isothermal-conjugate system of lines on the former correspond to the spherical representation of the corresponding isothermal-conjugate system on its conjugate.*

From the general equation of the lines of curvature

$$(ED' - FD)du^2 + (ED'' - GD)du\, dv + (FD'' - GD')dv^2 = 0,$$

it is seen that for both $S$ and $S_1$ the lines of curvature are given by

$$F du^2 + (G - E)du\, dv - F dv^2 = 0. \tag{16}$$

Hence *the lines of curvature correspond upon a spherical surface and its conjugate.*

Suppose that the isothermal-conjugate lines on $S$ are the lines of curvature; from (9) it is seen that we can put

$$E = a^2 \cosh^2\theta, \qquad G = a^2 \sinh^2\theta, \qquad F = 0 \tag{17}$$

or

$$E = a^2 \sinh^2\theta, \qquad G = a^2 \cosh^2\theta, \qquad F = 0 \tag{18}$$

according as the constant in (9) is positive or negative, $a^2$ being the absolute value of the constant.* Now in either case

$$D = D'' = a \sinh\theta \cosh\theta,$$

and equation (2) becomes

$$\frac{\partial^2\theta}{\partial u^2} + \frac{\partial^2\theta}{\partial v^2} + \sinh\theta \cosh\theta = 0.$$

* Lezioni, vol. II., p. 436; p. 472.

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One sees that the values (18) lead to the conjugate of the surface whose first fundamental coefficients are given by (17) and vice versa. But this is the transformation which BIANCHI has called the *transformation of* HAZZIDAKIS.

In order to show that the transformation which we have been considering in this section is the transformation of HAZZIDAKIS in a more general form, we have only to prove the theorem:

*The transformation of* HAZZIDAKIS *is the only one changing a spherical surface into one of the same curvature in such a way that the lines of curvature on the two surfaces correspond and the second quadratic forms are the same.*

Let $S$ and $S_1$ be two surfaces of curvature $1/a^2$ referred to their corresponding lines of curvature and let the second quadratic forms of the two surfaces be

$$D(du^2 + dv^2). \tag{19}$$

If no distinction is made with regard to homothetic surfaces, the most general integrals of equations (7) for $S$ and $S_1$ are

$$E - G = \pm a^2, \qquad E_1 - G_1 = \pm a^2. \tag{20}$$

Since the total curvatures of $S$ and $S_1$ are equal and (19) is the second quadratic form of both surfaces, we must have also

$$E_1 G_1 = EG. \tag{21}$$

In conformity with this we put

$$E_1 = \lambda E, \qquad G_1 = \frac{1}{\lambda} G.$$

Substituting these values in the second of equations (20) and taking account of the first, we get

$$\lambda^2 E \mp a^2 \lambda + (\pm a^2 - E) = 0,$$

of which the solutions are

$$\lambda = \pm 1, \qquad \pm \frac{G}{E}.$$

Since $E_1$ and $G_1$ must be positive, the only possibility is

$$E_1 = G, \qquad G_1 = E,$$

which proves the theorem. Hence the HAZZIDAKIS transform, or conjugate, of a spherical surface may be defined to be the surface with the same total curvature and second quadratic form and such that the lines of curvature on the two surfaces are in correspondence.

We shall find that on the two surfaces $S$ and $S_1$ the characteristic lines also

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are in correspondence. For, the general equation of the characteristic lines upon a surface referred to a conjugate system is *

$$D(GD - ED'')du^2 - 4FDD''dudv + D''(ED'' - GD)dv^2 = 0.$$

When the system is such that the second quadratic form is (19), this reduces to

$$(G - E)du^2 - 4Fdudv + (E - G)dv^2 = 0, \tag{22}$$

which in consequence of (13) is the equation of these lines on $S_1$ also. Hence the *characteristic lines correspond upon two conjugate spherical surfaces.*

§ 3. *Surfaces of revolution and surfaces of constant mean curvature.*

The spherical surfaces of revolution, referred to their meridians and parallels, lend themselves readily to the transformation of HAZZIDAKIS. BIANCHI † shows that these surfaces are defined by the equations

$$x = r\cos w, \qquad y = r\sin w, \qquad z = \phi(r), \tag{23}$$

where

$$r = \kappa\cos u, \qquad z = \int\sqrt{1 - \kappa^2\sin^2 u}\,du, \tag{24}$$

in which $\kappa$ denotes a constant. When $\kappa$ is equal to one, $S$ is a sphere of unit radius. The other surfaces of this kind are divided into two classes according as $\kappa$ is less or greater than one. From (23) and (24) it is found that the linear element of $S$ has the form

$$ds^2 = \frac{dr^2}{\kappa^2 - r^2} + r^2dw^2 \tag{25}$$

and for the second quadratic form we have

$$\Phi = r\sqrt{1 - \kappa^2 + r^2}\left[\frac{dr^2}{(\kappa^2 - r^2)(1 - \kappa^2 + r^2)} + dw^2\right]. \tag{26}$$

As the case of the sphere does not afford any particular interest, we pass to the consideration of the two remaining cases in turn.

1°. $\kappa < 1$. In order to have $D = D''$ we change the parameters, putting

$$dt = \frac{dr}{\sqrt{(1 - \kappa^2 + r^2)(\kappa^2 - r^2)}}.$$

Upon integration we have $r = \kappa\, cnt$, with $\kappa$ the modulus. When this change is effected upon (25) and (26), they become

* *Three particular systems of lines*, l. c., p. 429.

† L. c., vol. 1, p. 233 ; German translation, p. 197.

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$$ds^2 = dn^2 t\, dt^2 + \kappa^2 cn^2 t\, dw^2 \tag{27}$$

$$\Phi = \kappa\, cnt\, dnt\, (dt^2 + dw^2). \tag{28}$$

2°. $\kappa_1 > 1$. For this case we put

$$\frac{dt}{\kappa_1} = \frac{dr}{\sqrt{(1-\kappa_1^2+r^2)(\kappa^2-r^2)}},$$

which gives by integration $r^2 = \kappa_1^2 - sn^2 t$, and the modulus is $1/\kappa_1$. Now the linear element and second quadratic form, after putting $w_1 = \kappa_1 w$, reduce to

$$ds^2 = \frac{cn^2 t}{\kappa_1^2}\, dt^2 + dn^2 t\, dw_1^2, \tag{29}$$

$$\Phi = \frac{cnt \cdot dnt}{\kappa_1}(dt^2 + dw_1^2). \tag{30}$$

If we take for $\kappa_1$ the inverse of $\kappa$, the elliptic functions have the same modulus. Again, if we take the same $t$ for both cases and to a line $w = \text{const}$, on one of the first case make correspond a line $w_1 = \text{const}$, on a surface of the second kind, the forms (28) and (30) are the same, and (30) is obtained by interchanging the coefficients in (28) and vice versa. Hence:

*Given a surface of revolution of constant positive curvature corresponding to a certain $\kappa$; its conjugate is the similar surface corresponding to $1/\kappa$.*

BONNET has shown that there is a surface of constant mean curvature $1/2a$ parallel to a surface of total curvature $1/a^2$ and at the distance $a$. If the parametric curves be the lines of curvature for these surfaces and the fundamental functions for the former surface be denoted by $e$, $g$, $d$, $d''$, the following relations hold:*

$$e = g = \lambda a\left[a\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)-2\right],$$

$$d = \frac{e}{a} - \lambda\left[a\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)-1\right], \qquad d'' = \frac{g}{a} - \lambda\left[a\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)-1\right],$$

where

$$\lambda = d = d''.$$

Let $S$ and $S_1$ be two surfaces of unit total curvature which are the transforms of one another and denote by $S$ and $S_1$ the two surfaces of constant mean curvature respectively which are associated with the former, as remarked by the theorem of BONNET. Now $a$ in the above formulæ has the value unity. If the functions for $S_1$ be denoted by $E_1$, $G_1$, $D_1$, $D_1''$, the above relations lead to the following in consequence of the relation between $S$ and $S_1$:

$$e = g = e_1 = g_1, \qquad d = d_1'', \qquad d'' = d_1.$$

* KNOBLAUCH, Einleitung in die allgemeine Theorie der krummen Flächen, p. 234.

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We have then the theorem:

*Given any surface of constant mean curvature, there is another surface of this kind applicable to it in such a way that lines of curvature correspond; when the latter are parametric and the parameters are suitably chosen*

$$D = D_1'', \qquad D'' = D_1.$$

BONNET * has shown that every surface of constant mean curvature is applicable to an infinity of surfaces of the same mean curvature with preservation of the principal radii. It has just been seen that to every surface of this kind there is applicable a surface of the same mean curvature with preservation of the lines of curvature and equality of the principal radii in inverse order. Hence the theorem:

*Given a surface $S$ of constant mean curvature, there are two families of surfaces of the same mean curvature applicable to it; for the surfaces of the one family the principal radii are equal to the corresponding radii of $S$, and for the surfaces of the other family the principal radii are equal to those of $S$ in reverse order.*

## § 4. *A new transformation of spherical surfaces.*

Let $E$, $F$, $G$ be three functions satisfying equations (7). These equations are satisfied also by

$$E_1 = \frac{E+G-2F}{2}, \quad F_1 = \frac{E-G}{2}, \quad G_1 = \frac{E+G+2F}{2}. \tag{31}$$

We find that

$$E_1 G_1 - F_1^2 = EG - F^2, \tag{32}$$

so that we are led to inquire whether there exists a surface $S_1$ whose first fundamental coefficients are given by (31) and whose second quadratic form is the same as for $S$. In order that this be true, it is necessary and sufficient that these functions satisfy the GAUSS equation (13′), which in consequence of (6), (31) and (32) can be given the form

$$\begin{aligned} 4Ha^2 = \Big\{ & \frac{\partial}{\partial v}\log(E+G-2F)\frac{\partial}{\partial u}\left(\frac{E-G}{H}\right) - \frac{\partial}{\partial u}\left[\frac{1}{H}\frac{\partial}{\partial u}(E+G+2F)\right] \\ & - \frac{\partial}{\partial v}\left[\frac{1}{H}\frac{\partial}{\partial v}(E+G+2F)\right] - \frac{\partial}{\partial u}\log(E+G-2F)\frac{\partial}{\partial v}\left(\frac{E-G}{H}\right)\Big\}. \end{aligned} \tag{33}$$

* *Mémoire sur la théorie des surfaces applicables sur une surface donnée*, Journal de l'École Polytechnique, Cahier 42, p. 77.

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The GAUSS equation for $S$ can be reduced to either of the equivalent forms

$$(34)\quad 2a^2H = \frac{1}{E}\frac{\partial E}{\partial v}\frac{\partial}{\partial u}\left(\frac{F}{H}\right) - \frac{\partial}{\partial u}\left(\frac{1}{H}\frac{\partial G}{\partial u}\right) - \frac{\partial}{\partial v}\left(\frac{1}{H}\frac{\partial G}{\partial v}\right) - \frac{1}{E}\frac{\partial E}{\partial u}\frac{\partial}{\partial v}\left(\frac{F}{H}\right).$$

$$(35)\quad 2a^2H = \frac{1}{G}\frac{\partial G}{\partial u}\frac{\partial}{\partial v}\left(\frac{F}{H}\right) - \frac{\partial}{\partial v}\left(\frac{1}{H}\frac{\partial E}{\partial v}\right) - \frac{\partial}{\partial u}\left(\frac{1}{H}\frac{\partial E}{\partial u}\right) - \frac{1}{G}\frac{\partial G}{\partial v}\frac{\partial}{\partial u}\left(\frac{F}{H}\right).$$

If these equations be added and the result be subtracted from (33), we will be brought to an equation which vanishes identically in consequence of (7). Hence:

*Given a spherical surface referred to an isothermal-conjugate system of lines and let the parameters be so chosen that the second fundamental functions $D$ and $D''$ are equal; there exists a second spherical surface with the same second quadratic form and the coefficients of its linear element are given by* (31). For convenience we shall call the new surface $S_1$ the *adjoint* of $S$. When the expressions (31) are substituted in the equation of the characteristic directions on $S_1$, namely (22)

$$(G_1 - E_1)du^2 - 4F_1 du dv + (E_1 - G_1)dv^2 = 0,$$

this equation reduces to (16). Hence the characteristic lines on $S_1$ correspond to the lines of curvature on $S$. And the equation of the lines of curvature on $S_1$ reduces to (22). Hence *upon a spherical surface and its adjoint the lines of curvature of one surface correspond to the characteristic lines upon the other.*

For the spherical representation of $S_1$ we have

$$\mathcal{E}_1 = \frac{G_1}{a^2}, \qquad \mathcal{F}_1 = \frac{-F_1}{a^2}, \qquad \mathcal{G}_1 = \frac{E_1}{a^2},$$

which in consequence of (31) and (11) may be written

$$\mathcal{E}_1 = \frac{\mathcal{E} + \mathcal{G} - 2\mathcal{F}}{2}, \qquad \mathcal{F}_1 = \frac{\mathcal{E} - \mathcal{G}}{2}, \qquad \mathcal{G}_1 = \frac{\mathcal{E} + \mathcal{G} + 2\mathcal{F}}{2}.$$

Hence the coefficients of the spherical representation of corresponding isothermal-conjugate lines on a spherical surface and its adjoint bear the same relation to one another as the first fundamental functions for these surfaces.

For the sake of convenience we shall refer to the transformation of HAZZIDAKIS and the one treated in the preceding section as the transformations $H$ and $I$ respectively. First of all we remark that throughout any number of operations of $H$ and $I$ separately or conjointly the second quadratic form is invariant. Since the relation between a surface and its transform by either $H$ or $I$ is completely reciprocal, it follows from (13) and (31) that the same sur-

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face is obtained when $H$ and $I$ are applied consecutively as when their order is reversed. And also that one is brought back to the original surface by any one of the sets of combined transformations

$$HH, \qquad HII, \qquad IHI, \qquad IIH.$$

Denote by $T$ any transformation of a spherical surface into a second surface of this kind which preserves the total curvature and second quadratic form and interchanges lines of curvature and characteristic lines. Then $IT$ leaves lines of curvature unaltered and consequently is either identity or an $H$ transformation. Therefore $T$ is either an $I$ transformation or a combination of an $I$ and one or more $H$ transformations. Hence the theorem:

*The only transformation which changes a spherical surface into a spherical surface of the same curvature and the same second quadratic form and interchanges the lines of curvature and characteristic lines is the $I$ transformation or a combination of the latter and Hazzidakis transformations.*

## § 5. *Surfaces of negative curvature.*

In this section we study surfaces of negative curvature in a manner similar to the preceding. We have shown * that upon such a surface there is an infinity of conjugate systems such that, when these lines are taken as parametric, the parameters can be so chosen that

$$D = -D'', \qquad D' = 0.$$

The GAUSS equation can be written

$$K = \frac{1}{H}\left[\frac{\partial}{\partial v}\left(\frac{H}{E}\left\{\begin{matrix}11\\2\end{matrix}\right\}\right) - \frac{\partial}{\partial u}\left(\frac{H}{E}\left\{\begin{matrix}12\\2\end{matrix}\right\}\right)\right], \tag{36}$$

and proceeding as in § 1 we find that if we put

$$K = -\frac{D^2}{H^2} = -1, \tag{37}$$

the CODAZZI equations can be put in the form

$$2\frac{\partial F}{\partial u} = \frac{\partial E}{\partial v} + \frac{\partial G}{\partial v}, \qquad 2\frac{\partial F}{\partial v} = \frac{\partial E}{\partial u} + \frac{\partial G}{\partial u}. \tag{38}$$

From this it is seen that $F$ and $E + G$ are solutions of

$$\frac{\partial^2 \theta}{\partial u^2} - \frac{\partial^2 \theta}{\partial v^2} = 0. \tag{39}$$

* *Isothermal-conjugate Systems*, American Journal of Mathematics, vol. 25 (1903), p. 220.



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When either of these functions is known, the other is given by a quadrature from (38) and then it remains to find $E$ or $G$ from (36). This shows that every solution of (39) leads to surfaces of constant negative curvature. The most general form which $F$ can have is

$$F = \frac{\Phi(u+v) + \Psi(u-v)}{2},$$

and then

$$E + G = \Phi(u+v) - \Psi(u-v).$$

It is clear that equations (38) are satisfied by

$$E_1 = G, \qquad G_1 = E, \qquad F_1 = F. \tag{40}$$

When these values of $E$, $G$, $F$ are substituted in (36) with $K$ replaced by $-1$, and it is noted that they satisfy (38), we get

$$\left[\frac{\partial}{\partial u}\left(\frac{H_1}{G_1}\left\{\begin{matrix}22\\1\end{matrix}\right\}_1\right) - \frac{\partial}{\partial v}\left(\frac{H_1}{G_1}\left\{\begin{matrix}12\\1\end{matrix}\right\}_1\right)\right] = 1,$$

where the CHRISTOFFEL symbols $\left\{\begin{matrix}rs\\t\end{matrix}\right\}_1$ are formed with respect to

$$E_1 du^2 + 2F_1 du\, dv + G_1 dv^2. \tag{41}$$

But the left-hand side of the above expression is equal to the total curvature of the surface with the linear element (41)* Hence when these coefficients have the values (40), the surface $S_1$ has positive curvature, so that for pseudospherical surfaces a transformation analogous to the transformation of HAZZIDAKIS does not exist. The linear element of the spherical representation of $S$ is readily found to be†

$$d\sigma^2 = G du^2 + 2F du dv + E dv^2,$$

which is the same as (41) when the coefficients in the latter have the values given by (40); this accounts for the above result.

It is of interest to determine whether there is a surface with total curvature equal to one whose linear element takes this form when the parametric lines form an isothermal conjugate system. The conjugate of this surface would have for linear element

$$ds^2 = E du^2 - 2F du dv + G dv^2,$$

so that $E$, $F$, $G$ would have to satisfy equations

$$2\frac{\partial F}{\partial u} = \frac{\partial G}{\partial v} - \frac{\partial E}{\partial v}, \qquad 2\frac{\partial F}{\partial v} = \frac{\partial E}{\partial u} - \frac{\partial G}{\partial u}, \tag{42}$$

* BIANCHI, l. c., vol. I., p. 77; German translation, p. 53.

† Ibid, p. 150; p. 120.

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as well as (38). For these to be satisfied it is necessary that $E$ be a function of $u$ alone and $G$ of $v$ alone. In consequence of (42), we have for the forms of these functions

$$E = \alpha u^2 + 2\gamma u + \epsilon, \quad F = \alpha uv + \beta u + \gamma v + \delta, \quad G = \alpha v^2 + 2\beta v + \kappa,$$

where $\alpha$, $\beta$, $\gamma$, $\delta$, $\epsilon$ and $\kappa$ are constants. For no values of these constants does the formula (36) make $K$ equal to $-1$, which shows the non-existence of such a transformation.

One sees also that if $E$, $G$, $F$ are solutions of equations (38) so also are

$$E_1 = \frac{E - G + 2F}{2}, \qquad G_1 = \frac{G - E + 2F}{2}, \qquad F_1 = \frac{E + G}{2}.$$

But now

$$E_1 G_1 - F_1^2 = -(EG - F^2);$$

hence the transform is imaginary.

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# VOLUMES AND AREAS*

BY

N. J. LENNES

Professor HILBERT's *Festschrift* (1899) contains a theory of areas of plane polygons independent of every axiom of continuity, and dependent on the remaining axioms of plane geometry, viz., those of incidence, order, parallelism, congruence. The main results obtained by HILBERT in this connection are the following:

(*a*) The measure of area of a triangle [(base × altitude)/2] is independent of the selection of the base.

(*b*) The measure of area of any polygon is independent of any particular decomposition. [The measure of area of a polygon being defined as the sum of the measures of area of a set of triangles into which the polygon may be decomposed].

(*c*) If two polygons have equal measures of area it does not follow that there exists a decomposition of them into finite sets of polygons which are congruent in pairs. If such decomposition does exist the polygons are said to be decompositionally congruent.

(*d*) If two polygons have equal measures of area then it is always possible to add polygons decompositionally congruent such that the resulting polygons shall be decompositionally congruent.†

The measures of volume are expressed in terms of elements which satisfy certain conditions imposed upon a segment calculus. Among other conditions they satisfy such conditions of order as are usually adapted for scalar quantities, viz.:

* Presented to the Society (Chicago) December 29, 1904. Received for publication January 3, 1905.

† We notice here that HILBERT's theory requires a slight modification. HILBERT defines as follows: "Sind $P$ und $Q$ zwei inhaltsgleiche Polygone so mus es nach der Definition zwei flächengleiche [decompositionally congruent] Polygone $P'$ und $Q'$ geben, so dass das aus $P$ und $P'$ zusammengesetzte Polygon mit dem aus $Q$ und $Q'$ zusammengesetzten Polygon flächengleich ausfällt." *Festschrift*, p. 45.

In developing the theory it becomes necessary to adjoin polygons which are entirely predetermined, but this is not always possible. Following is an example of two polygons which cannot be thus adjoined: A regular sharp pointed star with a large number of sides and a regular convex polygon of a large number of sides.

If we regard the sum of a number of polygons not as one polygon formed by adjoining these polygons but as an aggregate of polygons which may be decomposed separately there is no such difficulty.



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If $a$, $b$ and $c$ are any such scalar quantities then

(1) Of the three relations $a = b$, $a < b$, $b < a$ one and only one is valid.

(2) If $a = b$ and $b = c$ then $a = c$.

(3) If $a < b$ and $b < c$ then $a < c$.

The proposition listed as $(d)$ makes it possible to define equal and unequal areas in terms of geometric congruence of a finite number of polygons in such manner that it shall correspond to the definition stated in terms of measure of area.

In space the situation is different. M. DEHN has proved * the following condition necessary in order that two polyhedrons shall be decompositionally congruent:

If two polyhedrons $P'$ and $P''$ are decompositionally congruent then there exists a linear homogeneous function $f(\pi'_1, \pi'_2, \cdots, \pi''_1, \pi''_2, \cdots)$ with integral coefficients all different from zero such that

$$f(\pi'_1, \pi'_2, \cdots, \pi''_1, \pi''_2, \cdots) \equiv 0 \pmod{2R}$$

(where $R$ denotes a right angle) in which $\pi'_1, \pi'_2, \cdots$ are the plane angles of the dihedral angles of $P'$ and $\pi''_1, \pi''_2, \cdots$ are the plane angles of the dihedral angles of $P''$.

In the article cited, M. DEHN further shows that this condition is also necessary in order that it shall be possible to adjoin to $P'$ and $P''$ polyhedrons decompositionally congruent so that the resulting polyhedrons shall be decompositionally congruent. Since it is not difficult to find two polyhedrons such that this condition is not satisfied no matter what may be their measures of volume, it follows that in respect to the volumes of such polyhedrons the theory must be essentially different from the theory of areas of polygons.

More recently,† S. O. SCHATUNOVSKY has discussed the measure of volumes of polyhedrons. He considers a scalar function $\mu$ (base $\times$ alt.) of a tetrahedron ($\mu$ is ultimately given the value $\frac{1}{3}$) which is called the measure of volume of the tetrahedron. The measure of volume of any polyhedron is defined as the sum of the measures of volume of the tetrahedrons into which the polyhedron may be decomposed. A proof is then given that this measure of volume of a polyhedron is independent of any particular decomposition of the polyhedron. Hence to every polyhedron $P$ there corresponds a unique segment (number), denoted by $M(P)$, which is the measure of volume of the polyhedron. Evidently these measures of volume satisfy the conditions of order enumerated above.

It follows that if two polyhedrons $P_1$ and $P_2$ are decompositionally congruent then $M(P_1) = M(P_2)$. If $P_1$ and $P_2$ are decomposable into sets of polyhe-

* Mathematische Annalen, vol. 55 (1902), p. 465.

† Mathematische Annalen, vol. 57 (1903), p. 496.

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drons $P_1'$ and $P_2'$ such that the elements of $P_1'$ are congruent in pairs to part of the element of $P_2'$ then $M(P_1) < M(P_2)$.

The converses of these propositions proved by SCHATUNOVSKY do not follow without the use of the archimedian axiom. From the hypothesis that $M(P_1) = kM(P_2)$, $k$ being any segment (number) whatever different from zero, we can draw no conclusion whatever as to whether or not there exist $P_1'$ and $P_2'$ such that the elements of $P_1'$ are congruent in pairs to a part or all of the elements of $P_2'$ or vice versa.

The proof of this statement is identical with that suggested * by HILBERT † in proof of the proposition that two parallelograms of equal base and altitude cannot be proved decompositionally congruent without the use of the archimedian axiom. Hence it does not appear possible without the use of the archimedian axiom so to define volume of polyhedrons in terms of geometric congruence of a finite number of polyhedrons that it shall correspond to the measure of volume.

The object of this paper is to state a theorem provable only by means of the archimedian axiom and to base upon it a definition of volume in terms of congruence which shall correspond to the current definition in terms of measure of volume.

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* A detailed proof runs as follows:

Consider an equilateral triangle $ABC$ whose sides are unity. It follows readily from the axioms at our disposal that any decomposition (into polygons) of this triangle will result in a set of polygons such that a segment connecting any two points in one of them will be equal to, or less than, unity.

On a non-archimedian line consider three points $A$, $B$, $C'$, in the order $ABC'$, such that $AB = 1$ while the point $C'$ cannot be reached from $B$ by laying off unit segments any finite number of times. At $C'$ erect a perpendicular to the line $AC'$ and on this perpendicular lay off a segment $C'D = \frac{1}{2}\sqrt{3}$ ($\frac{1}{2}$ the altitude of the equilateral triangle described above). Connect $AD$ and $BD$.

The following lemma follows directly from the theory of proportion: Let $M$, $N$, $M'$, $N'$ be four points on the segment $BC'$ such that $MN = M'N'$. Erect perpendiculars to the line $BC'$ at these points. These perpendiculars intersect the sides $AD$ and $BD$ of the triangle $ABD$ forming quadrilaterals. The quadrilateral two of whose sides lie on the perpendiculars at $M'$ and $N'$ is congruent to part of the quadrilateral two of whose sides lie on the perpendiculars at $M$ and $N$, provided at least one of the points $M$ and $N$ lie between $B$ and both $M'$ and $N'$.

Any decomposition of the triangle $ABD$ into polygons which shall be congruent in part with polygons of a decomposition of $ABC$ must be such that a segment connecting any two points in one of them is equal to or less than unity. Hence any such polygon lies between a pair of lines perpendicular to $BC'$ and not more than a unit distance apart. It follows from the lemma that any set of $n$ such polygons may be obtained by suitably decomposing that part of the triangle $ABD$ which lies between the perpendiculars to $AC'$ at $A$ and at a point $E$ between $B$ and $C'$ such that $BE = n(AB)$. By this process we cannot reach a point $P$ between $B$ and $C'$ such that $PC' \leqq n(AB)$ for any value whatever of $n$. Hence any decomposition of the triangle $ABC$ and $ABD$ into polygons of which any number $n$ are congruent in pairs will always have a set of polygons resulting from the decomposition of $ABD$, not included in these $n$ polygons, the sum of whose measures of area is greater than the measure of all of that part of the triangle $ABD$ which lies between a perpendicular to the line $BC'$ at $P$ and the perpendicular to that line at $C'$. Consequently $ABD$ cannot be exhausted even by an infinite limiting process, in spite of the fact that its measure of area is only one half that of $ABC$.

† *Festschrift*, p. 42; TOWNSEND's translation, p. 60.

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THEOREM. *If $M(P_1)$ denotes the measure of volume of a polyhedron $P_1$ and $M(P_2)$ the measure of volume of a polyhedron $P_2$ and if $M(P_2) < M(P_1)$ then there exists a decomposition of $P_1$ and $P_2$ into sets of polyhedrons $P_1'$ and $P_2'$ such that the elements of $P_2'$ are congruent in pairs with part* of the elements of $P_1'$.*

*Proof*: Denote by $S_1$ and $S_2$ the areas of the surfaces of $P_1$ and $P_2$ respectively, by $l_1$ and $l_2$ the total lengths of their edges and by $v_1$ and $v_2$ the numbers of their vertices. Consider a division of space into a set $[c]$ of equal cubes.† Denote by $[c_1]$ a subset of the set $[c]$ such that every cube of $[c_1]$ has at least one point in common with $P_1$ and by $[c_2]$ a subset of $[c]$ such that every cube of $[c_2]$ has at least one point in common with $P_2$. Denote further by $[c_1']$ all cubes each of which has at least one point in common with the boundary of $P_1$ and by $[c_2']$ all cubes each of which has a point in common with the boundary of $P_2$. Denote the diagonals of such cubes by $k$. Then all cubes of the set $[c]$ which have a point in common with a segment of length $l$ lie within a rectangular parallelopiped of length $l + 2k$ its other dimensions being $2k$. Hence the total volume of such cubes is less than $4k^2(l + 2k)$. Therefore the total volume of all cubes of the set $[c]$ which have a point in common with an edge of the polyhedron $P_1$ is less than $4k^2 l_1 + 8k^3 v$. It is readily seen that if we add to this $2ks_1$ we shall have a sum greater than the measure of volume of the set $[c_1]$.

Denote by $f_1(k)$ the expressions $4k^2 l_1 + 8k^3 v_1 + 2ks_1$, and by $f_2(k)$ the similar expression $4k^2 l_2 + 8k^3 v_2 + 2ks_2$, which is greater than the measure of volume of the set $[c_2']$.

Let $M(P_1) - M(P_2) = \sigma$. Take $k$ so that $f_1(k) + f_2(k) < \sigma$.‡ Then

$$f_1(k) + f_2(k) < M(P_1) - M(P_2)$$

or

$$M(P_2) + f_2(k) < M(P_1) - f_1(k)$$

and therefore

$$M[c_2] < \{M[c_1] - M[c_1']\}.$$

Hence for such values of $k$ there is a larger number of cubes of the set $[c]$ which contain interior points of $P_1$ without containing any point of its boundary than there is in the complete set $[c_2]$. Hence there exists a decomposition of $P_1$ and $P_2$ into $P_1'$ and $P_2'$ such that the elements of $P_2'$ are congruent in pairs to part of the elements of $P_1'$ which was to be proven.

We arrange a relation between any two polyhedrons $P_1$, $P_2$ as to volume, so that in every case at least one of the relations

* By *part* is meant proper subset (not denoting the whole of $P_1'$).

† This division is effected by means of parallel planes. The parallel line axiom is also used in forming the calculus of segments here used.

‡ It is at this point that the archimedian axiom is used.

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$$\text{vol}(P_1) > \text{vol}(P_2), \qquad \text{vol}(P_1) = \text{vol}(P_2), \qquad \text{vol}(P_1) < \text{vol}(P_2)$$

holds by the following

DEFINITION. *The relation*

$$\text{vol}(P_1) > \text{vol}(P_2) \qquad \textit{or} \qquad \text{vol}(P_2) < \text{vol}(P_1)$$

*implies that the two polyhedrons $P_1, P_2$ are decomposable into sets $P_1', P_2'$ of polyhedrons such that part of the elements of $P_1'$ are in a one-to-one way congruent to all of the elements of $P_2'$. The relation*

$$\text{vol}(P_1) = \text{vol}(P_2)$$

*implies that neither of the relations*

$$\text{vol}(P_1) > \text{vol}(P_2), \qquad \text{vol}(P_2) < \text{vol}(P_2)$$

*holds.*

It follows from the above theorem that if $M(P_1) = M(P_2)$, then in the sense of the definition just given $\text{vol}(P_1) = \text{vol}(P_2)$ and conversely that if $\text{vol}(P_1) = \text{vol}(P_2)$ then $M(P_1) = M(P_2)$.

We have thus obtained in terms of congruence a definition of equal and unequal volumes such that if $a$ and $b$ represent the volume of any two polyhedrons then between $a$ and $b$, there exists one and only one of the following relations

$$a = b, \qquad a < b, \qquad b < a.$$

Obviously these relations are transitive, as required by the statement on page 487.

THE UNIVERSITY OF CHICAGO,
*October* 31, 1904.

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# ON A PROBLEM INCLUDING THAT OF SEVERAL BODIES AND ADMITTING OF AN ADDITIONAL INTEGRAL*

BY

EDGAR ODELL LOVETT

In the problem of three bodies BERTRAND† introduced certain quadratic functions of the coördinates of the bodies and of quantities proportional to the projections of the velocities on the axes of coördinates. BOUR‡ showed that BERTRAND's variables satisfy a certain system $(S)$ of ordinary differential equations of the first order, and pointed out that the problem of three bodies may be considered as a particular solution of a more general problem whose equations are those of $S$ and of which a certain integral $D$ is known.

It is the object of the following note to write out the extension of these results to the case of any number of bodies.

Given a system of $n+1$ bodies consisting of a fixed body $(0, 0, 0; \mu)$ and $n$ others $(x_i, y_i, z_i; m_i)$, mutually attracting one another by central forces varying directly as the masses and as any arbitrary function of the distance; to determine the motion of the $n$ bodies about the fixed center we arrive at a system of $6n$ differential equations of the first order in the canonical form:

$$(1)\qquad \begin{cases} \dfrac{dx_i}{dt} = -\dfrac{\partial F}{\partial \xi_i}, & \dfrac{dy_i}{dt} = -\dfrac{\partial F}{\partial \eta_i}, & \dfrac{dz_i}{dt} = -\dfrac{\partial F}{\partial \zeta_i}, \\ \dfrac{d\xi_i}{dt} = \dfrac{\partial F}{\partial x_i}, & \dfrac{d\eta_i}{dt} = \dfrac{\partial F}{\partial y_i}, & \dfrac{d\zeta_i}{dt} = \dfrac{\partial F}{\partial z_i}, \end{cases} \qquad (i=1, 2, \cdots, n),$$

where $\xi_i$, $\eta_i$, $\zeta_i$ are proportional to the projections of the velocities of the bodies on the axes of coördinates, and the function $F$ is of the form

$$(2)\qquad F = U - \sum_{i=1}^{n} \frac{\xi_i^2 + \eta_i^2 + \zeta_i^2}{2m_i},$$

the force-function being designated by $U$.

* Presented to the Society April 29, 1905. Received for publication in revised form, March 19, 1905.

† BERTRAND, *Mémoire sur l'intégration des équations différentielles de la mécanique*, Journal de Liouville, ser. 1, vol. 17 (1852), pp. 393–436.

‡ BOUR, *Mémoire sur le problème des trois corps*, Journal de l'École Polytechnique, vol. 21 (1856), pp. 35–58.



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Let new variables

$$(3)\qquad \begin{cases} q_{ij} = x_i x_j + y_i y_j + z_i z_j, & q_{ji} = q_{ij}, \\ r_{ij} = \xi_i \xi_j + \eta_i \eta_j + \zeta_i \zeta_j, & r_{ji} = r_{ij}, \\ s_{ij} = x_i \xi_j + y_i \eta_j + z_i \zeta_j, & s_{ji} \neq s_{ij} \end{cases} \qquad (i, j = 1, 2, 3, \cdots, n),$$

be introduced. These variables are of the same form as those employed by BERTRAND in the memoir cited. They are $n(2n+1)$ in number and are not all distinct. The relations among them may be set up by the aid of the following well known theorem of the theory of determinants: If $k$ is a given number and $i, j$ two numbers which may take all values from 1 to $n$, the determinant of $n^2$ elements whose general element is

$$(4)\qquad a_{ij} = \sum_{h=1}^{k} \alpha_{i,h} \beta_{j,h}$$

is equal to the product of the determinant of the $\alpha$'s by that of the $\beta$'s for $k = n$, and is identically equal to zero for all values of $k$ less than $n$.

From this theorem and the forms (3) it readily appears that the symmetrical determinant

$$(5)\qquad \Delta = \begin{vmatrix} q_{ij} & s_{ij} \\ s_{ij} & r_{ij} \end{vmatrix} \qquad (i, j = 1, 2, \cdots, n),$$

where $q_{ij}$ represents the square of $n^2$ elements obtained by giving to $i, j$ the values $1, 2, \cdots, n$, and all its minors down to and including all the $\frac{1}{2}\binom{2n}{4}\{\binom{2n}{4}+1\}$ which are determinants of the fourth order vanish, and that no one of the $\frac{1}{2}\binom{2n}{3}\{\binom{2n}{3}+1\}$ which are of the third order vanishes.

These $\frac{1}{2}\binom{2n}{4}\{\binom{2n}{4}+1\}$ conditions among $n(2n+1)$ quantities are far too numerous, but they can be reduced to proper bounds by means of the following theorem given in 1869 by KRONECKER*: If in the determinant of the $n$th order

$$M = |a_{ij}| \qquad (i, j = 1, 2, \cdots, n),$$

the minor of the $m$th order

$$|a_{11}, a_{22}, \cdots, a_{mm}| \qquad (m < n)$$

does not vanish, and the minors of the $(m+1)$th order

$$|a_{11}, a_{22}, \cdots, a_{mm}, a_{ik}| \quad (i, k = m+1, m+2, \cdots, n)$$

do vanish, then all the $(m+1)$th order subdeterminants of $M$ vanish.

* KRONECKER, *Bemerkungen zur Determinanten-Theorie*, Crelle's Journal, vol. 72 (1870), pp. 152–175.

My colleague Dr. O. D. KELLOGG gave me a proof of the above theorem, believing the theorem to be new. Later I found that KRONECKER had published it as just cited.

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Accordingly the vanishing of all the $\frac{1}{2}\binom{2n}{4}\{\binom{2n}{4}+1\}$ fourth order subdeterminants of the above symmetrical determinant $\Delta$ is a consequence of the vanishing of $(n-1)(2n-3)$ properly chosen independent fourth order subdeterminants, and this choice can be made in $\frac{1}{2}\binom{2n}{3}\{\binom{2n}{3}+1\}$ ways. Then by the aid of these independent relations $(n-1)(2n-3)$ of the variables (3) can be eliminated if they be employed in problem (1); there would remain $6n-3$ independent variables which would be sufficient since a loss of three from the original $6n$ independent variables can be accounted for by change in orientation. On making $n=2$ in $\Delta$ we have BOUR's determinant $D$, the vanishing of which expresses the single relation among BERTRAND's ten variables (3) in the problem of three bodies.

In the variables (3) the force-function $U$ becomes

$$U=\sum_{i=1}^{n}\mu m_i f(\sqrt{q_{ii}})-\sum_{i=1}^{n}\sum_{j=1}^{n}m_i m_j f(\sqrt{q_{ii}+q_{jj}-2q_{ij}}); \tag{6}$$

accordingly the partial derivatives of $F$ are of the form

$$\frac{\partial F}{\partial x_i}=\mu_i x_i+\sum_{j=1}^{n}\mu_{ij}x_j, \qquad \frac{\partial F}{\partial \xi_i}=-\frac{\xi_i}{m_i}, \tag{7}$$

where the quantities

$$\left\{\begin{aligned} \mu_i &= \mu m_i\frac{f'(\sqrt{q_{ii}})}{\sqrt{q_{ii}}}-\sum_{j=1}^{n}\mu_{ij}, \\ \mu_{ij} &= m_i m_j\frac{f'(\sqrt{q_{ii}+q_{jj}-2q_{ij}})}{\sqrt{q_{ii}+q_{jj}-2q_{ij}}}=\mu_{ji}, \end{aligned}\right. \tag{8}$$

are coefficients depending on the forces and expressed in terms of the $q$'s alone.

Then in virtue of (1) the variables (3) satisfy the following system of ordinary differential equations:

$$\left\{\begin{aligned} \frac{dq_{ij}}{dt} &= \frac{s_{ij}}{m_j}+\frac{s_{ji}}{m_i}; \\ \frac{dr_{ij}}{dt} &= \mu_i s_{ij}+\mu_j s_{ji}+\mu_{ij}(s_{ii}+s_{jj})+\sum_{k=1}^{n}\mu_{jk}s_{ki}+\sum_{l=1}^{n}\mu_{il}s_{lj} \\ &\qquad\qquad (i,j=1,2,\cdots,n); \\ \frac{ds_{ij}}{dt} &= \mu_j q_{ij}+\mu_{ij}q_{ii}+\frac{r_{ij}}{m_i}+\sum_{k=1}^{n}\mu_{jk}q_{ik}; \end{aligned}\right. \tag{9}$$

these equations are the generalizations of BOUR's equations in the problem of three bodies.

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It may now be shown without difficulty that the determinant $\Delta$ equated to a constant gives an integral of the equations (9). This can be done perhaps most simply on remarking that $\Delta$ does not contain the $\mu$'s. Let $\phi$ be a function of all the $q$'s, $r$'s and $s$'s; if it is an integral not containing the $\mu$'s its total derivative with regard to the time

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\frac{\partial\phi}{\partial q_{ij}}\frac{dq_{ij}}{dt}+\frac{\partial\phi}{\partial r_{ij}}\frac{dr_{ij}}{dt}+\frac{\partial\phi}{\partial s_{ij}}\frac{ds_{ij}}{dt}\right\} \tag{10}$$

should vanish independently of the $\mu$'s when the total derivatives are replaced by their values (9).

From the absolute term of the equation thus formed we have the equation *

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\left(\frac{s_{ij}}{m_j}+\frac{s_{ji}}{m_i}\right)\phi_{q_{ij}}+\frac{r_{ij}}{m_i}\phi_{s_{ij}}\right\}=0; \tag{11}$$

from the coefficients of the $\mu_i$ the following $n$ equations:

$$b_i \equiv 2w_i\phi_{v_i}+u_i\phi_{w_i}+\sum_{j=1}^{n}(s_{ij}\phi_{r_{ij}}+q_{ij}\phi_{s_{ji}})=0 \qquad (i=1,2,\cdots,n); \tag{12}$$

and finally from the terms in which the $\mu_{ij}$ appear the following $\frac{1}{2}n(n-1)$ equations:

$$\begin{aligned} d_{ij}\equiv d_{ji}\equiv{}& 2s_{ji}\phi_{v_i}+2s_{ij}\phi_{v_j}+q_{ij}(\phi_{w_i}+\phi_{w_j})+(w_i+w_j)\phi_{r_{ij}}\\ &+u_i\phi_{s_{ij}}+u_j\phi_{s_{ji}}+\sum_{k=1}^{n}(s_{ik}\phi_{r_{jk}}+s_{jk}\phi_{r_{ik}}\\ &+q_{jk}\phi_{s_{ki}}+q_{ki}\phi_{s_{kj}})=0 \qquad (i,j=1,2,\cdots,n), \end{aligned} \tag{13}$$

where for brevity we have put

$$\begin{aligned} q_{ii}&=u_i,\\ r_{ii}&=v_i,\\ s_{ii}&=w_i. \end{aligned} \tag{14}$$

Combining these $\frac{1}{2}n(n+1)+1$ equations (11), (12), (13) in all possible pairs, by Poisson's operation, we obtain the following complete system of $n(2n+1)$ linear partial differential equations of the first order:

* In a previous communication to Professor E. W. Brown this equation was immediately broken up into the $n$ equations $a_i=0$ which follow above. He pointed out to me that this led to confusion since the $\xi$'s contain the masses. The correction was made, but curiously enough it left the resulting complete system unchanged.

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$$
(15)\quad \begin{cases}
a_i \equiv 2w_i\phi_{u_i} + v_i\phi_{w_i} + \sum_{j=1}^{n}(s_{ji}\phi_{q_{ij}} + r_{ij}\phi_{s_{ij}}) = 0; \quad b_i = 0; \\
c_i \equiv 2u_i\phi_{u_i} - 2v_i\phi_{v_i} + \sum_{j=1}^{n}(q_{ij}\phi_{q_{ij}} - r_{ij}\phi_{r_{ij}} + s_{ij}\phi_{s_{ij}} - s_{ji}\phi_{s_{ji}}) = 0; \quad d_{ij} = 0; \\
e_{ij} \equiv 2q_{ij}\phi_{u_i} - 2r_{ij}\phi_{v_j} + s_{ji}(\phi_{w_i} - \phi_{w_j}) + u_j\phi_{q_{ij}} - v_i\phi_{r_{ij}} \\
\qquad + (w_j - w_i)\phi_{s_{ij}} + \sum_{k=1}^{n}(q_{jk}\phi_{q_{ik}} - r_{ki}\phi_{r_{jk}} + s_{jk}\phi_{s_{ik}} - s_{ki}\phi_{s_{kj}}) = 0; \\
f_{ij} \equiv 2s_{ij}\phi_{u_i} + 2s_{ji}\phi_{u_j} + r_{ij}(\phi_{w_i} + \phi_{w_j}) + (w_i + w_j)\phi_{q_{ij}} + v_j\phi_{s_{ij}} \\
\qquad + v_i\phi_{s_{ji}} + \sum_{k=1}^{n}(s_{kj}\phi_{q_{ik}} + s_{ki}\phi_{q_{jk}} + r_{jk}\phi_{s_{ik}} + r_{ki}\phi_{s_{jk}}) = 0; \\
d_{ji} = d_{ij}, \qquad e_{ji} \neq e_{ij}, \qquad f_{ji} = f_{ij} \quad (i, j = 1, 2, \cdots, n).
\end{cases}
$$

These equations (15) are the generalizations of those given by GRAVÉ * for the case $n = 2$.

On replacing $\phi$ by $\Delta$ in equations (15) it is at once seen that they are identically satisfied in virtue of the well-known theorem of the theory of determinants which states that the sum of the products of the elements of any line of a determinant by the algebraic complements of the minors of the corresponding elements of a parallel line is zero.

We have then in equations (9) a problem including the problem of several bodies and admitting of the integral $\Delta =$ constant.

In virtue of the existence of this solution the determinant of the system (15) of $n(2n+1)$ equations in $n(2n+1)$ variables vanishes.

It may be added that the equations (15) admit of another integral which is a quadratic function of the integrals of areas in the $n$-body problem.

The reader will have little difficulty in verifying that the $n(2n+1)$ operators

$$
(16)\qquad a_i,\ b_i,\ c_i,\ d_{ij},\ e_{ij},\ f_{ij}
$$

constitute a continuous group of transformations in LIE's sense. It is hoped to study this group in detail in a subsequent note.

PRINCETON, NEW JERSEY.

* GRAVÉ, *Sur le problème des trois corps*, Nouvelles Annales de Mathématiques, ser. 3, vol. 15 (1896), pp. 537–547.

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# ON THE STABILITY OF THE MOTION OF A VISCOUS LIQUID*

BY

FRANCIS ROBERT SHARPE

§1. *Introduction.* REYNOLDS † has established an energy criterion for the stability of the motion of a viscous liquid, and has applied it to the case of a liquid moving between two parallel planes. For a liquid of density $\rho$ and viscosity $\mu$, moving between two planes $2b$ apart with mean velocity $U$ he found that the motion is unstable when

$$\frac{2b\rho U}{\mu} > 517.$$

Using the principle that the critical velocity is inversely proportional to the hydraulic mean depth he inferred that for a cylindrical pipe of radius $a$ the critical velocity is given by

$$\frac{2a\rho U}{\mu} > 1034,$$

and he compared this with his experimental value 1900. In the following paper I discuss directly the case of the cylindrical pipe and find that the motion is unstable when

$$\frac{2a\rho U}{\mu} > 470.$$

I also find, by using a different solution of the equation of continuity and the boundary conditions, that the motion between two parallel planes is unstable when

$$\frac{2b\rho U}{\mu} > 167,$$

instead of 517 as found by REYNOLDS.

§ 2. *Rectilinear flow through a pipe.* When the motion through a pipe is symmetrical the equations of motion, expressed in cylindrical coördinates $r, \theta, z$, are ‡

* Presented to the Society February 25, 1905. Received for publication May 5, 1905.

† REYNOLDS, Philosophical Transactions of the Royal Society, vol. 186 (1895) A, pp. 123–164.

‡ LOVE, *Mathematical Theory of Elasticity*, vol. 1, p. 217.



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$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}\right)=\frac{\partial p_{rr}}{\partial r}+\frac{\partial p_{zr}}{\partial z}+\frac{p_{rr}-p_{\theta\theta}}{r}, \tag{1}$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}\right)=\frac{\partial p_{rz}}{\partial r}+\frac{\partial p_{zz}}{\partial z}+\frac{p_{rz}}{r}. \tag{2}$$

The extensions are* $\partial u/\partial r$, $u/r$, $\partial w/\partial z$, and the shear $\partial u/\partial z+\partial w/\partial r$. Hence†

$$p_{rr}=-p+2\mu\frac{\partial u}{\partial r},\qquad p_{\theta\theta}=-p+2\mu\frac{u}{r},\qquad p_{zz}=-p+2\mu\frac{\partial w}{\partial z},$$

$$p_{zr}=p_{rz}=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right).$$

The equation of continuity is‡

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}=0, \tag{3}$$

and the boundary conditions are

$$u=w=0 \qquad \text{when} \qquad r=a. \tag{4}$$

The simplest solution of these equations is§

$$u=\frac{\partial w}{\partial t}=\frac{\partial w}{\partial z}=\frac{\partial p}{\partial r}=0,\qquad \frac{\partial p}{\partial z}=\text{constant},$$

$$w=-\frac{1}{4\mu}\frac{\partial p}{\partial z}(a^2-r^2),$$

and the mean velocity of flow is

$$U=-\frac{a^2}{8\mu}\frac{\partial p}{\partial z}.$$

§ 3. *Steady non-rectilinear motion.* For some minimum value of $U$, to be determined, permanent eddying motion is possible, that is, the rectilinear motion is unstable. The mean motion is still rectilinear but $u$ is no longer zero at any particular instant. We assume then that

$$u=u',\qquad w=\bar{w}+w', \tag{5}$$

where $\bar{w}$ is a function of $r$, and $u'$ and $w'$ are periodic functions of $r$ and $z$ whose mean values are zero.

The method used by REYNOLDS is to calculate the average rate of increase of the motion $u=u'$, $w=w'$, which is the motion relative to the mean motion $u=0$, $w=\bar{w}$, for a volume of liquid extending to the boundaries of the liquid.

* LOVE, loc. cit., p. 216.

† LAMB, *Hydrodynamics*, p. 512, 521.

‡ LAMB, loc. cit., p. 134.

§ LAMB, loc. cit., p. 521.

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§ 4. *The equations of mean motion.* Using a bar to denote a mean value we have, on substituting (5) in (1), (2) and (3),

$$\rho\left\{\frac{\partial u'}{\partial t}+u'\frac{\partial u'}{\partial r}+(\bar{w}+w')\frac{\partial u'}{\partial z}\right\}=\frac{\partial(\bar{p}_{rr}+p'_{rr})}{\partial r}+\frac{\partial(\bar{p}_{zr}+p'_{zr})}{\partial z}+\frac{p_{rr}+p'_{rr}-\bar{p}_{\theta\theta}-p'_{\theta\theta}}{r},$$

$$\rho\left\{\frac{\partial w'}{\partial t}+u'\frac{\partial w'}{\partial r}+(\bar{w}+w')\frac{\partial w'}{\partial z}+u'\frac{\partial \bar{w}}{\partial r}\right\}=\frac{\partial(\bar{p}_{rz}+p'_{rz})}{\partial r}+\frac{\partial(\bar{p}_{zz}+p'_{zz})}{\partial z}+\frac{\bar{p}_{rz}+p'_{rz}}{r}.$$

$$\frac{\partial u'}{\partial r}+\frac{u'}{r}+\frac{\partial w'}{\partial z}=0. \tag{6}$$

Multiplying (6) by $\rho u'$ and $\rho w'$ respectively and adding to the preceding equations,

$$\rho\left\{\frac{\partial u'}{\partial t}+\frac{\partial(u'u')}{\partial r}+\frac{\partial(u'w')}{\partial z}+\bar{w}\frac{\partial u'}{\partial z}+\frac{u'u'}{r}\right\}=\frac{\partial(\bar{p}_{rr}+p'_{rr})}{\partial r}+\frac{\partial(\bar{p}_{zr}+p'_{zr})}{\partial z}+\frac{\bar{p}_{rr}+p'_{rr}-\bar{p}_{\theta\theta}-p'_{\theta\theta}}{r},$$

$$\rho\left\{\frac{\partial w'}{\partial t}+\frac{\partial(u'w')}{\partial r}+\frac{\partial(w'w')}{\partial z}+\bar{w}\frac{\partial w'}{\partial z}+u'\frac{\partial \bar{w}}{\partial r}+\frac{u'w'}{r}\right\}=\frac{\partial(\bar{p}_{rz}+p'_{rz})}{\partial r}+\frac{\partial(p_{zz}+p'_{zz})}{\partial z}+\frac{\bar{p}_{rz}+p'_{rz}}{r}.$$

Taking mean values, we have the equations of mean motion

$$\rho\left\{\frac{\partial(\overline{u'u'})}{\partial r}+\frac{\overline{u'u'}}{r}\right\}=\frac{\partial\bar{p}_{rr}}{\partial r}+\frac{\partial\bar{p}_{zr}}{\partial z}+\frac{\bar{p}_{rr}}{r}, \tag{7}$$

$$\rho\left\{\frac{\partial(\overline{u'w'})}{\partial r}+\frac{\overline{u'w'}}{r}\right\}=\frac{\partial\bar{p}_{rz}}{\partial r}+\frac{\partial\bar{p}_{zz}}{\partial z}+\frac{\bar{p}_{rz}}{r}=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2\bar{w}}{\partial r^2}+\frac{1}{r}\frac{\partial\bar{w}}{\partial r}\right). \tag{8}$$

Multiplying (8) by $r$ and integrating

$$r\rho\overline{u'w'}=-\frac{r^2}{2}\frac{\partial p}{\partial z}+\mu r\frac{\partial\bar{w}}{\partial r}+C.$$

Since this holds for $r=0$, we have $C=0$. Hence

$$\mu\frac{\partial w}{\partial r}=\frac{r}{2}\frac{\partial p}{\partial z}+\rho\overline{u'w'}. \tag{9}$$

§ 5. *The energy of the relative motion.* Subtracting (7) and (8) from the preceding equations we have for the relative motion,

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$$\rho\left\{\frac{\partial u'}{\partial t}+\frac{\partial}{\partial r}(u'u'-\overline{u'u'})+\bar{w}\frac{\partial u'}{\partial z}+\frac{\partial(u'w')}{\partial z}+\frac{u'u'-\overline{u'u'}}{r}\right\}$$
$$=\frac{\partial p'_{rr}}{\partial r}+\frac{\partial p'_{zr}}{\partial z}+\frac{p'_{rr}-p'_{\theta\theta}}{r},$$

$$\rho\left\{\frac{\partial w'}{\partial t}+u'\frac{\partial w'}{\partial r}+\frac{\partial}{\partial r}(u'w'-\overline{u'w'})+\bar{w}\frac{\partial w'}{\partial z}+u'\frac{\partial \bar{w}}{\partial r}+\frac{u'w'-\overline{u'w'}}{r}\right\}$$
$$=\frac{\partial p'_{rz}}{\partial r}+\frac{\partial p'_{zz}}{\partial z}+\frac{p'_{rz}}{r}.$$

Multiplying these equations by $u'$, and $w'$ respectively, adding, and taking mean values,

$$\left(\frac{\partial}{\partial t}+\bar{w}\frac{\partial}{\partial z}\right)\tfrac{1}{2}\rho(u'^2+\bar{w}'^2)+\rho\overline{u'w'}\frac{\partial \bar{w}}{\partial r}=$$
$$\text{mean value of } u'\left(\frac{\partial p'_{rr}}{\partial r}+\frac{\partial p'_{zr}}{\partial z}+\frac{p'_{rr}-p'_{\theta\theta}}{r}\right)+w'\left(\frac{\partial p'_{rz}}{\partial r}+\frac{\partial p'_{zz}}{\partial z}+\frac{p'_{rz}}{r}\right).$$

Integrating over the section of the pipe and by parts on the right hand side, dividing by $2\pi$, and remembering that $u'$, $w'$ vanish on the boundary, we obtain the equation

$$\int_0^a\left(\frac{\partial}{\partial t}+\bar{w}\frac{\partial}{\partial z}\right)\tfrac{1}{2}\rho(\bar{u}'^2+\bar{w}'^2)\cdot rdr=\int_0^a-\rho\overline{u'w'}\frac{\partial \bar{w}}{\partial r}\cdot rdr$$
$$-\mu\int_0^a\left[2\left(\frac{\partial u'}{\partial r}\right)^2+2\left(\frac{\partial w'}{\partial z}\right)^2+2\frac{u'^2}{r^2}+\left(\frac{\partial u'}{\partial z}+\frac{\partial w'}{\partial r}\right)^2\right]rdr,$$

where, in the second integral on the right, mean values are to be taken.

The left hand side of this equation is the average rate of increase of the energy of the relative motion of a length $1/2\pi$ of liquid in the pipe.

Hence we have REYNOLD'S criterion for the possibility of permanent eddying motion expressed in cylindrical coördinates, namely,

$$(10)\qquad -\int_0^a\rho\overline{u'w'}\frac{\partial \bar{w}}{\partial r}\cdot rdr\geqq\mu\int_0^a\left[2\left(\frac{\partial u'}{\partial r}\right)^2+2\left(\frac{\partial w'}{\partial z}\right)^2\right.$$
$$\left.+2\frac{u'^2}{r^2}+\left(\frac{\partial u'}{\partial z}+\frac{\partial w'}{\partial r}\right)^2\right]rdr,$$

where, in the integrand on the right, mean values are to be taken.

The interpretation of equations (9) and (10) is that in order to maintain the relative as well as the mean motion an increased pressure gradient is necessary and the additional energy is dissipated in the relative motion, the right-hand



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integrand of (10) being STOKES's dissipation function expressed in cylindrical coördinates.

§ 6. *Determination of a possible relative motion.* The solution of the continuity equation (6) with the boundary conditions $u' = w' = 0$, for $r = a$, which leads to a minimum value of $U$ is

$$u' = LA\frac{\pi r}{2a}(\sin p + \sin 3p)\sin\frac{\pi Lz}{2a} - LB\frac{\pi r}{2a}(\sin 2p + \tfrac{1}{2}\sin 4p)\cos\frac{\pi Lz}{2a},$$

$$w' = A\left\{2(\sin p + \sin 3p) + \frac{\pi r}{2a}(\cos p + 3\cos 3p)\right\}\cos\frac{\pi Lz}{2a}$$

$$+ B\left\{2(\sin 2p + \tfrac{1}{2}\sin 4p) + \frac{\pi r}{2a}(2\cos 2p + 2\cos 4p)\right\}\sin\frac{\pi Lz}{2a},$$

as is easily seen on substitution, $p$ being $\pi r/2a$.

These give

$$\overline{u'w'} = \frac{LAB\pi^2 r^2}{32a^2}(3\sin p + 3\sin 3p - \sin 5p - \sin 7p).$$

Hence from (9) we find, neglecting $\overline{u'w'}^2$, the left-hand side of equation (10) equal to

$$-\frac{LAB\pi^2\rho U}{64a^2\mu}\int_0^a r^4(3\sin p + 3\sin 3p - \sin 5p - \sin 7p)\,dr = .215LABa\rho U.$$

Substituting the values of $u'$ and $w'$ in the right hand side of (10), taking mean values, and integrating, we obtain the condition

$$\frac{.215LABa\rho U}{\mu} \geqq (.304A^2 + .127B^2)L^4$$
$$+ (4.26A^2 + 6.67B^2)L^2 + 35.7A^2 + 16.34B^2.$$

Hence for a minimum value of $U$ we have to minimize

$$\left(\frac{.304}{\lambda} + .127\lambda\right)L^3 + \left(\frac{4.26}{\lambda} + 6.67\lambda\right)L + \left(\frac{35.7}{\lambda} + 16.34\lambda\right)\frac{1}{L}$$

where $\lambda = B/A$.

Differentiating with respect to $L$ and $\lambda$ we find

$$3(.304 + .127\lambda^2)L^4 + (4.26 + 6.67\lambda^2)L^2 - 35.7 - 16.34\lambda^2 = 0,$$

$$(-.304 + .127\lambda^2)L^4 + (-4.26 + 6.67\lambda^2)L^2 - 35.7 + 16.34\lambda^2 = 0.$$

Eliminating $\lambda^2$,

$$.4633L^8 + 20.55L^6 + 151.67L^4 - 2333.35 = 0.$$

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Hence $L = 1.798$, $\lambda = 1.158$, approximately. The minimum value of $U$ is therefore given by

$$\frac{2a\rho U}{\mu} = 470.$$

REYNOLDS' experiments gave

$$\frac{2a\rho U}{\mu} = 1900.$$

REYNOLDS' result for the flow between two parallel planes a distance $2b$ apart is

$$\frac{2b\rho U}{\mu} = 517,$$

and because the hydraulic mean depth for a cylinder radius $a$ is $\frac{1}{2}a$ and for two parallel planes is $b$ he inferred that for a cylinder

$$\frac{2a\rho U}{\mu} = 1034.$$

§ 7. *Flow between parallel planes.* When the motion is the same in planes parallel to the $xy$ plane the equations of motion are*,

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y},$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = \frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{yy}}{\partial y},$$

where

$$p_{xx} = -p + 2\mu\frac{\partial u}{\partial x}, \; p_{yy} = -p + 2\mu\frac{\partial u}{\partial y},$$

$$p_{yx} = p_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right).$$

The equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

In the case of liquid flowing between two parallel planes $y = \pm b$ the boundary conditions are $u = v = 0$ when $y = \pm b$.

The simplest solution of these equations is

$$v = \frac{\partial u}{\partial x} = \frac{\partial p}{\partial y} = 0, \qquad \frac{\partial p}{\partial x} = \text{constant}, \qquad u = \frac{1}{2\mu}\frac{\partial p}{\partial x}(y^2 - b^2),$$

and the mean velocity of flow is

$$U = -\frac{b^2}{3\mu}\frac{\partial p}{\partial x}.$$

* LAMB, *Hydrodynamics*, pp. 512–520.

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Using the method of the preceding problem we find for the equation of mean motion

$$\rho \frac{\partial(u'v')}{\partial y} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 \bar{u}}{\partial y^2}, \tag{11}$$

and the criterion of stability,

$$-\int_{-b}^{b} \rho \overline{u'v'} \frac{\partial \bar{u}}{\partial y} dy \geqq \mu \int_{-b}^{+b} \left\{ 2\left(\frac{\partial u'}{\partial x}\right)^2 + 2\left(\frac{\partial v'}{\partial y}\right)^2 + \left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right)^2 \right\} dy. \tag{12}$$

§ 8. *Determination of a possible relative motion.* The solution of the continuity equation, with the boundary conditions, which leads to a minimum value of $U$ is

$$u' = A(\sin p + \sin 3p)\cos\frac{\pi L x}{2b} + B(2\sin 2p + 4\sin 4p)\sin\frac{\pi L x}{2b},$$

$$v' = -LA(\cos p + \tfrac{1}{3}\cos 3p)\sin\frac{\pi L x}{2b} + LB(\cos 2p + \cos 4p)\cos\frac{\pi L x}{2b},$$

where $p = \pi y/2b$.

These give

$$4\overline{u'v'} = -LAB\{\tfrac{11}{3}\sin p + 6\sin 3p + \tfrac{8}{3}\sin 5p + \tfrac{1}{3}\sin 7p\}.$$

To determine $\partial\bar{u}/\partial y$ we have, on integrating (11),

$$\rho\overline{u'v'} = -y\frac{\partial p}{\partial x} + \mu\frac{\partial \bar{u}}{\partial y} + C.$$

Now $\overline{u'v'}$ is an odd function of $y$ and so is $\partial\bar{u}/\partial y$ since the mean motion is symmetrical with respect to the planes $y = \pm b$. Hence $C$ is zero and approximately

$$\frac{\partial \bar{u}}{\partial y} = \frac{y}{\mu}\frac{\partial p}{\partial x} = -\frac{3Uy}{b^2}.$$

Hence the left hand side of (12) reduces to

$$\frac{-18.6LAB\rho U}{\pi^2}.$$

Substituting the values of $u'$ and $v'$ in the right hand side of (12), taking mean values, and integrating, we have the condition

$$\frac{-18.6LAB\rho U}{\pi^2}$$

$$\geqq \frac{\mu b}{2}\left\{\frac{\pi^2 L^4}{4b^2}(\tfrac{10}{9}A^2 + 2B^2) + \frac{2\pi^2 L^2}{4b^2}(2A^2 + 20B^2) + \frac{\pi^2}{4b^2}(10A^2 + 272B^2)\right\},$$

or

$$\frac{3b\rho U}{\mu} \geqq \left(\frac{\pi}{2}\right)^4 \frac{(\tfrac{10}{9}A^2 + 2B^2)L^4 + (4A^2 + 40B^2)L^2 + 10A^2 + 272B^2}{-3.1LAB}.$$

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REYNOLDS takes the solution

$$u' = A(\cos p + 3\cos 3p)\cos\frac{\pi Lx}{2b} + B(2\cos 2p + 2\cos 4p)\sin\frac{\pi Lx}{2b},$$

$$v' = LA(\sin p + \sin 3p)\sin\frac{\pi Lx}{2b} - LB(\sin 2p + \tfrac{1}{2}\sin 4p)\cos\frac{\pi Lx}{2b},$$

and obtains the condition

$$\frac{3bU\rho}{\mu} \geqq \left(\frac{\pi}{2}\right)^4 \frac{(2A^2 + \tfrac{5}{4}B^2)L^4 + (20A^2 + 16B^2)L^2 + 82A^2 + 80B^2}{-1.325LAB}.$$

The coefficients of $A^2$ and $B^2$ being nearly equal, he determines the minimum value of $U$ by assuming $A = B$.

It is interesting to notice that his value for $v'$ vanishes midway between the planes and $u'$ is there a maximum whereas the values here adopted give a maximum value of $v'$ midway between the planes, which seem to be more in accordance with experiments.

Using the method of the preceding problem we find a minimum value of $U$ given by

$$\frac{3bU\rho}{\mu} = 41\left(\frac{\pi}{2}\right)^4,$$

when $L = 1.398$ and $A/B = 4.03$.

Hence our final result is that the motion is unstable when

$$\frac{2bU\rho}{\mu} \geqq 167.$$

REYNOLDS' result is 517.

CORNELL UNIVERSITY, *April*, 1905.

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# ÜBER DIE VOLLSTÄNDIG REDUCIBLEN GRUPPEN, DIE ZU EINER GRUPPE LINEARER HOMOGENER SUBSTITUTIONEN GEHÖREN*

VON

ALFRED LOEWY

## *Einleitung.*

Unter einer Gruppe† $\mathfrak{G}$ linearer homogener Substitutionen verstehen wir eine Gesamtheit linearer homogener Substitutionen von der Vollständigkeit, dass das Produkt irgend zweier Substitutionen sowie die reciproke Substitution jeder Substitution von $\mathfrak{G}$ in $\mathfrak{G}$ enthalten ist. Die Abgeschlossenheit einer Gesamtheit linearer homogener Substitutionen inbezug auf die Productbildung ist für den Gruppencharakter nicht ausreichend. Die Substitutionen von $\mathfrak{G}$ sollen keiner Annahme unterworfen sein, die Gruppe kann also continuirlich oder discontinuirlich sein, jedoch soll durch die Voraussetzung, dass $\mathfrak{G}$ stets das wirkliche Produkt irgend zweier seiner Substitutionen enthält, ausgeschlossen sein, dass $\mathfrak{G}$ eine Congruenzgruppe ist. Eine Gruppe $\mathfrak{G}$ linearer homogener Substitutionen in $n$ Variablen heisst reducibel, wenn man $\nu < n$ lineare homogene Combinationen der Variablen mit constanten Coefficienten finden kann, die durch jede Substitution von $\mathfrak{G}$ nur unter einander transformirt werden. Anders ausgedrückt, eine Gruppe $\mathfrak{G}$ linearer homogener Substitutionen in $n$ Variablen heisst reducibel, wenn man eine constante Matrix $T$ von nicht verschwindender Determinante angeben kann, dass alle Matrices der zu $\mathfrak{G}$ ähnlichen Gruppe $\bar{\mathfrak{G}} = T\mathfrak{G}T^{-1}$ gleichzeitig von der besonderen Form:

$$\begin{matrix} \mathfrak{G}_{11} & 0 \\ \mathfrak{G}_{21} & \mathfrak{G}_{22} \end{matrix}$$

werden und $\mathfrak{G}_{11}$ einen Inbegriff von Matrices mit $\nu < n$ Zeilen und Colonnen bedeutet. Die angewandte symbolische Bezeichnung sagt aus: man kann die Gruppe $\mathfrak{G}$ durch Einführung neuer Variablen in die Gruppe $\bar{\mathfrak{G}}$ transformiren, so dass jede Substitution der transformirten Gruppe $\bar{\mathfrak{G}}$ von der besonderen Form

* Presented to the Society (Chicago) April 22, 1905. Received for publication December 26, 1904.

† Für Gruppen sowie Gesamtheiten von Matrices benütze ich deutsche, für eine einzelne Matrix lateinische Buchstaben.



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$$y_i = \sum_{j=1}^{\nu} g_{ij} y_j', \qquad y_k = \sum_{j=1}^{n} g_{kj} y_j' \qquad (i=1, \cdots, \nu;\ k=\nu+1, \cdots, n)$$

wird.

Eine jede reducible Gruppe $\mathfrak{G}$ linearer homogener Substitutionen lässt sich durch Einführung linearer homogener Functionen der Variablen mit constanten Coefficienten stets derartig in eine ähnliche Gruppe $\mathfrak{A} = R\mathfrak{G}R^{-1}$ transformiren, wobei $R$ eine constante Matrix ist, dass gleichzeitig alle Matrices der Substitutionen der Gruppe $\mathfrak{A}$, oder, wie ich statt dessen kurz sage, die Gruppe $\mathfrak{A}$ von der besonderen Form:

$$\begin{matrix} \mathfrak{a}_{11} & 0 & 0 & 0 & \cdots & 0 \\ \mathfrak{a}_{21} & \mathfrak{a}_{22} & 0 & 0 & \cdots & 0 \\ \mathfrak{a}_{31} & \mathfrak{a}_{32} & \mathfrak{a}_{33} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \mathfrak{a}_{\lambda 1} & \mathfrak{a}_{\lambda 2} & \mathfrak{a}_{\lambda 3} & \mathfrak{a}_{\lambda 4} & \cdots & \mathfrak{a}_{\lambda\lambda} \end{matrix}$$

wird. Sind die $\lambda$ mit $\mathfrak{G}$ isomorphen Gruppen linearer homogener Substitutionen, die durch die Matrices $\mathfrak{a}_{11}, \mathfrak{a}_{22}, \cdots, \mathfrak{a}_{\lambda\lambda}$ bestimmt werden, was stets zu erreichen und worauf für das Folgende besonderes Gewicht zu legen ist, irreducible Gruppen, so sagen wir: *$\mathfrak{G}$ ist unter Hervorhebung seiner irreduciblen Bestandteile oder Teilgruppen in eine ähnliche Gruppe transformirt.* Die irreduciblen Gruppen $\mathfrak{a}_{11}, \mathfrak{a}_{22}, \cdots, \mathfrak{a}_{\lambda\lambda}$ heissen *die irreduciblen Bestandteile oder Teilgruppen von $\mathfrak{G}$, die sich bei der Darstellung von $\mathfrak{G}$ in der Form $\mathfrak{A}$ ergeben.*

Für die irreduciblen Bestandteile von $\mathfrak{G}$ habe ich den auch für die Theorie der linearen homogenen Differentialgleichungen wichtigen Satz hergeleitet, dass, wie auch immer $\mathfrak{G}$ unter Hervorhebung seiner irreduciblen Bestandteile in eine ähnliche Gruppe transformirt wird, die irreduciblen Bestandteile von $\mathfrak{G}$, falls man ähnliche Gruppen als nicht verschieden ansieht, bis auf die Reihenfolge eindeutig bestimmt sind.*

Unter den Gruppen linearer homogener Substitutionen verdienen diejenigen besondere Aufmerksamkeit, die ich nach einem brieflichen Vorschlage von Herrn W. BURNSIDE † als *vollständig reducibel* bezeichne. Eine Gruppe $\mathfrak{G}$ linearer homogener Substitutionen heisst vollständig reducibel, wenn man wenigstens eine

* A LOEWY, *Über die Reducibilität der Gruppen linearer homogener Substitutionen*, Transactions of the American Mathematical Society, vol. 4 (1903), p. 44. Eine Anwendung dieses Satzes auf höhere complexe Zahlen giebt Herr EPSTEEN in seiner Arbeit: *Semireducible hypercomplex number systems*, Transactions, vol. 4, p. 444.

† Inzwischen hat Herr W. BURNSIDE diese Bezeichnung auch in seinen Arbeiten, *On the reduction of a group of homogeneous linear substitutions of finite order*, Acta mathematica, vol. 28, 369–387, und *On the representation of a group of finite order as an irreducible group of linear substitutions and the direct establishment of the relations between the group-characteristics*, Proceedings of the London Mathematical Society (2), vol. 1 (1903), 117–123.

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constante Matrix $R$ von nicht verschwindender Determinante finden kann, dass die zu $\mathfrak{G}$ ähnliche Gruppe $R\mathfrak{G}R^{-1}$ von der besonderen Form

$$\{\mathfrak{a}_{11}, \mathfrak{a}_{22}, \cdots, \mathfrak{a}_{\lambda\lambda}\}: \quad \begin{matrix} \mathfrak{a}_{11} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \mathfrak{a}_{22} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \mathfrak{a}_{33} & 0 & \cdots & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & \cdots & \mathfrak{a}_{\lambda\lambda} \end{matrix}$$

wird, also, abgesehen von der Diagonale, die nur irreducible Bestandteile enthält, lauter Nullmatrices stehen. Eine vollständig reducible Gruppe ist also auf folgende Art charakterisirt: Man kann durch Einführung neuer Variablen stets erreichen, dass nach der Transformation die Variablen in Systeme zerfallen, die Variablen eines jeden Systemes sich nur unter einander transformiren und die Gruppen linearer homogener Substitutionen, die jedes Variablensystem für sich allein definirt, irreducibel sind. *In der Bezeichnung "vollständig reducibel" sollen auch die irreduciblen Gruppen eingeschlossen sein; bei den irreduciblen Gruppen ist in dem zuletzt hingeschriebenen Schema* $\lambda = 1$.

Bei einer grossen Klasse von Gruppen folgt aus ihrem Charakter unmittelbar ihre vollständige Reducibilität. Eine derartige Klasse von Gruppen habe ich in den Transactions* mitgeteilt, und Herr L. E. DICKSON† hat hieran anschliessend das von mir erhaltene Resultat in weitgehendster Weise verallgemeinert. Auf Grund der Dicksonschen Arbeit kann mit Hülfe des Begriffes der vollständigen Reducibilität, wenn wir ihn noch auf Gruppen linearer homogener Substitutionen mit Coefficienten aus einem Körper (vgl. S. 509) erweitern, der folgende Satz ausgesprochen werden: Ist $F$ irgend ein Rationalitätsbereich, der kein endlicher Körper ist, und $F(\rho_0)$ der aus $F$ durch Adjunction einer Wurzel $\rho_0$ einer Normalgleichung r-ten Grades im Rationalitätsbereiche $F$ hervorgehende Rationalitätsbereich, so ist jede Gruppe linearer homogener Substitutionen mit Coefficienten aus $F$, die in $F$ irreducibel ist, in dem weiteren Rationalitätsbereiche $F(\rho_0)$ vollständig reducibel und zwar kann sie, wenn sie nicht auch in $F(\rho_0)$ irreducibel ist, in eine Gruppe $\{\mathfrak{G}_{11}, \mathfrak{G}'_{11}, \mathfrak{G}''_{11}, \cdots, \mathfrak{G}_{11}^{(r-1)}\}$ linear transformirt werden. $\mathfrak{G}_{11}, \mathfrak{G}'_{11}, \cdots, \mathfrak{G}_{11}^{(r-1)}$ bedeuten in $F(\rho_0)$ irreducible Gruppen linearer homogener Substitutionen, deren Coefficienten nicht ausnahmslos $F$ angehören. Die Substitutionen von $\mathfrak{G}_{11}^{(s)}(s = 1, 2, \cdots, r-1)$ werden erhalten, indem man in den Coefficienten jeder Matrix der Substitutionen der Gruppe $\mathfrak{G}_{11}$ die Wurzel $\rho_0$ durch $\rho_s$ ersetzt; hierbei ist angenommen, dass die Normalgleichung r-ten Grades in $F$, die $\rho_0$ zur Wurzel hat, ausser $\rho_0$ noch die Wurzeln $\rho_1, \rho_2, \cdots, \rho_{r-1}$ besitzt.

* A. LOEWY, *Über die Reducibilität der reellen Gruppen linearer homogener Substitutionen*, Transactions, vol. 4 (1903), 171–177.

† L. E. DICKSON, *On the reducibility of linear groups*, Transactions, vol. 4 (1903), 434–436.

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Auf vollständig reducible Gruppen linearer homogener Substitutionen ist auch Herr I. SCHUR* in seiner ausgezeichneten Dissertation geführt worden. Er hat sich dort das Problem gestellt und auch völlig erledigt, das wir folgendermassen formuliren: Alle Gruppen $T(\mathfrak{G})$ linearer homogener Substitutionen zu finden, die der allgemeinen Gruppe $\mathfrak{G}$ linearer homogener Substitutionen in $m$ Variablen isomorph sind, und bei denen die Substitutionscoefficienten ganze rationale homogene Functionen $n$-ter Dimension der $m^2$ Coefficienten der Gruppe $\mathfrak{G}$ sind. Von Herrn SCHUR's Resultaten führen wir an: Jede Gruppe $T(\mathfrak{G})$ ist vollständig reducibel (Vgl. S. 71). Die Gesamtheit aller irreduciblen Bestandteile der Gruppen $T(\mathfrak{G})$ ist, wenn man ähnliche Gruppen als nicht verschieden ansieht, endlich und zwar gleich der Anzahl $k$ der Zerlegungen der ganzen Zahl $n$ in höchstens $m$ gleiche oder verschiedene positive Summanden. Zwei Gruppen $T(\mathfrak{G})$ sind dann und nur dann ähnlich, wenn stets in zwei entsprechenden Matrices der zwei isomorphen Gruppen die Summen der Glieder ihrer Hauptdiagonalen einander gleich sind. Die Ergebnisse des § 1 und § 2 meiner Arbeit, *Zur Gruppentheorie mit Anwendungen auf die Theorie der linearen homogenen Differentialgleichungen*, Transactions, Bd. 5, S. 61, sind nur ganz specielle Fälle der Schurschen Theorie. Aus SCHUR's Untersuchungen ergiebt sich auch, dass man auf S. 65 in Formel (6) meines Aufsatzes $\Pi_r A$ noch weiter linear derartig transformiren kann, dass $A_{21} = 0$ wird; dies ist mir bisher nicht mit so einfachen Hülfsmitteln, wie ich sie am angeführten Orte anwandte, zu beweisen gelungen.

Im § 1 der vorliegenden Arbeit beweise ich über vollständig reducible Gruppen das folgende Theorem: *Jede Gruppe linearer homogener Substitutionen, die eine definite Hermitesche Form in sich transformirt, ist vollständig reducibel.*

Nach einen von Herrn MOORE und mir † in der gleichen Woche veröffentlichten Satze gehört zu jeder endlichen Gruppe linearer homogener Substitutionen eine invariante definite Hermitesche Form. Aus dem mitgeteilten Theoreme folgt daher der von Herrn FROBENIUS ‡ im Verlaufe seiner tiefgehenden Untersuchungen über die endlichen Gruppen linearer homogener Substitutionen erhaltene und auch hiervon unabhängig von Herrn MASCHKE § gefundene Fun-

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* I. SCHUR, *Über eine Klasse von Matrizen, die sich einer gegebenen Matrix zuordnen lassen*, Berlin, 1901.

† A. LOEWY, Comptes Rendus, vol. 123 (1896), 168–171; E. H. MOORE, Mathematische Annalen, Bd. 50, S. 213–219.

‡ G. FROBENIUS, *Über die Darstellung der endlichen Gruppen durch lineare Substitutionen*, I und II, Sitzungsberichte der Preussischen Akademie der Wissenschaften, 1897 u. 1899. Vgl. Jahrgang 1899, S. 483.

§ H. MASCHKE, *Beweis des Satzes, dass diejenigen endlichen linearen Substitutionsgruppen, in welchen einige durchgehends verschwindende Coefficienten auftreten, intransitiv sind*, Mathematische Annalen, Band 52, S. 363. Einen neuen Beweis hat W. BURNSIDE in der S. 002 zuerst citirten Arbeit gegeben.

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damentalsatz: *Jede endliche Gruppe linearer homogenen Substitutionen ist vollständig reducibel.*

Nicht jede reducible Gruppe linearer homogener Substitutionen ist vollständig reducibel. Ich leite in der vorliegenden Arbeit im § 2 und den folgenden Paragraphen — die Lectüre des § 1 ist für das Verständnis der folgenden Paragraphen entbehrlich — den für die Theorie der unendlichen Gruppen linearer homogener Substitutionen, wie mir scheint, fundamentalen Satz her: *Zu jeder Gruppe $\mathfrak{G}$ linearer homogener Substitutionen gehören $\mu$ vollständig reducible Gruppen linearer homogener Substitutionen; diese sind ebenso wie die irreduciblen Bestandteile von $\mathfrak{G}$, wenn man ähnliche Gruppen als nicht verschieden ansieht, völlig eindeutig bestimmt. Die $\mu$ vollständig reduciblen, zu $\mathfrak{G}$ gehörigen Gruppen sind, was für die irreduciblen Bestandteile von $\mathfrak{G}$ nicht zutrifft, auch ihrer Reihenfolge nach eindeutig festgelegt. Man kann also von der ersten, zweiten, u. s. w. bis $\mu$-ten vollständig reduciblen zu $\mathfrak{G}$ gehörigen Gruppe sprechen. Die Gesamtzahl der Variablen der $\mu$ Gruppen ist gleich der Variablenzahl von $\mathfrak{G}$. Ist $\mu = 1$, so ist die Gruppe $\mathfrak{G}$ vollständig reducibel.* Zu den $\mu$ vollständig reduciblen Gruppen, die zu $\mathfrak{G}$ gehören, gelangt man auf folgende Weise: Jede Gruppe $\mathfrak{G}$ linearer homogener Substitutionen kann durch eine constante Matrix $P$ von nicht verschwindender Determinante in eine ähnliche Gruppe $\mathfrak{A}^* = P\mathfrak{G}P^{-1}$ der besonderen Form:

$$(\mathfrak{A}^*): \quad \begin{matrix} \mathfrak{A}^*_{11} & 0 & 0 & 0 & \cdots & 0 \\ \mathfrak{A}^*_{21} & \mathfrak{A}^*_{22} & 0 & 0 & \cdots & 0 \\ \mathfrak{A}^*_{31} & \mathfrak{A}^*_{32} & \mathfrak{A}^*_{33} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \mathfrak{A}^*_{\mu 1} & \mathfrak{A}^*_{\mu 2} & \mathfrak{A}^*_{\mu 3} & \cdots & \cdots & \mathfrak{A}^*_{\mu\mu} \end{matrix}$$

transformirt werden, so dass die Gesamtheit von Matrices $\mathfrak{A}^*_{11}$, bez. $\mathfrak{A}^*_{22}, \mathfrak{A}^*_{33}, \cdots, \mathfrak{A}^*_{\mu\mu}$ vollständig reducible Gruppen definiren. $\mathfrak{A}^*_{11}, \mathfrak{A}^*_{22}, \cdots, \mathfrak{A}^*_{\mu\mu}$ lege ich als gewisse Maximalgruppen ihrer Art fest. (Vgl. § 5.) Eine Transformation von $\mathfrak{G}$ in eine ähnliche Gruppe, wie sie eben geschildert wurde, bezeichne ich als *eine Transformation von $\mathfrak{G}$ in eine ähnliche Gruppe unter Hervorhebung aller auf einander folgender grösster vollständig reducibler Gruppen.*

Eine Darstellung der Gruppe $\mathfrak{G}$ in der Form $\mathfrak{A}^*$, also unter Hervorhebung aller auf einander folgender grösster vollständig reducibler Gruppen, ist durchaus nicht eindeutig. Es gilt nun folgender Satz: *Wie auch immer eine Gruppe $\mathfrak{G}$ linearer homogener Substitutionen unter Hervorhebung aller auf einander folgender grösster vollständig reducibler Gruppen in eine ähnliche transformirt wird, so sind die vollständig reduciblen Gruppen, die sich bei einer derartigen Darstellung von $\mathfrak{G}$ ergeben, den vollständig reduciblen Gruppen, die sich*

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*bei irgend einer anderen Darstellung von $\mathfrak{G}$ unter Hervorhebung aller auf einander folgender grösster vollständig reducibler Gruppen ergeben,* DER REIHE NACH *zugeordnet, so dass zwei zugeordnete vollständig reducible Gruppen gleichviele Variablen haben und ähnliche Gruppen sind.*

Im § 6 setzen wir voraus, dass eine Gruppe $\mathfrak{G}$ linearer homogener Substitutionen vorgelegt sei, bei der alle Coefficienten sämtlicher Substitutionen Zahlen aus einem unendlichen Zahlkörper oder Rationalitätsbereiche $\Omega$ sind. Den Begriff der Reducibilität einer Gruppe linearer homogener Substitutionen mit Coefficienten aus $\Omega$ bezüglich des Körpers $\Omega$ habe ich im § 4 meiner Arbeit, *Über die Reducibilität der Gruppen linearer homogener Substitutionen*, Transactions, Bd. 4, S. 58 ff., auseinandergesetzt. Die lineare homogene Substitutionsgruppe $\mathfrak{G}$ in $n$ Variablen, deren Substitutionscoefficienten ausnahmslos dem Körper $\Omega$ angehören, ist bezüglich $\Omega$ reducibel, wenn man $m < n$ lineare homogene Functionen der Variablen mit constanten Coefficienten finden kann, welche durch eine jede Substitution der Gruppe nur unter sich transformirt werden und dabei ausschliesslich Transformationen mit Coefficienten aus $\Omega$ erleiden. Eine Gruppe $\mathfrak{G}$ linearer homogener Substitutionen mit Coefficienten aus $\Omega$ nenne ich *bezüglich des Körpers $\Omega$ vollständig reducibel*, wenn man wenigstens eine Matrix $R$ von nicht verschwindender Determinante finden kann, dass die zu $\mathfrak{G}$ ähnliche Gruppe $R\mathfrak{G}R^{-1}$ von der besonderen Form $\{\mathfrak{a}_{11}, \mathfrak{a}_{22}, \cdots, \mathfrak{a}_{\lambda\lambda}\}$ wird, $\mathfrak{a}_{11}, \mathfrak{a}_{22}, \cdots, \mathfrak{a}_{\lambda\lambda}$ bezüglich des Körpers $\Omega$ irreducible Gruppen linearer homogener Substitutionen mit Coefficienten aus $\Omega$ bedeuten und entsprechend dem benützten Symbole rechts wie links von jedem bezüglich $\Omega$ irreduciblen Bestandteile $\mathfrak{a}_{ii}(i = 1, 2, \cdots, \lambda)$ Nullmatrices stehen. Eine Gruppe linearer homogener Substitutionen mit Coefficienten aus $\Omega$, die bezüglich $\Omega$ vollständig reducibel ist, ist also auf folgende Weise charakterisirt: Man kann durch Einführung neuer Variablen stets erreichen, dass nach der Transformation die Variablen eines jeden Systems sich nur unter einander transformiren, die Gruppen linearer homogener Substitutionen, die jedes Variablensystem für sich allein definirt, nur Coefficienten aus $\Omega$ haben und bezüglich $\Omega$ irreducible Gruppen sind. In der Bezeichnung "bezüglich $\Omega$ vollständig reducibel" sollen auch die bezüglich $\Omega$ irreduciblen Gruppen mit Coefficienten aus $\Omega$ eingeschlossen sein. Im § 6 wird die Erweiterung der im § 2–§ 5 gegebenen Untersuchungen auf Gruppen mit Coefficienten aus $\Omega$ behandelt.

## § 1. *Gruppen, die eine definite Hermitesche Form in sich transformiren.*

Für die in der Überschrift genannten Gruppen soll der auf Seite 507 der Einleitung angegebene Satz bewiesen werden.* Nach Voraussetzung transformirt

* Inzwischen hat Herr I. SCHUR einen etwas einfacheren Beweis des Herrn FROBENIUS veröffentlicht. SCHUR, Sitzungsberichte Preuss. Akad. der Wiss., April 6, 1905, S. 416. Nachschrift vom August 1905.

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die Gruppe $\mathfrak{G}$ linearer homogener Substitutionen eine definite Hermitesche Form $H_1$ in sich. Ist die Gruppe $\mathfrak{G}$ eine irreducible Gruppe, so gehört $\mathfrak{G}$ zur Klasse der vollständig reduciblen Gruppen, und der Satz ist bewiesen. Sei daher $\mathfrak{G}$ eine reducible Gruppe. In diesem Falle giebt es eine constante Matrix $T$ von nicht verschwindender Determinante, dass die Matrices aller Substitutionen der transformirten Gruppe $\overline{\mathfrak{G}} = T\mathfrak{G}T^{-1}$ gleichzeitig von der Form:

$$(\overline{\mathfrak{G}}): \quad \begin{matrix} \overline{\mathfrak{G}}_{11} & 0 \\ \overline{\mathfrak{G}}_{21} & \overline{\mathfrak{G}}_{22} \end{matrix}$$

werden; hierbei bedeuten $\overline{\mathfrak{G}}_{11}$, $\overline{\mathfrak{G}}_{21}$, $\overline{\mathfrak{G}}_{22}$ Matrices mit $\nu$ Zeilen und $\nu$ Colonnen, bez. $n-\nu$ Zeilen und $\nu$ Colonnen, bez. $n-\nu$ Zeilen und $n-\nu$ Colonnen. Da die Substitutionen von $\mathfrak{G}$ nach Voraussetzung eine Hermitesche Form $H_1$ invariant lassen, existirt auch für die zu $\mathfrak{G}$ ähnliche Gruppe $\overline{\mathfrak{G}}$ eine invariante Hermitesche Form. Sie sei mit $H$ bezeichnet. Man findet sie aus der Hermiteschen Form $H_1$ durch Einführung der neuen Variablen der Gruppe $\overline{\mathfrak{G}}$ in $H_1$. Ist $\overline{G}$ die Matrix einer beliebigen Transformation aus $\overline{\mathfrak{G}}$ und bedeutet stets *der obere Index* 0 *das Ersetzen aller Terme durch die conjugirt imaginären*, so ist nach der Frobeniusschen* Symbolik:

$(\overline{G}^0)'H\overline{G} = H$, und es wird: $H_1 = (T^0)'HT$. In den obigen Formeln bedeuten $(T^0)'$ bez. $(\overline{G}^0)'$ die zu $T^0$ bez. $\overline{G}^0$ transponirten oder conjugirten Matrices. Da $H_1$ eine definite Hermitesche Form ist, folgt, dass auch die aus $H_1$ durch Transformation hervorgehende Hermitesche Form $H$, die durch die Gruppe $\overline{\mathfrak{G}}$ in sich transformirt wird, definit sein muss. $H$ möge lauten:

$$H = \sum_{i=1}^{i=n} \sum_{k=1}^{k=n} \alpha_{ik} z_i^0 z_k \qquad (\alpha_{ik}^0 = \alpha_{ki}).$$

Wir führen auf $H$ die Substitution:

$$(1) \qquad \begin{aligned} z_i &= y_i && (i=1, 2, \cdots, \nu), \\ z_{\nu+k} &= \lambda_{\nu+k,1} y_1 + \lambda_{\nu+k,2} y_2 + \cdots + \lambda_{\nu+k,\nu} y_\nu + y_{\nu+k} && (k=1, 2, \cdots, n-\nu), \end{aligned}$$

aus. Wir finden dann als Coefficienten von $y_i y_{\nu+k}^0$ $(i \leqq \nu,\ k \leqq n-\nu)$:

$$\alpha_{\nu+k,i} + \alpha_{\nu+k,\nu+1}\lambda_{\nu+1,i} + \alpha_{\nu+k,\nu+2}\lambda_{\nu+2,i} + \cdots + \alpha_{\nu+k,n}\lambda_{n,i}.$$

In der verwandten Substitution (1) bestimmen wir die Grössen $\lambda_{\nu+k,i}$ derartig, dass der soeben hingeschriebene Coefficient von $y_i y_{\nu+k}^0$ Null wird. Die für die $(n-\nu)\nu$ Grössen $\lambda_{ik}$ getroffene Wahl ist auch stets möglich; denn bei einer definiten Hermiteschen Form:

* G. FROBENIUS, *Über lineare Substitutionen und bilineare Formen*, Journal für Mathematik, Bd. 84, S. 1; ferner G. FROBENIUS, *Über die principale Transformation der Thetafunctionen mehrerer Variablen*, *ibid.*, Bd. 95, S. 264. Vgl. auch A. LOEWY, *Über bilineare Formen mit conjugirt imaginären Variablen*, Nova Acta der Kai-Leopold. Carol. Deutschen Akademie der Naturforscher, Bd. 71 (1898), No. 8.

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$$\sum_{i=1}^{i=n}\sum_{k=1}^{k=n} \alpha_{ik} z_i^0 z_k,$$

müssen alle Hauptunterdeterminanten von Null verschieden sein, mithin ist auch die Determinante:

$$\begin{vmatrix} \alpha_{\nu+1,\,\nu+1} & \cdots & \alpha_{\nu+1,\,n} \\ \alpha_{\nu+2,\,\nu+1} & \cdots & \alpha_{\nu+2,\,n} \\ \cdot & \cdot & \cdot \\ \alpha_{n,\,\nu+1} & \cdots & \alpha_{n,\,n} \end{vmatrix}$$

ungleich Null.

Wählt man die Substitutionscoefficienten für die Substitution (1), wie angegeben, so geht $H$ in die neue Hermitesche Form:

$$(2) \qquad \sum_{i=1}^{i=\nu}\sum_{k=1}^{k=\nu} \beta_{ik} y_i^0 y_k + \sum_{i=\nu+1}^{i=n}\sum_{k=\nu+1}^{k=n} \beta_{ik} y_i^0 y_k$$

über. Da die neue Hermitesche Form (2) aus der definiten Hermiteschen Form $H$ durch Transformation hervorgeht, so ist sie ebenso wie $H$ definit.

Durch Ausführung der Substitution (1) wird die rechts oben bei $(\overline{\mathfrak{G}})$ stehende Nullmatrix nicht zerstört, und $\overline{\mathfrak{G}}$ geht in die ähnliche Gruppe $\overline{\mathfrak{G}}$ der Form:

$$(\overline{\mathfrak{G}}): \qquad \begin{matrix} \overline{\mathfrak{G}}_{11} & 0 \\ \overline{\mathfrak{G}}_{21} & \overline{\mathfrak{G}}_{22} \end{matrix}$$

über; die Gruppe $\overline{\mathfrak{G}}$ transformirt die Hermitesche Form (2) in sich.

Durch lineare homogene Transformation der Variablen $y_1, y_2, \cdots, y_\nu$ für sich und gesonderte lineare homogene Transformation von $y_{\nu+1}, y_{\nu+2}, \cdots, y_n$ kann wegen des definiten Charakters der Hermiteschen Form (2) bewirkt werden, dass:

$$\sum_{i=1}^{i=\nu}\sum_{k=1}^{k=\nu} \beta_{ik} y_i^0 y_k \qquad \text{in} \qquad x_1 x_1^0 + x_2 x_2^0 + \cdots + x_\nu x_\nu^0$$

und

$$\sum_{i=\nu+1}^{i=n}\sum_{k=\nu+1}^{k=n} \beta_{ik} y_i^0 y_k \qquad \text{in} \qquad x_{\nu+1} x_{\nu+1}^0 + x_{\nu+2} x_{\nu+2}^0 + \cdots + x_n x_n^0$$

übergehen, wir also die Hermitesche Form $\sum_{i=1}^{i=n} x_i x_i^0$ erhalten.

Durch Einführung der neuen Variablen geht $\overline{\mathfrak{G}}$ in eine neue, zu $\overline{\mathfrak{G}}$ und daher auch zu $\mathfrak{G}$ ähnliche in den Variablen $x$ geschriebene Gruppe über; sie sei mit $\overline{\overline{\mathfrak{G}}}$ bezeichnet. Durch Einführung der neuen Variablen wird die Nullmatrix, die bei $\overline{\mathfrak{G}}$ rechts oben auftrat, nicht zerstört; daher wird $\overline{\overline{\mathfrak{G}}}$ von der Form:

$$(\overline{\overline{\mathfrak{G}}}): \qquad \begin{matrix} \overline{\overline{\mathfrak{G}}}_{11} & 0 \\ \overline{\overline{\mathfrak{G}}}_{21} & \overline{\overline{\mathfrak{G}}}_{22}. \end{matrix}$$

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Wir behaupten, dass jetzt auch $\bar{\bar{\mathfrak{G}}}_{21}$ eine Nullmatrix ist. Die Gruppe $\bar{\bar{\mathfrak{G}}}$ führt die Hermitesche Form

$$E = \sum_{i=1}^{i=n} x_i x_i^0$$

in sich über. Daher muss, wenn $\bar{\bar{G}}$ eine beliebige Substitution aus $\bar{\bar{\mathfrak{G}}}$ ist, nach der Frobenius'schen Symbolik:

$$(\bar{\bar{G}}^0)' \bar{\bar{G}} = E$$

oder folglich

$$\bar{\bar{G}} = ((\bar{\bar{G}}^0)')^{-1}$$

sein. Diese Gleichung aber besagt, dass jeder Coefficient der Substitution $\bar{\bar{G}}$ gleich dem conjugirt imaginären Werte der ihm zugehörigen Unterdeterminante $n-1$ ten Grades dividirt durch den conjugirt imaginären Wert der Substitutionsdeterminante ist. Die Unterdeterminanten, die aber zu Coefficienten der ersten $\nu$ Colonnen und letzten $n-\nu$ Zeilen gehören, sind Null, da jede derselben die $\nu(n-\nu)$ Nullen der rechts oben stehenden Nullmatrix enthält; folglich ist $\bar{\bar{G}}_{21}$ Null zu setzen.*

Sind $\bar{\bar{\mathfrak{G}}}_{11}$ und $\bar{\bar{\mathfrak{G}}}_{22}$ irreducible Gruppen linearer homogener Substitutionen, so ist unser Satz bewiesen. Sind $\bar{\bar{\mathfrak{G}}}_{11}$ und $\bar{\bar{\mathfrak{G}}}_{22}$ oder ist eine dieser Gruppen reducibel, so führt jede von ihnen eine definite Hermitsche Form, nämlich

$$x_1 x_1^0 + x_2 x_2^0 + \cdots + x_\nu x_\nu^0$$

bez.

$$x_{\nu+1} x_{\nu+1}^0 + x_{\nu+2} x_{\nu+2}^0 + \cdots + x_n x_n^0,$$

in sich über. Die Teilgruppen sind folglich von demselben Charakter wie die ursprünglichen. Eine eventuelle wiederholte Anwendung des gleichen Verfahrens wie oben erweist daher sicher die Richtigkeit unseres Satzes, indem sie schliesslich zu irreduciblen Teilgruppen mit sich gesondert transformirenden Variablen führt.

Im Voraufgehenden habe ich den gleichen Gedankengang benützt, der Herrn MASCHKE in der oben citirten Arbeit dazu führt, die vollständige Reducibilität der endlichen Gruppen linearer homogener Substitutionen zu zeigen. Die von Herrn MASCHKE verwandten Schlüsse musste ich jedoch etwas modificiren, da seine Darstellung ein kleines Versehen enthält, auf dass ich aufmerksam zu machen nicht unterlassen kann. Herr MASCHKE will nachweisen, dass man *jede* Hermitesche Form:

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* Vgl. hierzu das von Herrn STICKELBERGER im § 2 der Programmabhandlung des Züricher Polytechnikums, *Ueber reelle orthogonale Substitutionen* (1877), bewiesene Lemma: "Wenn in einem (nicht notwendig reellen) orthogonalen System alle Coefficienten verschwinden, welche $m$ Zeilen mit $n-m$ Colonnen gemeinsam haben, so verschwinden auch alle diejenigen, welche die übrigen $n-m$ Zeilen mit den übrigen $m$ Colonnen gemeinsam haben."

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$$\sum_{i=1}^{i=n}\sum_{k=1}^{k=n} \alpha_{ik} z_i^0 z_k,$$

durch eine lineare homogene Substitution von nicht verschwindender Determinante:

$$z_i = \sum_{k=1}^{k=\nu} \lambda_{ik} y_k \qquad (i=1, 2, \cdots, \nu),$$

$$z_i = y_i + \sum_{k=1}^{k=\nu} \lambda_{ik} y_k \qquad (i=\nu+1, \nu+2, \cdots, n),$$

in die Form:

$$\sum_{i=1}^{i=\nu}\sum_{k=1}^{k=\nu} \beta_{ik} y_i^0 y_k + \sum_{i=\nu+1}^{i=n}\sum_{k=\nu+1}^{k=n} \beta_{ik} y_i^0 y_k$$

transformiren kann. Dies trifft nicht zu, wie das einfache Beispiel $z_1 z_1^0 + z_1 z_2^0 + z_2 z_1^0$ lehrt; durch $z_1 = \lambda_{11} y_1$, $z_2 = y_2 + \lambda_{21} y_1$ kann die Form nie so transformirt werden, dass der Coefficient von $y_1 y_2^0$, der $\lambda_{11}$ lautet, verschwindet. Thatsächlich liegt in der Festsetzung von Herrn MASCHKE auf Seite 368, Zeile 8: "Sei dies die Determinante $|\lambda_{ik}|$ für $i, k = 1, 2, \cdots, r$"* eine Bestimmung, die man nicht nach Willkür machen kann.

## § 2.

Die Gruppe $\mathfrak{G}$ linearer homogener Substitutionen in $n$ Variablen sei reducibel und durch die Matrix $S$ von nicht verschwindender Determinante in die ähnliche Gruppe $\mathfrak{A} = S\mathfrak{G}S^{-1}$ transformirt, so dass diese die Form:

$$(\mathfrak{A}): \quad \begin{matrix} \mathfrak{A}_{11} & 0 \\ \mathfrak{A}_{21} & \mathfrak{A}_{22} \end{matrix}$$

annimmt; dabei bedeute $\mathfrak{A}_{11}$ einen Inbegriff von Matrices mit $\nu < n$ Zeilen und Colonnen, eine weitere Voraussetzung wird über $\mathfrak{A}_{11}$ in diesem § nicht gemacht. $\mathfrak{A}_{21}$ bez. $\mathfrak{A}_{22}$ bezeichnen einen Inbegriff von Matrices mit $n - \nu$ Zeilen und $\nu$ bez. $n - \nu$ Colonnen. $\mathfrak{G}$ möge ferner in die ähnliche Gruppe $\mathfrak{B}$ der Form:

$$(\mathfrak{B}): \quad \begin{matrix} \mathfrak{b}_{11} & 0 \\ \mathfrak{b}'_{21} & \mathfrak{b}'_{22} \end{matrix}$$

transformirt sein; hierbei bedeute $\mathfrak{b}_{11}$ eine Gesamtheit von Matrices mit $g_1 < n$ Zeilen und Colonnen und demnach $\mathfrak{b}'_{21}$ bez. $\mathfrak{b}'_{22}$ einen Inbegriff von Matrices mit $n - g_1$ Zeilen und $g_1$ bez. $n - g_1$ Colonnen. Über $\mathfrak{b}_{11}$ sei die für das Folgende wesentliche Voraussetzung gemacht, dass $\mathfrak{b}_{11}$ *eine irreducible Gruppe linearer homogener Substitutionen in $g_1$ Variablen* definire.

Die Substitutionen von $\mathfrak{A}$ mögen

* Herr MASCKKE hat statt meines $\nu$ den Buchstaben $r$.

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$$(1) \qquad y_i = \sum_{s=1}^{s=n} a_{is}^{(t)} y'_s \qquad (i=1, 2, \cdots, n),$$

lauten; $t$ habe hierbei die Werte $1, 2, 3, \cdots$ und charakterisire die einzelnen Substitutionen der Gruppe $\mathfrak{A}$.*

Da die Gruppe $\mathfrak{A}$ die Form $(\mathfrak{A})$ hat, werden sich in (1) die ersten $\nu$ Variablen in der Form

$$(2) \qquad y_i = \sum_{k=1}^{\nu} a_{ik}^{(t)} y'_k \qquad (i=1, \cdots, \nu)$$

transformiren; es wird also für $i \leqq \nu$, $k > \nu$ stets $a_{ik}^{(t)} = 0$ sein.

Die Substitutionen von $\mathfrak{B}$ mögen:

$$(3) \qquad z_i = \sum_{s=1}^{s=n} b_{is}^{(t)} z'_s \qquad (i=1, 2, \cdots, n)$$

lauten; der obere Index $t$, der die Werte $1, 2, 3, \cdots$ annehmen möge, soll dabei die einzelnen Substitutionen von $\mathfrak{B}$ angeben.

Die ersten $g_1$ Variablen, welche die Substitutionen der nach Voraussetzung irreduciblen Gruppe $\mathfrak{b}_{11}$ bestimmen, werden Substitutionen der Form:

$$(4) \qquad z_i = \sum_{k=1}^{g_1} b_{ik}^{(t)} z'_k \qquad (i=1, \cdots, g_1;\ t=1, 2, \cdots;\ g_1 < n)$$

erleiden.

Die Gruppe $\mathfrak{G}$ ist den Gruppen $\mathfrak{A}$ und $\mathfrak{B}$ ähnlich; daher werden die zwei Gruppen $\mathfrak{A}$ and $\mathfrak{B}$ unter einander ähnlich sein, d. h. es wird eine Transformation:

$$(5) \qquad z_i = q_{i1} y_1 + q_{i2} y_2 + \cdots + q_{in} y_n \qquad (i=1, 2, \cdots, n),$$

von nicht verschwindender Determinante $|q_{ik}|$ existiren, die mit der zu ihr cogredienten Substitution:

$$(5') \qquad z'_i = \sum_{s=1}^{s=n} q_{is} y'_s \qquad (i=1, 2, \cdots, n)$$

alle Substitutionen der Gruppe $\mathfrak{A}$ in die entsprechenden von $\mathfrak{B}$ überführt. Es wird also die symbolische Gleichung $\mathfrak{B} = Q\mathfrak{A}Q^{-1}$ statthaben. Die sich entsprechenden Substitutionen von $\mathfrak{A}$ und $\mathfrak{B}$ sind durch gleiche Werte des oberen Index $t$ charakterisirt.

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* Eine solche Charakterisirung der einzelnen Substitutionen von $\mathfrak{A}$ durch einen oberen Index $t$ ist eigentlich nur zulässig, wenn $\mathfrak{A}$ eine endliche oder abzählbare unendliche Menge von Substitutionen enthält. Besitzt $\mathfrak{A}$ eine nicht abzählbare Menge von Substitutionen, so kann man die einzelnen Substitutionen von $\mathfrak{A}$ nicht derartig durch einen oberen Index $t$ charakterisiren. Unsere Schlüsse gelten auch für unendliche Gruppen linearer homogener Substitutionen mit einer nicht abzählbaren Menge von Substitutionen, denn unsere Betrachtungen hängen *gar nicht von der Charakterisirung der Substitutionen durch den Index t ab.* Wir haben nur der Bequemlichkeit wegen im Texte den oberen Index $t$ benützt, um auf die Zuordnung der entsprechenden Substitutionen von $\mathfrak{A}$ und $\mathfrak{B}$ gelegentlich hinweisen zu können.

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Wir wollen folgenden Hilfsatz beweisen: *Sind* $\mathfrak{A}$ *und* $\mathfrak{B}$ *ähnliche Gruppen linearer homogener Substitutionen der Form* ($\mathfrak{A}$) *und* ($\mathfrak{B}$) *und werden durch irgend eine Transformation* (5) *von nicht verschwindender Determinante und die zu ihr cogrediente* (5′) *die Substitutionen der einen Gruppe in die der anderen übergeführt, so müssen, wenn* $\mathfrak{b}_{11}$ *eine irreducible Gruppe linearer homogener Substitutionen ist und zwischen* $y_1, y_2, \cdots, y_\nu$ *und* $z_1, z_2, \cdots, z_{g_1}$ *auch nur eine einzige lineare homogene Relation mit constanten Coefficienten besteht, in der überführenden Transformation* (5) *die ersten* $g_1$ *Gleichungen* ($g_1 < n$):

$$(5_1) \qquad z_i = \sum_{k=1}^{\nu} q_{ik} y_k \qquad (i = 1, \cdots, g_1)$$

*lauten, also für* $i \leqq g_1$, $k > \nu$ *müssen die Grössen* $q_{ik}$ *notwendig Null sein.*

Den Beweis zerlegen wir in zwei Teile:

(I) Angenommen, es bestehe für *eine* der Zahlen $1, 2, \cdots, g_1$, die ich mit $l$ bezeichne, in der Substitution (5) die Relation:

$$(6) \qquad z_l = q_{l1} y_1 + q_{l2} y_2 + \cdots + q_{l\nu} y_\nu .$$

Infolgedessen wird für die cogrediente Substitution:

$$(6') \qquad z'_l = q_{l1} y'_1 + q_{l2} y'_2 + \cdots + q_{l\nu} y'_\nu .$$

Nach Voraussetzung ist also $q_{l\nu+1} = q_{l\nu+2} = \cdots = q_{ln} = 0$.

Nach (6) und (2) wird:

$$(7) \qquad z_l = \sum_{s=1}^{s=\nu} q_{ls} a_{s1}^{(t)} y'_1 + \sum_{s=1}^{s=\nu} q_{ls} a_{s2}^{(t)} y'_2 + \cdots + \sum_{s=1}^{s=\nu} q_{ls} a_{s\nu}^{(t)} y'_\nu .$$

Aus (4) und (5′) ergiebt sich:

$$(8) \qquad z_l = \sum_{s=1}^{s=g_1} b_{ls}^{(t)} q_{s1} y'_1 + \sum_{s=1}^{s=g_1} b_{ls}^{(t)} q_{s2} y'_2 + \cdots + \sum_{s=1}^{s=g_1} b_{ls}^{(t)} q_{sn} y'_n .$$

Der Vergleich von (7) und (8) lehrt, dass die Coefficienten von $y'_{\nu+1}, y'_{\nu+2}, \cdots, y'_n$ in (8) verschwinden müssen, d. h. es müssen die Relationen bestehen:

$$(9) \qquad \sum_{s=1}^{s=g_1} b_{ls}^{(t)} q_{si} = 0 \qquad (i = \nu+1, \cdots, n;\ t = 1, 2, \cdots).$$

Wir betrachten die erste dieser Relationen ($i = \nu + 1$) und bedenken, dass infolge von (6) $q_{l\nu+1} = 0$ sein muss. Daher erhalten wir:

$$(9_1) \quad b_{l1}^{(t)} q_{1\nu+1} + b_{l2}^{(t)} q_{2\nu+1} + \cdots b_{ll-1}^{(t)} q_{l-1\nu+1} + b_{ll+1}^{(t)} q_{l+1\nu+1} + \cdots + b_{lg_1} q_{g_1\nu+1} = 0 .$$

Da $\mathfrak{b}_{11}$ eine irreducible Gruppe linearer homogener Substitutionen in $g_1$ Variablen ist, muss nach den Untersuchungen von Herrn MASCHKE * $t$ mindestens

*MASCHKE, a. a. O., S. 365, 366.



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gleich $g_1 - 1$ sein; denn jede Gruppe linearer homogener Substitutionen, bei der die Anzahl der Substitutionen gleich oder kleiner als die um 2 verminderte Zahl der Substitutionsvariablen ist, ist notwendig reducibel. Daher stellt $(9_1)$ ein System von $t \geqq g_1 - 1$ linearen homogenen Gleichungen für die $g_1 - 1$ Grössen $q_{1\nu+1}, q_{2\nu+1}, \cdots, q_{l-1\,\nu+1}, q_{l+1\,\nu+1}, \cdots, q_{g_1\nu+1}$ dar: $l$ ist dabei eine durch (6) ganz bestimmt fixirte Zahl aus der Reihe der Zahlen $1, 2, \cdots, g_1$.

Sollen nicht sämtliche der $g_1 - 1$ Grössen:

$$q_{1\nu+1}, q_{2\nu+1}, \cdots, q_{l-1\,\nu+1}, q_{l+1\nu+1}, \cdots, q_{g_1\nu+1},$$

verschwinden, so muss jede der Determinanten:

$$\begin{vmatrix} b_{l1}^{(i_1)} & b_{l2}^{(i_1)} & \cdots & b_{ll-1}^{(i_1)} & b_{ll+1}^{(i_1)} & \cdots & b_{lg_1}^{(i_1)} \\ b_{l1}^{(i_2)} & b_{l2}^{(i_2)} & \cdots & b_{ll-1}^{(i_2)} & b_{ll+1}^{(i_2)} & \cdots & b_{lg_1}^{(i_2)} \\ \vdots & & & & & & \\ b_{l1}^{(i_{g_1-1})} & b_{l2}^{(i_{g_1-1})} & \cdots & b_{ll-1}^{(i_{g_1-1})} & b_{ll+1}^{(i_{g_1-1})} & \cdots & b_{lg_1}^{(i_{g_1-1})} \end{vmatrix},$$

die man erhält, wenn man $i_1, i_2, \cdots, i_{g_1-1}$ irgend welche $g_1 - 1$ verschiedene Werte aus der Zahlenreihe, die $t$ durchläuft, beilegt, Null sein. Anders ausgedrückt: Man muss aus der $l$-ten Zeile der Substitutionsgruppe $\mathfrak{b}_{11}$ die $g_1 - 1$ Coefficienten:

$$b_{l1}, b_{l2}, \cdots, b_{ll-1}, b_{ll+1}, \cdots, b_{lg_1},$$

herausgreifen können, dass jede der Determinanten, gebildet aus den entsprechenden Coefficienten jeder möglichen Combination von je $g_1 - 1$ Substitutionen der Gruppe $\mathfrak{b}_{11}$, verschwindet. Tritt dies aber ein, so ist die Gruppe $\mathfrak{b}_{11}$, wie aus einem Satze von Herrn MASCHKE* folgt, reducibel. Infolgedessen bleibt nur übrig, dass die $g_1 - 1$ Grössen:

$$(10) \qquad q_{1\nu+1}, q_{2\nu+1}, \cdots, q_{l-1\nu+1}, q_{l+1\nu+1}, \cdots, q_{g_1\nu+1}$$

sämtlich gleichzeitig verschwinden.

Mit Hülfe der weiteren Gleichungen des Systems (9) wird auf genau die gleiche Weise der Nachweis geführt, dass:

$$(10) \qquad q_{1i} = q_{2i} = \cdots = q_{l-1i} = q_{l+1i} = \cdots = q_{g_1i} = 0 \quad (i = \nu+2, \cdots, n)$$

ist.

Unter Berücksichtigung von (10) und

$$q_{l\nu+1} = q_{l\nu+2} = \cdots = q_{ln} = 0$$

ergiebt sich daher, dass im Falle (I) die ersten $g_1$ Gleichungen von (5) notwendig die Form $(5_1)$ haben müssen.

(II). Wir setzen jetzt voraus, dass zwischen $y_1, y_2, \cdots, y_\nu, z_1, z_2, \cdots, z_{g_1}$ irgend eine lineare homogene Relation:

* H. MASCHKE, a. a. O., S. 364, 366.

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$$(11) \qquad d_1 y_1 + d_2 y_2 + \cdots + d_\nu y_\nu + e_1 z_1 + e_2 z_2 + \cdots + e_{g_1} z_{g_1} = 0$$

bestehe. Hierbei bedeuten $d_1, d_2, \cdots, d_\nu, e_1, e_2, \cdots, e_{g_1}$ Constante, von denen weder alle $d$ noch alle $e$ Null sind. Wir führen Fall (II) auf (I) zurück. Zu dem Zwecke setzen wir:

$$(12) \qquad \bar{z}_1 = e_1^{(1)} z_1 + e_2^{(1)} z_2 + \cdots + e_{g_1}^{(1)} z_{g_1}$$

und verstehen unter $e_1^{(1)}, e_2^{(1)}, \cdots, e_{g_1}^{(1)}$ dieselben Grössen wie unter $e_1, e_2, \cdots, e_{g_1}$. Die Änderung der Bezeichnung ist nur aus Gründen des Symmetrie vorgenommen. Wir führen $g_1 - 1$ neue weitere Variablen $\bar{z}_2, \bar{z}_3, \cdots, \bar{z}_{g_1}$ durch die Gleichungen:

$$(12) \qquad \bar{z}_k = \sum_{i=1}^{g_1} e_i^{(k)} z_i \qquad (k = 2, 3, \cdots, g_1),$$

ein, hierbei sind die $e_i^{(k)}$ ganz willkürliche Constante, die nur so gewählt sein sollen, dass die Determinante $|e_k^{(i)}|$ $(i = 1, 2, \cdots, g_1;\ k = 1, 2, \cdots, g_1)$ von Null verschieden ist. Durch die Formel:

$$(12') \qquad \bar{z}_i' = e_1^{(i)} z_1' + e_2^{(i)} z_2' + \cdots + e_{g_1}^{(i)} z_{g_1}' \qquad (i = 1, 2, \cdots, g_1)$$

definiren wir die sich mit $\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_{g_1}$ cogredient transformirenden Variablen $\bar{z}_1', \bar{z}_2', \cdots, \bar{z}_{g_1}'$. Führt man in der Gruppe $\mathfrak{B}$ für $z_1, z_2, \cdots, z_{g_1}$ die Variablen $\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_{g_1}$ ein, ersetzt ferner $z_1', z_2', \cdots, z_{g_1}'$ nach (12′) durch $\bar{z}_1', \bar{z}_2', \cdots, \bar{z}_{g_1}'$, behält hingegen $z_{g_1+1}, z_{g_1+2}, \cdots, z_n$ sowie $z_{g_1+1}', z_{g_1+2}', \cdots, z_n'$ bei, so geht die Gruppe $\mathfrak{B}$ in eine ähnliche Gruppe $\bar{\mathfrak{B}}$ über; die letztere ist dann in den Variablen $\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_{g_1}, z_{g_1+1}, z_{g_1+2}, \cdots, z_n$ bez. den Variablen $\bar{z}_1', \bar{z}_2', \cdots, \bar{z}_{g_1}', z_{g_1+1}', z_{g_1+2}', \cdots, z_n'$ geschrieben. Bedeutet:

$$(13) \qquad q_{ik}' = e_1^{(i)} q_{1k} + e_2^{(i)} q_{2k} + \cdots + e_{g_1}^{(i)} q_{g_1 k} \quad (i = 1, 2, \cdots, g_1;\ k = 1, 2, \cdots, n),$$

so ergiebt sich aus (5) und (12):

$$(14) \qquad \begin{aligned} \bar{z}_i &= q_{i1}' y_1 + q_{i2}' y_2 + \cdots + q_{in}' y_n \qquad (i = 1, \cdots, g_1), \\ z_i &= q_{i1} y_1 + q_{i2} y_2 + \cdots + q_{in} y_n \qquad (i = g_1 + 1, \cdots, n). \end{aligned}$$

Die hierzu cogrediente Substitution ist nach (5′) und (12′) gegeben durch:

$$(14') \qquad \begin{aligned} \bar{z}_i' &= q_{i1}' y_1' + q_{i2}' y_2' + \cdots + q_{in}' y_n' \qquad (i = 1, \cdots, g_1), \\ z_i' &= q_{i1} y_1' + q_{i2} y_2' + \cdots + q_{in} y_n' \qquad (i = g_1 + 1, \cdots, n). \end{aligned}$$

Die Formeln (14) und (14′) vermitteln den Übergang von der Gruppe $\mathfrak{A}$ zu der ihr ähnlichen Gruppe $\bar{\mathfrak{B}}$. In $\bar{\mathfrak{B}}$ transformiren sich die Variablen $\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_{g_1}$ nach der irreduciblen Gruppe $\bar{\mathfrak{b}}_{11}$, die aus der irreduciblen Gruppe $\mathfrak{b}_{11}$ infolge der Transformation (12) hervorgeht. Ersetzt man in (11) die Variablen $z_1, z_2, \cdots, z_{g_1}$ durch ihre Werte nach (5), so erhält man durch Nullsetzen der Factoren von $y_1, y_2, \cdots, y_n$ das Gleichungssystem:

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$$(11_1) \qquad d_i + e_1 q_{1i} + e_2 q_{2i} + \cdots + e_{g_1} q_{g_1 i} = 0 \qquad (i = 1, 2, \cdots, n),$$

wobei $d_{\nu+1} = d_{\nu+2} = \cdots = d_n = 0$ ist. Beachtet man (13), so wird die Relation: $\bar{z}_1 = -(d_1 y_1 + d_2 y_2 + \cdots + d_\nu y_\nu)$ durch Benützung der ersten $\nu$ Gleichungen $(11_1)$ übergehen in:

$$(14_1) \qquad \bar{z}_1 = q'_{11} y_1 + q'_{12} y_2 + \cdots + q'_{1\nu} y_\nu.$$

Da $\bar{\mathfrak{b}}_{11}$ eine irreducible Gruppe linearer homogener Substitutionen ist und die Gleichung $(14_1)$ besteht, muss nach dem in (I) dieses § bewiesenen Resultate:

$$(15) \qquad q'_{1i} = q'_{2i} = \cdots = q'_{g_1 i} = 0 \qquad (i = \nu+1, \cdots, n)$$

werden. Unter Beachtung von (13) lauten die ersten $g_1$ Gleichungen von (15):

$$e_1^{(i)} q_{1\nu+1} + e_2^{(i)} q_{2\nu+1} + \cdots + e_{g_1}^{(i)} q_{g_1 \nu+1} = 0 \qquad (i = 1, 2, \cdots, g_1).$$

Da die Determinante

$$|e_k^{(i)}| \qquad (i = 1, 2, \cdots, g_1;\ k = 1, 2, \cdots, g_1)$$

von Null verschieden ist, müssen die Grössen

$$q_{1\nu+1},\ q_{2\nu+1},\ \cdots,\ q_{g_1\nu+1}$$

sämtlich verschwinden. Ebenso schliesst man aus den Gleichungen (15) weiter, dass:

$$q_{1i} = q_{2i} = \cdots = q_{g_1 i} = 0 \qquad (i = \nu+2, \cdots, n)$$

wird.

Hiermit ist gezeigt, dass bei den über die Gruppen $\mathfrak{A}$ und $\mathfrak{B}$ gemachten Voraussetzungen, wenn $y_1, y_2, \cdots, y_\nu, z_1, z_2, \cdots, z_{g_1}$ nicht linear unabhängig sind, in der Transformation (5) die ersten $g_1$ Variablen sich notwendig nach $(5_1)$ transformiren.

## § 3.

Die Gruppe $\mathfrak{G}$ linearer homogener Substitutionen in $n$ Variablen sei in die ähnliche Gruppe $\mathfrak{A}^*$ der Form:

$$(\mathfrak{A}^*) \qquad \begin{matrix} \mathfrak{A}^*_{11} & 0 \\ \mathfrak{A}^*_{21} & \mathfrak{A}^*_{22} \end{matrix}$$

transformirt, dabei sollen $\mathfrak{A}^*_{11}, \mathfrak{A}^*_{21}, \mathfrak{A}^*_{22}$ einen Inbegriff von Matrices mit $\nu$ Zeilen und $\nu$ Colonnen, bez. $n - \nu$ Zeilen und $\nu$ Colonnen, bez. $n - \nu$ Zeilen und $n - \nu$ Colonnen bedeuten.

*Definirt* $\mathfrak{A}^*_{11}$, *für sich betrachtet, eine vollständig reducible Gruppe linearer homogener Substitutionen,* und existirt *keine* Matrix $L$ von nicht verschwindender Determinante, welche die Gruppe $\mathfrak{G}$ in eine ähnliche $L\mathfrak{G}L^{-1}$ der Form:

$$\begin{matrix} \mathfrak{A}^*_{11} & 0 & 0 \\ 0 & \mathfrak{a}_{\varepsilon+1,\varepsilon+1} & 0 \\ \mathfrak{A}^*_{31} & \mathfrak{A}^*_{32} & \mathfrak{A}^*_{33} \end{matrix}$$

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überführt, wobei $\mathfrak{a}_{\epsilon+1,\,\epsilon+1}$ eine irreducible Gruppe linearer homogener Substitutionen in $f_{\epsilon+1}(f_{\epsilon+1} > 0)$ Variablen bedeute und demnach $\mathfrak{A}^*_{31}$, $\mathfrak{A}^*_{32}$, $\mathfrak{A}^*_{33}$ Matrices mit $n - \nu - f_{\epsilon+1}$ Zeilen und $\nu$ bez. $f_{\epsilon+1}$ bez. $n - \nu - f_{\epsilon+1}$ Colonnen vorstellen, so sagen wir: *die Gruppe* $\mathfrak{G}$ *ist unter Hervorhebung ihrer grössten vollständig reduciblen Gruppe* $\mathfrak{A}^*_{11}$ *in die Gruppe* $\mathfrak{A}^*$ *transformirt worden.*

Hat man eine Gruppe $\mathfrak{G}$ linearer homogener Substitutionen, so kann man offenbar durch wiederholte Einführung linearer homogener Functionen der Variablen mit constanten Coefficienten die Gruppe $\mathfrak{G}$ mit Hülfe einer Matrix $S$ von nicht verschwindender Determinante in eine ähnliche Gruppe $\mathfrak{A} = S\mathfrak{G}S^{-1}$ transformiren, so dass diese ähnliche Gruppe die Form:

$$(16)\quad (\mathfrak{A}):\quad \begin{matrix} \mathfrak{a}_{11} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \mathfrak{a}_{22} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \mathfrak{a}_{33} & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & \mathfrak{a}_{\epsilon-1,\,\epsilon-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \mathfrak{a}_{\epsilon,\,\epsilon} & 0 \\ A_{\epsilon+1,1} & A_{\epsilon+1,2} & A_{\epsilon+1,3} & A_{\epsilon+1,4} & \cdots & A_{\epsilon+1,\,\epsilon-1} & A_{\epsilon+1,\,\epsilon} & A_{\epsilon+1,\,\epsilon+1} \end{matrix}$$

annimt. $\mathfrak{a}_{11}, \mathfrak{a}_{22}, \cdots, \mathfrak{a}_{\epsilon\epsilon}$ sollen hierbei irreducible Bestandteile sein, deren Variablen sich ausnahmslos gesondert für sich transformiren (rechts und links von jedem $\mathfrak{a}_{ii}$ stehen lauter Nullmatrices). Unter Umständen kann $\epsilon = 1$ sein. Ist es *unmöglich*, die Gruppe $\mathfrak{A}$ durch weitere Einführung neuer Variablen in die Form

$$\begin{matrix} \mathfrak{a}_{11} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \mathfrak{a}_{22} & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \mathfrak{a}_{\epsilon,\,\epsilon} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \mathfrak{a}_{\epsilon+1,\,\epsilon+1} & 0 \\ A_{\epsilon+2,1} & A_{\epsilon+2,2} & A_{\epsilon+2,3} & \cdots & A_{\epsilon+2,\,\epsilon} & A_{\epsilon+2,\,\epsilon+1} & A_{\epsilon+2,\,\epsilon+2} \end{matrix}$$

zu bringen, so ist $(\mathfrak{A})$ *eine Darstellung der Gruppe* $\mathfrak{G}$ *unter Hervorhebung der grössten vollständig reduciblen Gruppe* $\{\mathfrak{a}_{11}, \mathfrak{a}_{22}, \cdots, \mathfrak{a}_{\epsilon\epsilon}\}$. *Die soeben hingeschriebene, vollständig reducible Gruppe bezeichnen wir mit* $\mathfrak{A}_{11}$ und demgemäss benützen wir auch für die Darstellung $(\mathfrak{A})$ dieses § die Bezeichnung:

$$(16_1)\qquad (\mathfrak{A}):\quad \begin{matrix} \mathfrak{A}_{11} & 0 \\ \mathfrak{A}_{21} & \mathfrak{A}_{22}. \end{matrix}$$

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Ich suche hierdurch Anlehnung an die Bezeichnung des § 2, nur ist in diesem § $\mathfrak{A}_{11}$ eine vollständig reducible Gruppe, wohingegen im § 2 keine derartige Bedingung statthatte.

Die Transformation einer Gruppe $\mathfrak{G}$ in eine ähnliche unter Hervorhebung einer grössten vollständig reduciblen Gruppe ist durchaus nicht eindeutig. Die Gruppe $\mathfrak{G}$ sei unter Hervorhebung der grössten vollständig reduciblen Gruppe in analoger Weise wie in die Gruppe $\mathfrak{A}$ auch in die ähnliche Gruppe $\mathfrak{B}$ transformirt. $\mathfrak{B}$ möge die Form

$$(17)\quad (\mathfrak{B}):\begin{matrix} \mathfrak{b}_{11} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \mathfrak{b}_{22} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \mathfrak{b}_{33} & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & \mathfrak{b}_{\epsilon'-1,\,\epsilon'-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \mathfrak{b}_{\epsilon',\,\epsilon'} & 0 \\ B_{\epsilon'+1,1} & B_{\epsilon'+1,2} & B_{\epsilon'+1,3} & B_{\epsilon'+1,4} & \cdots & B_{\epsilon'+1,\,\epsilon'-1} & B_{\epsilon'+1,\,\epsilon'} & B_{\epsilon'+1,\,\epsilon'+1} \end{matrix}$$

haben. $\mathfrak{b}_{11}, \mathfrak{b}_{22}, \cdots, \mathfrak{b}_{\epsilon'\epsilon'}$ bedeuten hierbei irreducible Bestandteile von $\mathfrak{G}$. Die Darstellung $(\mathfrak{B})$ werde ich auch mit

$$(17_1)\qquad (\mathfrak{B}):\begin{matrix} \mathfrak{B}_{11} & 0 \\ \mathfrak{B}_{21} & \mathfrak{B}_{22} \end{matrix}$$

bezeichnen. Hierbei bedeutet $\mathfrak{B}_{11}$ die vollständig reducible Gruppe $\{\mathfrak{b}_{11}, \mathfrak{b}_{22}, \cdots, \mathfrak{b}_{\epsilon'\epsilon'}\}$. Um Anschluss an den § 2 zu gewinnen, soll die Gruppe $\mathfrak{B}$ der Form $(\mathfrak{B})$ auch mit:

$$(17_2)\qquad (\mathfrak{B}):\begin{matrix} \mathfrak{b}_{11} & 0 \\ \mathfrak{b}'_{21} & \mathfrak{b}'_{22} \end{matrix}$$

bezeichnet werden.

Die Gruppen $\mathfrak{A}$ und $\mathfrak{B}$, die durch $(16_1)$ und $(17_2)$ dargestellt werden, sind ähnliche Gruppen, ferner ist $\mathfrak{b}_{11}$ eine irreducible Gruppe. Wir können daher die Ergebnisse des § 2 anwenden. Wir tun dies und benützen hierbei die gleichen Bezeichnungen wie im § 2. Infolge der Ähnlichkeit der zwei Gruppen $\mathfrak{A}$ und $\mathfrak{B}$ giebt es Übergangsformeln (5) und (5′), welche die Substitutionen der zwei Gruppen in einander überführen. Angenommen, es bestehe zwischen $z_1, z_2, \cdots, z_{g_1}, y_1, y_2, \cdots, y_\nu$ nach Voraussetzung auch nur eine einzige lineare homogene Relation mit constanten Coefficienten, so müssen die Gleichungen $(5_1)$ bestehen, d. h. jede der Variablen $z_1, z_2, \cdots, z_{g_1}$ ist von $y_1, y_2, \cdots, y_\nu$ linear abhängig. Mithin sind nur zwei Fälle möglich: entweder sind $z_1, z_2, \cdots, z_{g_1}$ von $y_1, y_2, \cdots, y_\nu$ gleichzeitig linear abhängig oder gleichzeitig linear unabhängig.

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Die Unmöglichkeit des zweiten Falles ergiebt sich daraus, dass in diesem § die Gruppe $\mathfrak{G}$ unter Hervorhebung ihrer grössten vollständig reduciblen Gruppe $\mathfrak{A}_{11}$ in die Form ($\mathfrak{A}$) übergeführt sein soll. Angenommen nämlich, jede der Variablen $z_1, z_2, \cdots, z_{g_1}$ sei von den Variablen $y_1, y_2, \cdots, y_\nu$ linear unabhängig. In diesem Fall kann man eine zu $\mathfrak{A}$ und daher auch zu $\mathfrak{G}$ ähnliche Gruppe auf folgende Weise bilden: Wir halten $y_1, y_2, \cdots, y_\nu$ bei, nehmen hierzu noch die nach Annahme von diesen $\nu$ Variablen sowie auch unter einander unabhängigen Variablen $z_1, z_2, \cdots, z_{g_1}$, die lineare homogene Functionen von $y_1, y_2, \cdots, y_n$ mit constanten Coefficienten sind, und führen zu diesen $\nu + g_1$ Variablen noch $n - \nu - g_1$ linear unabhängige Functionen von $y_1, y_2, \cdots, y_n$ mit constanten Coefficienten ein. Die Gesamtheit der Matrices der auf diese Weise gewonnenen, zu $\mathfrak{A}$ und $\mathfrak{G}$ ähnlichen Gruppe lässt sich dann in der Form:

$$\begin{matrix} \mathfrak{A}_{11} & 0 & 0 \\ 0 & \mathfrak{b}_{11} & 0 \\ \vartheta_{31} & \vartheta_{32} & \vartheta_{33} \end{matrix}$$

schreiben; dabei bedeuten $\vartheta_{31}, \vartheta_{32}, \vartheta_{33}$ einen Inbegriff von Matrices mit $n - \nu - g_1$ Zeilen und $\nu$ bez. $g_1$ bez. $n - \nu - g_1$ Colonnen. Die zuletzt erhaltene Darstellung steht im Widerspruch damit, dass $\mathfrak{A}$ eine Darstellung der Gruppe $\mathfrak{G}$ unter Hervorhebung der grössten vollständig reduciblen Gruppe $\mathfrak{A}_{11}$ sein soll. Mithin sind die Variablen in $\mathfrak{B}$, die sich auf $\mathfrak{b}_{11}$ beziehen, nicht von $y_1, y_2, \cdots, y_\nu$ linear unabhängig, und es gelten die Gleichungen ($5_1$).

In genau der gleichen Weise ergiebt sich, da die Gruppe $\mathfrak{B}$ ja auch die Form (17), hat, dass die Variablen von $\mathfrak{B}$, die zu den irreduciblen Bestandteilen $\mathfrak{b}_{22}, \mathfrak{b}_{33}, \cdots, \mathfrak{b}_{e'e'}$ gehören, nicht von $y_1, y_2, \cdots, y_\nu$ linear unabhängig sein können; denn durch blosse Variablenvertauschung kann jeder der irreduciblen Bestandteile $\mathfrak{b}_{22}, \mathfrak{b}_{33}, \cdots, \mathfrak{b}_{e'e'}$ an die Stelle von $\mathfrak{b}_{11}$ treten.

Nehmen wir an, dass sich $\mathfrak{B}_{11}$ auf $\nu'$ Variablen bezieht, so werden, weil wir für die zu $\mathfrak{b}_{11}$ gehörigen Variablen die Gültigkeit der Gleichungen ($5_1$) und für die zu $\mathfrak{b}_{22}, \mathfrak{b}_{33}, \cdots, \mathfrak{b}_{e'e'}$ gehörigen Variablen die Richtigkeit der zu ($5_1$) analogen Gleichungen nachgewiesen haben, die ersten $\nu'$ Gleichungen der Transformation (5):

$$z_i = q_{i1} y_1 + q_{i2} y_2 + \cdots + q_{i\nu} y_\nu \qquad (i = 1, \cdots, \nu') \tag{18}$$

lauten. $\nu'$ kann nicht grösser als $\nu$ sein, denn sonst würden die $\nu'$ unabhängigen Variablen $z_1, z_2, \cdots, z_{\nu'}$, die zur Gruppe $\mathfrak{B}_{11}$ gehören, in Dependenz sein. Hieraus folgt:

$$\nu' \leqq \nu. \tag{19}$$

$\mathfrak{B}$ und $\mathfrak{A}$ sind ähnliche Gruppen; dieser Begriff ist aber ein wechselseitiger. Ferner spielen $\mathfrak{A}_{11}$ und $\mathfrak{B}_{11}$ in $\mathfrak{A}$ und $\mathfrak{B}$ genau die gleiche Rolle. Infolge dieser

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Gleichberechtigung ist der Schluss: $\nu' \geqq \nu$ gestattet. Unter Berücksichtigung von (19) ergiebt sich daher: $\nu' = \nu$, d. h. in $(16_1)$ und $(17_1)$ haben $\mathfrak{A}_{11}$ und $\mathfrak{B}_{11}$ die gleiche Anzahl Zeilen und Colonnen.

Setzt man in (18) $\nu = \nu'$, so sieht man die Richtigkeit des folgenden Resultats ein: *Wird die Gruppe $\mathfrak{A}$ der Form $(16_1)$, wobei $\mathfrak{A}_{11}$ die grösste vollständig reducible Gruppe $\{\mathfrak{a}_{11}, \mathfrak{a}_{22}, \cdots, \mathfrak{a}_{\epsilon\epsilon}\}$ von $\mathfrak{G}$ ist, in eine ähnliche Gruppe $\mathfrak{B}$ der Form $(17_1)$, wobei $\mathfrak{B}_{11}$ die grösste vollständig reducible Gruppe $\{\mathfrak{b}_{11}, \mathfrak{b}_{22}, \cdots, \mathfrak{b}_{\epsilon'\epsilon'}\}$ bedeutet, übergeführt, so muss jede Matrix $Q$ von nicht verschwindender Determinante, welche die Überführung der einen Gruppe in die andere leistet, notwendig die Form:*

$$(20) \qquad \begin{matrix} Q_{11} & 0 \\ Q_{21} & Q_{22} \end{matrix}$$

*haben. Hierbei besitzt die Matrix $Q_{11}$ genau ebensoviele Zeilen und Colonnen wie die Gruppen $\mathfrak{A}_{11}$ und $\mathfrak{B}_{11}$.*

In (20) ist $Q_{11}$ die Matrix von (18). Die Determinante von $Q$ ist von Null verschieden; mithin kann infolge der Form (20), die $Q$ hat, auch die Determinante von $Q_{11}$ nicht Null sein. Hieraus folgt die *Ähnlichkeit der zwei Gruppen $\mathfrak{A}_{11}$ und $\mathfrak{B}_{11}$. Es ist $\mathfrak{B}_{11} = Q_{11}\mathfrak{A}_{11}Q_{11}^{-1}$.* Nach dem in der Einleitung angegebenen Satze (Transactions, vol. 4, p. 46) wird $\epsilon = \epsilon'$.

## § 4.

Die Gruppe $\mathfrak{G}$ linearer homogener Substitutionen in $n$ Variablen sei unter Hervorhebung der grössten vollständig reduciblen Gruppe $\mathfrak{A}_{11}^*$ in die ähnliche Gruppe $\mathfrak{A}^*$ der Form:

$$(21) \qquad (\mathfrak{A}^*): \begin{matrix} \mathfrak{A}_{11}^* & 0 \\ \mathfrak{A}_{21}^* & \mathfrak{A}_{22}^* \end{matrix}$$

transformirt. Ferner möge die Gruppe $\mathfrak{G}$ unter Hervorhebung der grössten vollständig reduciblen Gruppe $\mathfrak{B}_{11}^*$ in die ähnliche Gruppe:

$$(22) \qquad (\mathfrak{B}^*): \begin{matrix} \mathfrak{B}_{11}^* & 0 \\ \mathfrak{B}_{21}^* & \mathfrak{B}_{22}^* \end{matrix}$$

übergeführt sein.

Wir wollen uns in diesem Paragraphen von einer im vorigen gemachten Voraussetzung befreien. Wir hatten im vorigen § die Gruppe $\mathfrak{G}$ unter Hervorhebung der grössten vollständig reduciblen Gruppen $\mathfrak{A}_{11}$ und $\mathfrak{B}_{11}$ in die zwei ähnlichen Gruppen $\mathfrak{A}$ und $\mathfrak{B}$ der Form $(16_1)$ und $(17_1)$ transformirt angenommen und hierbei für $\mathfrak{A}_{11}$ und $\mathfrak{B}_{11}$ die Form $\{\mathfrak{a}_{11}, \mathfrak{a}_{22}, \cdots, \mathfrak{a}_{\epsilon\epsilon}\}$ bez. $\{\mathfrak{b}_{11}, \mathfrak{b}_{22}, \cdots, \mathfrak{b}_{\epsilon'\epsilon'}\}$ vorausgesetzt, man sollte also $\mathfrak{A}_{11}$ und $\mathfrak{B}_{11}$ den Charakter der vollständigen Reducibilität auf den ersten Blick ansehen. $\mathfrak{A}_{11}^*$ und $\mathfrak{B}_{11}^*$ sollen auch grösste voll-

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ständig reducible Gruppen von $\mathfrak{G}$ vorstellen, jedoch braucht ihr Charakter der vollständigen Reducibilität nicht unmittelbar evident zu sein.

Die Gruppe $\mathfrak{A}^*$ möge in den Variablen $y_1^*, y_2^*, \cdots, y_n^*$ bez. $y_1^{*\prime}, y_2^{*\prime}, \cdots, y_n^{*\prime}$, die Gruppe $\mathfrak{B}^*$ in den Variablen $z_1^*, z_2^*, \cdots, z_n^*$ bez. $z_1^{*\prime}, z_2^{*\prime}, \cdots, z_n^{*\prime}$ geschrieben sein.* Eine Substitution von nicht verschwindender Determinante, die mit ihrer cogredienten Substitution den Übergang der ähnlichen Gruppen $\mathfrak{A}^*$ und $\mathfrak{B}^*$ in einander vermittelt, sei mit $Q^*$ bezeichnet, und laute:

$$z_i^* = q_{i1}^* y_1^* + q_{i2}^* y_2^* + \cdots + q_{in}^* y_n^* \qquad (i = 1, \cdots, n). \tag{23}$$

Die sich auf die Gruppe $\mathfrak{A}_{11}^*$ in $\mathfrak{A}^*$ beziehenden Variablen seien die Variablen $y_1^*, y_2^*, \cdots, y_\nu^*$ bez. $y_1^{*\prime}, y_2^{*\prime}, \cdots, y_\nu^{*\prime}$. Nach Voraussetzung ist $\mathfrak{A}_{11}^*$ eine vollständig reducible Gruppe. Daher muss man ein neues Variablensystem $y_1, y_2, \cdots, y_\nu$ finden können, das sich durch $y_1^*, y_2^*, \cdots, y_\nu^*$ linear homogen mit constanten Coefficienten ausdrückt, so dass durch Einführung von $y_1, y_2, \cdots, y_\nu$ für $y_1^*, y_2^*, \cdots, y_\nu^*$ sowie der zu $y_1, y_2, \cdots, y_\nu$ cogredienten Variablen $y_1', y_2', \cdots, y_\nu'$ für $y_1^{*\prime}, y_2^{*\prime}, \cdots, y_\nu^{*\prime}$ die Gruppe $\mathfrak{A}_{11}^*$ in eine ähnliche Gruppe $\mathfrak{A}_{11}$ der Form $\{\mathfrak{a}_{11}, \mathfrak{a}_{22}, \cdots, \mathfrak{a}_{\epsilon\epsilon}\}$ übergeht; hierbei sollen $\mathfrak{a}_{11}, \mathfrak{a}_{22}, \cdots, \mathfrak{a}_{\epsilon\epsilon}$ irreducible Gruppen sein.

Führt man für die zwei Variablensysteme $y_1^*, y_2^*, \cdots, y_\nu^*$ und $y_1^{*\prime}, y_2^{*\prime}, \cdots, y_\nu^{*\prime}$, wie wir soeben angaben, die zwei neuen Variablensysteme $y_1, y_2, \cdots, y_\nu$ und $y_1', y_2', \cdots, y_\nu'$ ein, behält hingegen die Variablen $y_{\nu+1}^*, y_{\nu+2}^*, \cdots, y_n^*$ und $y_{\nu+1}^{*\prime}, y_{\nu+2}^{*\prime}, \cdots, y_n^{*\prime}$ bei, so geht die Gruppe $\mathfrak{A}^*$ der Form (21) in eine ähnliche Gruppe $\mathfrak{A}$ der Form:

$$\begin{matrix} \mathfrak{A}_{11} & 0 \\ \mathfrak{A}_{21} & \mathfrak{A}_{22} \end{matrix} \tag{24}$$

über. In (24) ist $\mathfrak{A}_{22}$ übrigens mit $\mathfrak{A}_{22}^*$ identisch; nur der Symmetrie wegen haben wir $\mathfrak{A}_{22}$ statt $\mathfrak{A}_{22}^*$ gesetzt. Die Gruppe $\mathfrak{A}$ erscheint in den Variablen $y_1, y_2, \cdots, y_\nu, y_{\nu+1}^*, y_{\nu+2}^*, \cdots, y_n^*$ bez. $y_1', y_2', \cdots, y_\nu', y_{\nu+1}^{*\prime}, y_{\nu+2}^{*\prime}, \cdots, y_n^{*\prime}$ geschrieben.

In analoger Weise transformiren wir die Gruppe $\mathfrak{B}^*$ in eine ähnliche Gruppe $\mathfrak{B}$. Die sich auf die Gruppe $\mathfrak{B}_{11}^*$ in $\mathfrak{B}^*$ (vgl. 22) beziehenden Variablen mögen $z_1^*, z_2^*, \cdots, z_{\nu'}^*$, bez. $z_1^{*\prime}, z_2^{*\prime}, \cdots, z_{\nu'}^{*\prime}$ lauten. $\mathfrak{B}_{11}^*$ ist eine vollständig reducible Gruppe. Man muss daher ein neues Variablensystem $z_1, z_2, \cdots, z_{\nu'}$ finden können, das sich linear homogen mit constanten Coefficienten durch $z_1^*, z_2^*, \cdots, z_{\nu'}^*$ ausdrückt, so dass durch Einführung der Variablen $z_1, z_2, \cdots, z_{\nu'}$ für $z_1^*, z_2^*, \cdots, z_{\nu'}^*$ sowie der zu $z_1, z_2, \cdots, z_{\nu'}$ cogredienten Variablen $z_1', z_2', \cdots, z_{\nu'}'$ für $z_1^{*\prime}, z_2^{*\prime}, \cdots, z_{\nu'}^{*\prime}$ die Gruppe $\mathfrak{B}_{11}^*$ in eine ähnliche Gruppe $\mathfrak{B}_{11}$ der Form $\{\mathfrak{b}_{11}, \mathfrak{b}_{22}, \cdots, \mathfrak{b}_{\epsilon'\epsilon'}\}$ übergeht. $\mathfrak{b}_{11}, \mathfrak{b}_{22}, \cdots, \mathfrak{b}_{\epsilon'\epsilon'}$ bedeuten irreducible Gruppen.

* Wir benützen für die Gruppen $\mathfrak{A}^*$ und $\mathfrak{B}^*$ die gleichen Variablen wie für die Gruppen $\mathfrak{A}$ und $\mathfrak{B}$ des vorigen §, nur haben die Variablen der Gruppen $\mathfrak{A}^*$ und $\mathfrak{B}^*$ noch einen Stern beigefügt erhalten.

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Die neu eingeführten Variablen $z_1, z_2, \cdots, z_{\nu'}$ mögen mit den alten $z_1^*, z_2^*, \cdots, z_{\nu'}^*$ durch die Transformation:

$$(25) \qquad z_i = m_{i1} z_1^* + m_{i2} z_2^* + \cdots + m_{i\nu'} z_{\nu'}^* \qquad (i = 1, \cdots, \nu')$$

zusammenhängen; die Grössen $m_{ik}$ sind dabei Constante von nicht verschwindender Determinante. Die zu (25) cogrediente Transformation lautet:

$$(25') \qquad z_i' = m_{i1} z_1^{*\prime} + m_{i2} z_2^{*\prime} + \cdots + m_{i\nu'} z_{\nu'}^{*\prime} \qquad (i = 1, \cdots, \nu').$$

Führt man in der Gruppe $\mathfrak{B}^*$ die durch (25) und (25′) definirten Variablen $z_1, z_2, \cdots, z_{\nu'}$ bez. $z_1', z_2', \cdots, z_{\nu'}'$ ein, behält hingegen die Variablen $z_{\nu'+1}^*, z_{\nu'+2}^*, \cdots, z_n^*$ bez. $z_{\nu'+1}^{*\prime}, z_{\nu'+2}^{*\prime}, \cdots, z_n^{*\prime}$ bei, so geht die Gruppe $\mathfrak{B}^*$ der Form (22) in eine ähnliche Gruppe $\mathfrak{B}$ der Form:

$$(\mathfrak{B}): \quad \begin{matrix} \mathfrak{B}_{11} & 0 \\ \mathfrak{B}_{21} & \mathfrak{B}_{22} \end{matrix}$$

über; hierbei ist übrigens $\mathfrak{B}_{22}^*$ mit $\mathfrak{B}_{22}$ identisch.

Die Gruppen $\mathfrak{A}^*$ und $\mathfrak{B}^*$, ferner $\mathfrak{A}^*$ und $\mathfrak{A}$, sowie $\mathfrak{B}^*$ und $\mathfrak{B}$ sind ähnliche Gruppen. Mithin ist die Gruppe $\mathfrak{A}$ ähnlich zu $\mathfrak{B}$. Da $\mathfrak{A}_{11}$ und $\mathfrak{B}_{11}$ grösste vollständig reducible Gruppen von $\mathfrak{A}$ und $\mathfrak{B}$ der Form $\{\mathfrak{a}_{11}, \mathfrak{a}_{22}, \cdots, \mathfrak{a}_{ee}\}$ bez. $\{\mathfrak{b}_{11}, \mathfrak{b}_{22}, \cdots, \mathfrak{b}_{e'e'}\}$ sind, so beziehen sich, wie im § 3 bewiesen wurde, die zwei Gruppen $\mathfrak{A}_{11}$ und $\mathfrak{B}_{11}$ auf gleichviele Variablen, folglich ist $\nu = \nu'$

Die Substitution, welche die in den Variablen

$$z_1, z_2, \cdots, z_\nu, z_{\nu+1}^*, z_{\nu+2}^*, \cdots, z_n^*$$

geschriebene Gruppe $\mathfrak{B}$ in die in den Variablen

$$y_1, y_2, \cdots, y_\nu, y_{\nu+1}^*, y_{\nu+2}^*, \cdots, y_n^*$$

geschriebene Gruppe $\mathfrak{A}$ überführt, möge lauten:

$$z_i = q_{i1} y_1 + q_{i2} y_2 + \cdots + q_{i\nu} y_\nu + q_{i\nu+1} y_{\nu+1}^* + q_{i\nu+2} y_{\nu+2}^* + \cdots + q_{in} y_n^* \qquad (i = 1, \cdots, \nu),$$

$$z_i^* = q_{i1} y_1 + q_{i2} y_2 + \cdots + q_{i\nu} y_\nu + q_{i\nu+1} y_{\nu+1}^* + q_{i\nu+2} y_{\nu+2}^* + \cdots + q_{in} y_n^* \quad (i = \nu+1, \cdots, n).$$

Nach dem Resultat, das wir im § 3 in der Formel (20) erhalten haben, muss

$$(26) \qquad q_{1i} = q_{2i} = \cdots = q_{\nu i} = 0 \qquad (i = \nu+1, \cdots, n),$$

sein. Aus (25) und (23) finden wir, wenn man noch $\nu = \nu'$ beachtet, dass:

$$q_{i\nu+1} = \sum_{s=1}^{s=\nu} m_{is} q_{s\nu+1}^* \qquad (i = 1, \cdots, \nu)$$

wird. Die Determinante $|m_{ik}|$ ist von Null verschieden; daher schliessen wir aus dem Verschwinden von $q_{1\nu+1}, q_{2\nu+1}, \cdots, q_{\nu\nu+1}$, dass:

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$$q^*_{1\nu+1} = q^*_{2\nu+1} = \cdots = q^*_{\nu\nu+1} = 0$$

wird. In genau analoger Weise ergiebt sich aus den weiteren Gleichungen (26), dass:

$$q^*_{1i} = q^*_{2i} = \cdots = q^*_{\nu i} = 0 \qquad (i = \nu + 2, \cdots, n)$$

wird. Die Gleichungen (23) lauten daher:

$$(27) \quad \begin{aligned} z^*_i &= q^*_{i1} y^*_1 + q^*_{i2} y^*_2 + \cdots + q^*_{i\nu} y^*_\nu && (i = 1, \cdots, \nu), \\ z^*_i &= q^*_{i1} y^*_1 + q^*_{i2} y^*_2 + \cdots + q^*_{i\nu} y^*_\nu + q^*_{i\nu+1} y^*_{\nu+1} + \cdots + q^*_{in} y^*_n && (i = \nu + 1, \cdots, n). \end{aligned}$$

Die Matrix der durch (27) gegebenen Substitution ist folglich in der Form:

$$(Q^*): \quad \begin{matrix} Q^*_{11} & 0 \\ Q^*_{21} & Q^*_{22} \end{matrix}$$

zu schreiben; hierbei ist $Q^*_{11}$ eine Matrix mit ebensovielen Zeilen und Colonnen, wie sie die Gruppen $\mathfrak{A}^*_{11}$ und $\mathfrak{B}^*_{11}$ haben, nämlich $\nu$. $Q^*_{22}$ ist folglich eine Matrix mit $n - \nu$ Zeilen und Colonnen. Da $Q^*$ eine nicht verschwindende Determinante hat, verschwinden auch die Determinanten der zwei Matrices $Q^*_{11}$ und $Q^*_{22}$ nicht, wie aus der Form $(Q^*)$ der Matrix $Q^*$ unmittelbar ersichtlich ist.

Die Ähnlichkeit der Gruppen $\mathfrak{A}^*$ und $\mathfrak{B}^*$, welche durch die Matrix $Q^*$ vermittelt wird, kann symbolisch durch die Gleichung:

$$(28) \qquad \mathfrak{B}^* = Q^* \mathfrak{A}^* (Q^*)^{-1} \qquad \text{oder} \qquad \mathfrak{B}^* Q^* = Q^* \mathfrak{A}^*$$

ausgedrückt werden. Da $\mathfrak{A}^*$ und $\mathfrak{B}^*$ die Form (21) und (22) haben und $Q^*$, wie wir eben sahen, analog gebaut ist, folgt aus (28) unmittelbar, dass:

$$(29) \qquad \mathfrak{B}^*_{11} Q^*_{11} = Q^*_{11} \mathfrak{A}^*_{11}$$

und

$$(30) \qquad \mathfrak{B}^*_{22} Q^*_{22} = Q^*_{22} \mathfrak{A}^*_{22}$$

wird. Da $Q^*_{11}$ und $Q^*_{22}$ von Null verschiedene Determinanten haben, so sagen die Gleichungen (29) und (30) aus, die Gruppe $\mathfrak{A}^*_{11}$ ist mit der Gruppe $\mathfrak{B}^*_{11}$ und die Gruppe $\mathfrak{A}^*_{22}$, die aus $\mathfrak{A}^*$ durch Streichen der ersten $\nu$ Zeilen und $\nu$ Colonnen hervorgeht, ist mit der Gruppe $\mathfrak{B}^*_{22}$, die sich durch Streichen der ersten $\nu$ Zeilen und $\nu$ Colonnen aus $\mathfrak{B}$ ergiebt, ähnlich.

Wir können daher folgenden Satz formuliren: *Hat man irgend eine Gruppe $\mathfrak{G}$ linearer homogener Substitutionen in zwei ähnliche Gruppen $\mathfrak{A}^*$ und $\mathfrak{B}^*$ unter Hervorhebung der grössten vollständig reduciblen Gruppen transformirt, so dass $\mathfrak{A}^*$ die Form:*

$$\begin{matrix} \mathfrak{A}^*_{11} & 0 \\ \mathfrak{A}^*_{21} & \mathfrak{A}^*_{22} \end{matrix}$$

*und $\mathfrak{B}^*$ die Form:*

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$$\begin{matrix} \mathfrak{B}_{11}^* & 0 \\ \mathfrak{B}_{21}^* & \mathfrak{B}_{22}^* \end{matrix}$$

*hat und bedeuten* $\mathfrak{A}_{11}^*$ *und* $\mathfrak{B}_{11}^*$ *grösste vollständig reducible Gruppen, so beziehen sich* $\mathfrak{A}_{11}^*$ *und* $\mathfrak{B}_{11}^*$ *auf die gleiche Anzahl von Variablen und sind ähnliche Gruppen. Das Gleiche trifft für die zwei Gruppen* $\mathfrak{A}_{22}^*$ *und* $\mathfrak{B}_{22}^*$ *zu. Jede Matrix* $Q^*$ *von nicht verschwindender Determinante, die bewirkt, dass*

$$\mathfrak{B}^* = Q^* \mathfrak{A}^* (Q^*)^{-1}$$

*wird, ist notwendig von der Form:*

$$\begin{matrix} Q_{11}^* & 0 \\ Q_{21}^* & Q_{22}^* \end{matrix}$$

$Q_{11}^*$ *ist hierbei eine Matrix mit gleichvielen Zeilen und Colonnen, wie sie* $\mathfrak{A}_{11}^*$ *und* $\mathfrak{B}_{11}^*$ *haben. Die Ähnlichkeit der Gruppen* $\mathfrak{B}_{11}^*$ *und* $\mathfrak{A}_{11}^*$ *sowie der Gruppen* $\mathfrak{B}_{22}^*$ *und* $\mathfrak{A}_{22}^*$ *wird durch die Matrices* $Q_{11}^*$ *bez.* $Q_{22}^*$ *bewirkt. Es ist* $\mathfrak{B}_{11}^* = Q_{11}^* \mathfrak{A}_{11}^* (Q_{11}^*)^{-1}$, $\mathfrak{B}_{22}^* = Q_{22}^* \mathfrak{A}_{22}^* (Q_{22}^*)^{-1}$.

Obgleich im Voraufgehenden schon enthalten, sprechen wir noch besonders das Theorem aus: *Wie auch immer eine Gruppe* $\mathfrak{G}$ *linearer homogener Substitutionen unter Hervorhebung ihrer grössten vollständig reduciblen Gruppe in eine ähnliche transformirt wird, so hat die grösste vollständig reducible Gruppe, die sich bei irgend einer derartigen Darstellung der Gruppe* $\mathfrak{G}$ *ergiebt, gleichviele Variablen wie die grösste vollständig reducible Gruppe, die sich bei irgend einer anderen Darstellung von* $\mathfrak{G}$ *ergiebt, und beide Gruppen sind ähnlich. Sieht man ähnliche Gruppen linearer homogener Substitutionen als nicht verschieden an, so ist die grösste vollständig reducible Gruppe von* $\mathfrak{G}$ *eindeutig bestimmt.*

## § 5.

Durch eine Matrix $P$ von nicht verschwindender Determinante sei die Gruppe $\mathfrak{G}$ linearer homogener Substitutionen in $n$ Variablen in eine ähnliche Gruppe $\mathfrak{A}^* = P\mathfrak{G}P^{-1}$ der Form:

$$(\mathfrak{A}^*): \quad \begin{matrix} \mathfrak{A}_{11}^* & 0 & 0 & 0 & \cdots & 0 \\ \mathfrak{A}_{21}^* & \mathfrak{A}_{22}^* & 0 & 0 & \cdots & 0 \\ \mathfrak{A}_{31}^* & \mathfrak{A}_{32}^* & \mathfrak{A}_{33}^* & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \mathfrak{A}_{\mu 1}^* & \mathfrak{A}_{\mu 2}^* & \mathfrak{A}_{\mu 3}^* & \mathfrak{A}_{\mu 4}^* & \cdots & \mathfrak{A}_{\mu\mu}^* \end{matrix}$$

transformirt.* $\mathfrak{A}_{11}^*$ soll die grösste vollständig reducible Gruppe von $\mathfrak{A}^*$ sein. $\mathfrak{A}_{22}^*$ soll die grösste vollständig reducible Gruppe der Gruppe:

* Die Bezeichnung ist gegen § 4 geändert.

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$$\begin{matrix} \mathfrak{A}^*_{22} & 0 & 0 & \cdots & 0 \\ \mathfrak{A}^*_{32} & \mathfrak{A}^*_{33} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \mathfrak{A}^*_{\mu 2} & \mathfrak{A}^*_{\mu 3} & \mathfrak{A}^*_{\mu 4} & \cdots & \mathfrak{A}^*_{\mu\mu} \end{matrix}$$

sein. $\mathfrak{A}^*_{33}$ soll die grösste vollständig reducible Gruppe der Gruppe:

$$\begin{matrix} \mathfrak{A}^*_{33} & 0 & 0 & \cdots & 0 \\ \mathfrak{A}^*_{43} & \mathfrak{A}^*_{44} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \mathfrak{A}^*_{\mu 3} & \mathfrak{A}^*_{\mu 4} & \mathfrak{A}^*_{\mu 5} & \cdots & \mathfrak{A}^*_{\mu\mu} \end{matrix}$$

vorstellen. Auf diese Art soll die Bildung weitergehen. Man kann die Gruppe $\mathfrak{G}$ offenbar stets durch wiederholte Einführung linearer homogener Functionen der ursprünglichen Variablen mit constanten Coefficienten, also durch eine Matrix $P$ von nicht verschwindender Determinante, auf die eben beschriebene Weise in eine ähnliche Gruppe $\mathfrak{A}^*$ transformiren. Hat man die Gruppe $\mathfrak{G}$ linearer homogener Substitutionen in eine ähnliche Gruppe transformirt, so dass alle in der Diagonale stehenden Gruppen auf die geschilderte Art festgelegte, grösste vollständig reducible Gruppen der auf einander folgend durch Streichung aus $\mathfrak{A}^*$ entstehenden Gruppen sind, so sagen wir: die *Gruppe* $\mathfrak{G}$ *ist unter Hervorhebung aller auf einander folgender grösster vollständig reducibler Gruppen in eine ähnliche Gruppe transformirt worden.* Wir nennen auch *die zu* $\mathfrak{G}$ *ähnliche Gruppe* $\mathfrak{A}^*$ *der oben beschriebenen Form eine Darstellung der Gruppe* $\mathfrak{G}$ *unter Hervorhebung aller auf einander folgender grösster vollständig reducibler Gruppen.* $\mathfrak{A}^*_{11}$ *bezeichnen wir als die erste,* $\mathfrak{A}^*_{22}$ *als die zweite,* $\mathfrak{A}^*_{33}$ *als die dritte, u. s. w.,* $\mathfrak{A}^*_{\mu\mu}$ *als die* $\mu$*-te zu* $\mathfrak{G}$ *gehörige vollständig reducible Gruppe, die sich bei der Darstellung der Gruppe* $\mathfrak{G}$ *in der Form* $(\mathfrak{A}^*)$ *ergiebt.*

$\mathfrak{G}$ möge ausser in die Form $(\mathfrak{A}^*)$ auch unter Hervorhebung aller auf einander folgender grösster vollständig reducibler Gruppen in die ähnliche Gruppe $\mathfrak{B}^*$ der Form:

$$(\mathfrak{B}^*): \quad \begin{matrix} \mathfrak{B}^*_{11} & 0 & 0 & 0 & \cdots & 0 \\ \mathfrak{B}^*_{21} & \mathfrak{B}^*_{22} & 0 & 0 & \cdots & 0 \\ \mathfrak{B}^*_{31} & \mathfrak{B}^*_{32} & \mathfrak{B}^*_{33} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \mathfrak{B}^*_{\mu'1} & \mathfrak{B}^*_{\mu'2} & \mathfrak{B}^*_{\mu'3} & \mathfrak{B}^*_{\mu'4} & \cdots & \mathfrak{B}^*_{\mu'\mu'} \end{matrix}$$

transformirt sein. $\mathfrak{B}^*_{11}, \mathfrak{B}^*_{22}, \cdots, \mathfrak{B}^*_{\mu'\mu'}$ bedeuten grösste vollständig reducible Gruppen der auf einander folgend aus $\mathfrak{B}^*$ durch Streichung entstehenden Gruppen.

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Wir wollen jetzt den auf Seite 508 der Einleitung ausgesprochenen Satz beweisen. Wir haben also den Nachweis zu führen, dass die zwei Gruppen $\mathfrak{A}^*_{11}$ und $\mathfrak{B}^*_{11}$, die zwei Gruppen $\mathfrak{A}^*_{22}$ und $\mathfrak{B}^*_{22}$, u. s. w., schliesslich die zwei Gruppen $\mathfrak{A}^*_{\mu\mu}$ und $\mathfrak{B}^*_{\mu'\mu'}$ ähnliche Gruppen sind, also auch $\mu = \mu'$ wird.

Die Gruppen $\mathfrak{A}^*$ und $\mathfrak{B}^*$ sind mit $\mathfrak{G}$ ähnlich, folglich sind sie auch unter einander ähnlich. Mithin existirt eine Matrix $K$ von nicht verschwindender Determinante, dass $\mathfrak{B}^* = K\mathfrak{A}^*K^{-1}$ wird. $\mathfrak{A}^*_{11}$ und $\mathfrak{B}^*_{11}$ sind grösste vollständig reducible Gruppen der ähnlichen Gruppen $\mathfrak{A}^*$ und $\mathfrak{B}^*$. Daher müssen nach dem letzten Satze des § 4 die Gruppen $\mathfrak{A}^*_{11}$ und $\mathfrak{B}^*_{11}$ ähnlich sein, und die Matrix $K$ muss notwendig die Form:

$$\begin{matrix} K_{11} & 0 \\ K'_{21} & K'_{22} \end{matrix} \tag{31}$$

haben. $K_{11}$ bedeutet eine Matrix mit ebensovielen Zeilen und Colonnen, wie jede der zwei ähnlichen Gruppen $\mathfrak{A}^*_{11}$ und $\mathfrak{B}^*_{11}$ Variablen hat. Bezeichnen wir die Anzahl der Variablen, auf die sich $\mathfrak{A}^*_{11}$ und demnach auch $\mathfrak{B}^*_{11}$ bezieht, mit $\nu_1$, so sind $K'_{21}$ und $K'_{22}$ Matrices mit $n - \nu_1$ Zeilen und $\nu_1$ bez. $n - \nu_1$ Colonnen. Nach dem schon citirten Satze des § 4 führt die Matrix $K_{11}$ die Gruppe $\mathfrak{A}^*_{11}$ in die Gruppe $\mathfrak{B}^*_{11}$ über, es ist $\mathfrak{B}^*_{11} = K_{11}\mathfrak{A}^*_{11}K_{11}^{-1}$. Das Gleiche leistet $K'_{22}$ für die zwei Gruppen, die aus $\mathfrak{A}^*$ und $\mathfrak{B}^*$ durch Fortlassen der ersten $\nu_1$ Zeilen und $\nu_1$ Colonnen aus allen Matrices der Gruppen $\mathfrak{A}^*$ und $\mathfrak{B}^*$ resultiren, d. h. die Matrix $K'_{22}$ transformirt die Gruppe:

$$\begin{matrix} \mathfrak{A}^*_{22} & 0 & 0 & \cdots & 0 \\ \mathfrak{A}^*_{32} & \mathfrak{A}^*_{33} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \mathfrak{A}^*_{\mu 2} & \mathfrak{A}^*_{\mu 3} & \mathfrak{A}^*_{\mu 4} & \cdots & \mathfrak{A}^*_{\mu\mu} \end{matrix} \tag{32}$$

in die ähnliche Gruppe:

$$\begin{matrix} \mathfrak{B}^*_{22} & 0 & 0 & \cdots & 0 \\ \mathfrak{B}^*_{32} & \mathfrak{B}^*_{33} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \mathfrak{B}_{\mu' 2} & \mathfrak{B}_{\mu' 3} & \mathfrak{B}_{\mu' 4} & \cdots & \mathfrak{B}_{\mu'\mu'}. \end{matrix} \tag{33}$$

$\mathfrak{A}^*_{22}$ und $\mathfrak{B}^*_{22}$ sind nach Voraussetzung grösste vollständig reducible Gruppen der zwei ähnlichen, durch (32) und (33) dargestellten Gruppen. Daher müssen nach dem im § 4 bewiesenen Satze die Gruppen $\mathfrak{A}^*_{22}$ und $\mathfrak{B}^*_{22}$ ähnlich sein, also im besonderen gleichviele Variablen haben; ferner muss die Matrix $K'_{22}$, welche die Gruppen (32) und (33) in einander transformirt, die Form:

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$$(34)\qquad \begin{matrix} K_{22} & 0 \\ K'_{32} & K'_{33} \end{matrix}$$

haben. Hierbei bedeutet $K_{22}$ eine Matrix, die ebensoviele Zeilen und Colonnen hat, wie jede der zwei ähnlichen Gruppen $\mathfrak{A}^*_{22}$ und $\mathfrak{B}^*_{22}$ Variablen besitzt. Hat jede der Gruppen $\mathfrak{A}^*_{22}$ und $\mathfrak{B}^*_{22}$ die Anzahl $\nu_2$ Variablen, so bedeuten $K'_{32}$ und $K'_{33}$ Matrices mit $n - \nu_1 - \nu_2$ Zeilen und $\nu_2$ bez. $n - \nu_1 - \nu_2$ Colonnen. Die Matrix $K_{22}$ führt die Gruppen $\mathfrak{A}^*_{22}$ und $\mathfrak{B}^*_{22}$ in einander über; es ist $\mathfrak{B}^*_{22} = K_{22}\mathfrak{A}^*_{22}K_{22}^{-1}$. Das Gleiche leistet die Matrix $K'_{33}$ für die zwei ähnlichen Gruppen:

$$(35)\qquad \begin{matrix} \mathfrak{A}^*_{33} & 0 & 0 & \cdots & 0 \\ \mathfrak{A}^*_{43} & \mathfrak{A}^*_{44} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \mathfrak{A}^*_{\mu 3} & \mathfrak{A}^*_{\mu 4} & \mathfrak{A}^*_{\mu 5} & \cdots & \mathfrak{A}^*_{\mu\mu} \end{matrix}$$

und

$$(36)\qquad \begin{matrix} \mathfrak{B}^*_{33} & 0 & 0 & \cdots & 0 \\ \mathfrak{B}^*_{43} & \mathfrak{B}^*_{44} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \mathfrak{B}^*_{\mu' 3} & \mathfrak{B}^*_{\mu' 4} & \mathfrak{B}^*_{\mu' 5} & \cdots & \mathfrak{B}^*_{\mu'\mu'} \end{matrix}$$

$\mathfrak{A}^*_{33}$ und $\mathfrak{B}^*_{33}$ sind grösste vollständig reducible Gruppen der zwei durch (35) und (36) dargestellten ähnlichen Gruppen. Daher sind die Gruppen $\mathfrak{A}^*_{33}$ und $\mathfrak{B}^*_{33}$ ähnlich; ferner muss die Matrix $K'_{33}$ notwendig von der Form:

$$(37)\qquad \begin{matrix} K_{33} & 0 \\ K'_{43} & K'_{44} \end{matrix}$$

sein. $K_{33}$ bedeutet eine Matrix mit gleichvielen Zeilen und Colonnen, wie jede der zwei ähnlichen Gruppen $\mathfrak{A}^*_{33}$ und $\mathfrak{B}^*_{33}$ Variablen hat. Die Matrix $K_{33}$ führt die zwei Gruppen $\mathfrak{A}^*_{33}$ und $\mathfrak{B}^*_{33}$ in einander über; es ist: $\mathfrak{B}^*_{33} = K_{33}\mathfrak{A}^*_{33}K_{33}^{-1}$. Fährt man so fort, so sieht man, dass $\mu = \mu'$ wird. Ferner ergiebt sich, dass stets die $i$-ten ($i = 1, 2, \cdots, \mu$) zu $\mathfrak{G}$ gehörigen vollständig reduciblen Gruppen $\mathfrak{A}^*_{ii}$ und $\mathfrak{B}^*_{ii}$, die sich bei den Darstellungen der Gruppe $\mathfrak{G}$ in den Formen $\mathfrak{A}^*$ und $\mathfrak{B}^*$ ergeben, ähnliche Gruppen sind. Hiermit ist der am Schluss der Einleitung ausgesprochene Satz völlig bewiesen.

An das Voraufgehende knüpfen wir noch eine Bemerkung *über die Form irgend einer Substitution $K$ von nicht verschwindender Determinante, welche die Gruppe $\mathfrak{A}^*$ in die ähnliche Gruppe $\mathfrak{B}^*$ überführt, so dass $\mathfrak{B}^* = K\mathfrak{A}^*K^{-1}$ wird. Die Matrix $K$ muss notwendig die Form:*

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$$
(38) \qquad \begin{matrix} K_{11} & 0 & 0 & \cdots & 0 \\ K_{21} & K_{22} & 0 & \cdots & 0 \\ K_{31} & K_{32} & K_{33} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ K_{\mu 1} & K_{\mu 2} & K_{\mu 3} & \cdots & K_{\mu\mu} \end{matrix}
$$

haben, also analog wie die Gruppen $\mathfrak{A}^*$ und $\mathfrak{B}^*$ gebaut sein. Zur Herleitung von (38) sind nur die Formeln (31), (34), (37) und die (34), (37) analog gebildeten, sich weiter ergebenden Formeln benützt worden; ferner sind die Matrices $K'_{21}$, $K'_{32}$, $K'_{43}$ und die sich dann weiter einstellenden Matrices $K'_{54}, K'_{65}, \cdots, K'_{\mu\mu-1}$ etwas anders geschrieben worden. Wir spalteten nämlich $K'_{21}$ in Matrices mit sovielen Zeilen, wie sie $K_{22}, K_{33}, \cdots, K_{\mu\mu}$ angeben; auf diese Weise erhalten wir $K_{21}, K_{31}, \cdots, K_{\mu 1}$. Ebenso spalteten wir $K'_{32}$ in Matrices mit sovielen Zeilen, wie sie die Matrices $K_{33}, K_{44}, \cdots, K_{\mu\mu}$ angeben; auf diese Weise gewinnen wir die Matrices $K_{32}, K_{42}, \cdots, K_{\mu 2}$. Analog wurden $K'_{43}, K'_{54}, \cdots, K'_{\mu\mu-1}$ behandelt.

Die Matrices $K_{11}, K_{22}, \cdots, K_{\mu\mu}$ transformiren die Gruppen $\mathfrak{A}^*_{11}, \mathfrak{A}^*_{22}, \cdots, \mathfrak{A}^*_{\mu\mu}$ der Reihe nach in die ähnlichen Gruppen $\mathfrak{B}^*_{11}, \mathfrak{B}^*_{22}, \cdots, \mathfrak{B}^*_{\mu\mu}$. Es ist symbolisch $\mathfrak{B}^*_{11} = K_{11}\mathfrak{A}^*_{11}K_{11}^{-1}$, $\mathfrak{B}^*_{22} = K_{22}\mathfrak{A}^*_{22}K_{22}^{-1}, \cdots, \mathfrak{B}^*_{\mu\mu} = K_{\mu\mu}\mathfrak{A}^*_{\mu\mu}K_{\mu\mu}^{-1}$.

## § 6.

Es sei irgend ein unendlicher Zahlkörper oder Rationalitätsbereich $\Omega$ gegeben. Unter einem Zahlkörper oder Rationalitätsbereich $\Omega$ verstehen wir im Folgenden ein System von unendlich vielen Zahlen, das von der Vollständigkeit ist, dass die Addition, Subtraction, Multiplication und Division (ausgenommen die Division durch Null) irgend welcher Zahlen des Systems nur wieder zu Zahlen desselben Systems führt. $\mathfrak{G}$ sei in diesem Paragraphen irgend eine Gruppe linearer homogener Substitutionen, bei der die Coefficienten sämtlicher Substitutionen dem Körper $\Omega$ angehören. Die in den §§ 2–5 angestellten Untersuchungen lassen sich erweitern, wenn man jetzt sein Augenmerk auf die zu $\mathfrak{G}$ gehörigen grössten inbezug auf den Körper $\Omega$ vollständig reduciblen Gruppen linearer homogener Substitutionen richtet. Im § 2 wurden zunächst Sätze aus der Arbeit von Herrn MASCHKE im 52ten Bande der Mathematischen Annalen verwandt. Zu ihrer Herleitung benützt Herr MASCHKE nur Operationen, die sich mittelst rationaler Verbindungen der Substitutionscoefficienten der vorgelegten Gruppe durchführen lassen. Die Maschkeschen Sätze sind daher auch für Gruppen mit Coefficienten aus $\Omega$ gültig. Man hat mithin die Resultate: Jede Gruppe linearer homogener Substitutionen mit Coefficienten aus $\Omega$, deren Ordnungszahl um wenigstens zwei kleiner als die Anzahl der Substitutionsvariablen ist, ist eine inbezug auf den Körper $\Omega$ reducible Gruppe.

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Kann man aus irgend einer Gruppe $\mathfrak{b}_{11}$ linearer homogener Substitutionen mit Coefficienten aus $\Omega$ aus einer Zeile, etwa der $l$-ten, $g_1 - 1$ Coefficienten:

$$b_{l1},\ b_{l2},\ \cdots,\ b_{l\,u-1},\ b_{l\,u+1},\ \cdots,\ b_{lg_1},$$

herausgreifen, dass jede der Determinanten, gebildet aus den entsprechenden Coefficienten jeder möglichen Combination von je $g_1 - 1$ Substitutionen der Gruppe $\mathfrak{b}_{11}$ verschwindet, so ist die Gruppe $\mathfrak{b}_{11}$ eine inbezug auf den Körper $\Omega$ reducible Gruppe linearer homogener Substitutionen.

Bei der Erweiterung der Betrachtungen des § 2 ist ferner zu beachten, dass zwei ähnliche Gruppen linearer homogener Substitutionen mit Coefficienten aus einem Körper $\Omega$ stets durch eine lineare homogene Substitution $Q$ von nicht verschwindender Determinante mit ausschliesslich nur dem Körper $\Omega$ angehörigen Coefficienten in einander übergeführt werden können; denn die Substitutionscoefficienten sind in diesem Falle nur durch Auflösen mit einander verträglicher linearer Gleichungen mit Coefficienten aus $\Omega$ zu finden. Geht man die Untersuchungen des § 2 durch, so sieht man, dass falls $\mathfrak{A}$ und $\mathfrak{B}$ Gruppen mit Coefficienten aus $\Omega$ sind und die überführende Substitution $Q$ ebenfalls nur Coefficienten aus $\Omega$ hat, alle im § 2 verwandten Operationen auch kein Heraustreten aus $\Omega$ erfordern, z. B. sind auch die Grössen $d_1, d_2, \cdots, d_\nu, e_1, e_2, \cdots, e_{g_1}$ der Gleichung (11), da sie im Fall ihrer Existenz durch die linearen Gleichungen $(11_1)$ mit Coefficienten aus $\Omega$ bestimmt sind, innerhalb des Körpers $\Omega$ bestimmbar. Wir gewinnen daher als Erweiterung des Seite 515 gegebenen Satzes das Resultat:

Sind $\mathfrak{A}$ und $\mathfrak{B}$ ähnliche Gruppen linearer homogener Substitutionen der Form $(\mathfrak{A})$ und $(\mathfrak{B})$ (vgl. S. 513) mit Coefficienten aus $\Omega$ und werden durch irgend eine Transformation (5) von nicht verschwindender Determinante mit Coefficienten aus $\Omega$ und die zu ihr cogrediente (5′) die Substitutionen der einen Gruppe in die der anderen übergeführt, so müssen, wenn $\mathfrak{b}_{11}$ eine bezüglich des Körpers $\Omega$ irreducible Gruppe linearer homogener Substitutionen ist und zwischen $y_1, y_2, \cdots, y_\nu$ und $z_1, z_2, \cdots, z_{g_1}$ auch nur eine einzige lineare homogene Relation mit constanten Coefficienten besteht, in der überführenden Transformation (5) die ersten $g_1$ Gleichungen $(g_1 < n)$:

$$(5_1) \qquad z_i = q_{i1}y_1 + q_{i2}y_2 + \cdots + q_{i\nu}y_\nu \qquad (i = 1, \cdots, g_1)$$

lauten, also für $i \leqq g_1$, $k > \nu$ müssen die Grössen $q_{ik}$ notwendig Null sein.

Nach Erweiterung des Resultats des § 2 definiren wir, was es heissen soll: *die Gruppe $\mathfrak{G}$ linearer homogener Substitutionen mit Coefficienten aus $\Omega$ ist unter Hervorhebung ihrer inbezug auf den Körper $\Omega$ grössten vollständig reduciblen Gruppe $\mathfrak{A}_{11}^*$ in die Gruppe $\mathfrak{A}^*$, deren Substitutionen auch nur ausschliesslich Coefficienten aus $\Omega$ besitzen sollen, transformirt.* Wir benützen die angegebene Bezeichnung, wenn die Gruppe $\mathfrak{G}$ linearer homogener Substi-



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tutionen in $n$ Variablen mit Coefficienten aus $\Omega$ in die ähnliche Gruppe $\mathfrak{A}^*$, die auch nur ausschliesslich Coefficienten aus $\Omega$ haben soll und die Form:

$$(\mathfrak{A}^*): \quad \begin{matrix} \mathfrak{A}_{11}^* & 0 \\ \mathfrak{A}_{21}^* & \mathfrak{A}_{22}^* \end{matrix}$$

hat, transformirt ist, dabei sollen $\mathfrak{A}_{11}^*$, $\mathfrak{A}_{21}^*$, $\mathfrak{A}_{22}^*$ einen Inbegriff von Matrices mit Coefficienten aus $\Omega$ mit $\nu$ Zeilen und $\nu$ Colonnen, bez. $n-\nu$ Zeilen und $\nu$ Colonnen, bez. $n-\nu$ Zeilen und $n-\nu$ Colonnen bedeuten, $\mathfrak{A}_{11}^*$ soll ferner eine inbezug auf den Körper $\Omega$ vollständig reducible Gruppe linearer homogener Substitutionen (vgl. wegen des Begriffes S. 509) definiren, und es soll keine Matrix $L$ von nicht verschwindender Determinante existiren, welche die Gruppe $\mathfrak{G}$ in eine ähnliche Gruppe $L\mathfrak{G}L^{-1}$ überführt, die auch nur ausschliesslich Coefficienten aus $\Omega$ hat und die Form:

$$\begin{matrix} \mathfrak{A}_{11}^* & 0 & 0 \\ 0 & \mathfrak{a}_{\epsilon+1,\,\epsilon+1} & 0 \\ \mathfrak{A}_{31}^* & \mathfrak{A}_{32}^* & \mathfrak{A}_{33}^* \end{matrix}$$

besitzt, wobei $\mathfrak{a}_{\epsilon+1,\,\epsilon+1}$ eine inbezug auf den Körper $\Omega$ irreducible Gruppe linearer homogener Substitutionen mit Coefficienten aus $\Omega$ in $f_{\epsilon+1}$ $(f_{\epsilon+1} > 0)$ Variablen bedeute, und demnach $\mathfrak{A}_{31}^*$, $\mathfrak{A}_{32}^*$, $\mathfrak{A}_{33}^*$ Matrices mit $n-\nu-f_{\epsilon+1}$ Zeilen und $\nu$ bez. $f_{\epsilon+1}$ bez. $n-\nu-f_{\epsilon+1}$ Colonnen vorstellen.

Geht man die Betrachtungen der § 3 und § 4 durch, so sieht man, dass man alle dort verwandten Operationen innerhalb des Körpers $\Omega$ ausführen kann. Daher erhält man als Erweiterung der im §4 gegebenen Sätze folgendes Resultat: *Hat man irgend eine Gruppe $\mathfrak{G}$ linearer homogener Substitutionen mit Coefficienten aus $\Omega$ in zwei ähnliche Gruppen $\mathfrak{A}^*$ und $\mathfrak{B}^*$, die auch nur Coefficienten aus $\Omega$ haben, unter Hervorhebung ihrer inbezug auf $\Omega$ grössten vollständig reduciblen Gruppen transformirt, so dass $\mathfrak{A}^*$ die Form:*

$$\begin{matrix} \mathfrak{A}_{11}^* & 0 \\ \mathfrak{A}_{21}^* & \mathfrak{A}_{22}^* \end{matrix}$$

*und $\mathfrak{B}^*$ die Form:*

$$\begin{matrix} \mathfrak{B}_{11}^* & 0 \\ \mathfrak{B}_{21}^* & \mathfrak{B}_{22}^* \end{matrix}$$

*hat, und bedeuten $\mathfrak{A}_{11}^*$ und $\mathfrak{B}_{11}^*$ inbezug auf den Körper $\Omega$ grösste vollständig reducible Gruppen von $\mathfrak{A}^*$ und $\mathfrak{B}^*$, so beziehen sich $\mathfrak{A}_{11}^*$ und $\mathfrak{B}_{11}^*$ auf die gleiche Anzahl von Variablen und sind ähnliche Gruppen. Das Gleiche trifft für die zwei Gruppen $\mathfrak{A}_{22}^*$ und $\mathfrak{B}_{22}^*$ mit Coefficienten aus $\Omega$ zu. Jede Matrix $Q^*$ von nicht verschwindender Determinante mit Coefficienten aus $\Omega$, die bewirkt, dass*

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$$\mathfrak{B}^* = Q^* \mathfrak{A}^* (Q^*)^{-1}$$

*wird, ist notwendig von der Form:*

$$\begin{matrix} Q_{11}^* & 0 \\ Q_{21}^* & Q_{22}^* \end{matrix}.$$

*$Q_{11}^*$ ist hierbei eine Matrix mit gleichvielen Zeilen und Colonnen, wie sie $\mathfrak{A}_{11}^*$ und $\mathfrak{B}_{11}^*$ haben. Die Ähnlichkeit der Gruppen $\mathfrak{B}_{11}^*$ und $\mathfrak{A}_{11}^*$ wird durch die Matrix $Q_{11}^*$, die von $\mathfrak{B}_{22}^*$ und $\mathfrak{A}_{22}^*$ durch die Matrix $Q_{22}^*$ bewirkt. Es ist:*

$$\mathfrak{B}_{11}^* = Q_{11}^* \mathfrak{A}_{11}^* (Q_{11}^*)^{-1}$$
$$\mathfrak{B}_{22}^* = Q_{22}^* \mathfrak{A}_{22}^* (Q_{22}^*)^{-1}.$$

Es gilt also das Resultat: *Wie auch immer eine Gruppe $\mathfrak{G}$ linearer homogener Substitutionen mit Coefficienten aus $\Omega$ unter Hervorhebung ihrer inbezug auf $\Omega$ grössten vollständig reduciblen Gruppe in eine ähnliche Gruppe mit Coefficienten aus $\Omega$ transformirt wird, so hat die inbezug auf $\Omega$ grösste vollständig reducible Gruppe, die sich bei irgend einer derartigen Darstellung der Gruppe $\mathfrak{G}$ ergiebt, gleichviele Variablen wie die inbezug auf $\Omega$ grösste vollständig reducible Gruppe, die sich bei irgend einer anderen derartigen Darstellung von $\mathfrak{G}$ ergiebt, und beide Gruppen sind ähnlich. Sieht man ähnliche Gruppen linearer homogener Substitutionen als nicht verschieden an, so ist die inbezug auf $\Omega$ grösste vollständig reducible Gruppe von $\mathfrak{G}$ eindeutig bestimmt.*

Eine Gruppe $\mathfrak{G}$ linearer homogener Substitutionen mit Coefficienten aus $\Omega$ kann analog wie im § 5 in eine ähnliche Gruppe mit Coefficienten aus $\Omega$ transformirt werden, so dass hierbei jetzt alle auf einander folgenden inbezug auf $\Omega$ grössten vollständig reduciblen Gruppen hervorgehoben werden. Wir bezeichnen eine derartige Transformation von $\mathfrak{G}$ in eine ähnliche Gruppe mit Coefficienten aus $\Omega$ als eine Darstellung der Gruppe $\mathfrak{G}$ *unter Hervorhebung aller auf einander folgender inbezug auf $\Omega$ grösster, vollständig reducibler Gruppen.*

Da zwei ähnliche Gruppen mit Coefficienten aus $\Omega$ stets durch eine lineare homogene Substitution von nicht verschwindender Determinante mit Coefficienten aus $\Omega$ in einander transformirbar sind, so bleiben auch alle Resultate des § 5 für Gruppen $\mathfrak{G}$ mit Coefficienten aus $\Omega$, wie wir sie jetzt betrachten, gültig.

*Zu jeder Gruppe $\mathfrak{G}$ linearer homogener Substitutionen mit Coefficienten aus $\Omega$ gehören $\mu$ inbezug auf $\Omega$ vollständig reducible Gruppen linearer homogener Substitutionen mit Coefficienten aus $\Omega$; diese sind, wenn man ähnliche Gruppen als nicht verschieden ansieht, sogar ihrer Reihenfolge nach eindeutig festgelegt. Man kann also von der ersten, zweiten, u. s. w. bis $\mu$-ten inbezug auf $\Omega$ vollständig reduciblen zu $\mathfrak{G}$ gehörigen Gruppe sprechen. Die Gesamtzahl der Variablen der $\mu$ Gruppen ist gleich der Variablenzahl von $\mathfrak{G}$.*

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# ON THE CAYLEY-VERONESE CLASS OF CONFIGURATIONS*

BY

WALTER B. CARVER

CAYLEY, in his paper *Sur quelques théorèmes de la géomètrie de position*,† first calls attention to the figures obtained by taking the section, by a plane or 3-dimensional space, of the complete $n$-point (viz., $n$ points, and the $\binom{n}{2}$ lines, $\binom{n}{3}$ planes, etc., determined by them) in a flat space of $\nu$ dimensions.

Later, VERONESE discusses more fully ‡ the nature of this class of configurations, treating in general the configurations thus obtained in $r$ dimensions. Both CAYLEY and VERONESE state that these same configurations can also be obtained as *projections* of higher-dimensional figures.

Among the posthumous papers of CAPORALI is a paper: *Sulla teoria delle configurazioni* § (1879), giving, without proof, a number of theorems concerning a certain class of *plane* configurations. Although there is no mention of space of higher dimensions in CAPORALI's paper, his configurations may be regarded as projections upon a plane, or sections by a plane, of simple higher-dimensional figures; and they belong, in fact, to the CAYLEY-VERONESE class indicated above. CAPORALI's theorems become almost self-evident when viewed from this standpoint.

In a paper *Ueber Polyedrale Configurationen*,‖ DE VRIES discusses a special sub-class of the Cayley-Veronese configurations, viz., the plane sections of complete $n$-points in ordinary space.

It is my purpose in this paper to state a number of general theorems for the Cayley-Veronese configurations in space of $r$-dimensions (extensions, for the most part, of CAPORALI's theorems for the plane case); to give a construction for the quadric polarity in the plane and in 3-dimensional space, based upon certain of these configurations; and finally to call attention to certain peculiar sets of conics in the plane, connected with plane configurations of this class. ¶

* Received for publication, August 23, 1904. Presented to the Society October 28, 1905.

† Crelle's Journal, vol. 31 (1846); Collected Papers, vol. 1, p. 317.

‡ *Behandlung der projectivischen Verhältnisse der Räume von verschiedenen Dimensionen durch das Princip des Projicirens und Schneidens*, Mathematische Annalen, vol. 19 (1882).

§ *Memorie di Geometria*, p. 262.

‖ Mathematische Annalen, vol. 34 (1889).

¶ There may be mentioned a few of the very many instances of the incidental occurrence of these configurations.

The well-known figure of two perspective triangles with their center and axis of perspection



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In this paper, $S_\nu$ denotes, as usual, a linear or flat space of $\nu$ dimensions (in points). The words "copoint," "coplane," etc., are used to specify the dual of the point, plane, etc., in the space $S_\nu$; thus, in $S_\nu$, a copoint is an $S_{\nu-1}$, and a complete $n$-copoint is the dual of the complete $n$-point. The number of combinations of $n$-things taken $\nu$ at a time will be denoted by $\binom{n}{\nu}$, with the usual convention that $\binom{n}{0} = \binom{n}{n} = 1$.

## § 1. *General Theorems.*

Let $\Gamma^\nu_{n,r}$ denote the configuration obtained in an $S_r$ by cutting a complete $n$-point in $S_\nu$ by the $S_r$. The incidence relations of the configuration * may be shown by means of the square matrix of the $r$th order

$$(\alpha_{pq}) \qquad (p, q = 0, 1, \cdots, r-1)$$

in which $\alpha_{pp}$ denotes the number of $S_p$'s in the configuration, and $\alpha_{pq}$ $(p \neq q)$ denotes the number of $S_p$'s incident with each $S_q$.† For the $\Gamma^\nu_{n,r}$ we have

$$\alpha_{pp} = \binom{n}{p+\nu-r+1},$$

$$\alpha_{pq} = \binom{q+\nu-r+1}{p+\nu-r+1} \qquad (q > p),$$

$$\alpha_{pq} = \binom{n-q-\nu+r-1}{p-q} \qquad (p > q).$$

If each of the $n$-points in $S_\nu$ is denoted by a single letter, any $S_p$ of the $\Gamma^\nu_{n,r}$ is denoted by a combination of $p + \nu - r + 1$ of these $n$ letters; and any particular $S_p$ and $S_q$ $(p > q)$ of the configuration are incident if the $q + \nu - r + 1$

belongs to this class. It was probably first mentioned by DESARGUES in 1636 (cf. ENESTRÖM, Bibliotheca Mathematica, 1885), and is discussed in the aforementioned paper of CAYLEY, and also in KANTOR's paper on $(3, 3)_{10}$ *Configurations*, Sitzungsberichte der K. Akad., Wien, vol. 84 (1881). VERONESE shows, in his memoirs, *Nuovi teoremi sull' hexagrammum mysticum*, Atti della R. Accad. dei Lincei, vol. 1 (1877), and *Interprétations géométriques de la theorie des substitutions de n-letters*, etc., Annali di Mathematica, vol. 11 (1881), that the sixty lines of the Pascal hexagram form six of these Desargues figures.

The configuration of two perspective tetrahedra with their center and plane of perspection occurs in KLEIN's memoir, *Zur Theorie der Liniencomplexe des ersten und zweiten Grades*, Mathematische Annalen, vol. 2 (1870); and also in RICHMOND's paper *On the figure of six points in space of four dimensions*, Quarterly Journal of Pure and Applied Mathematics, vol. 31 (1900).

WHITEHEAD treats of the figure of two perspective reference figures in space of any number of dimensions (*Universal Algebra*, vol. 1, p. 139).

CAYLEY, loc. cit., shows that a peculiar special case of the plane figure of fifteen lines and twenty points, obtained as a section of the figure of two perspective tetrahedra, occurs in the Pascal hexagram.

*Cf. VERONESE, Mathematische Annalen, vol. 19 (1882), p. 171; or *Grundzüge der Geometrie*, p. 615.

†This is the notation suggested by Professor MOORE, *Tactical Memoranda*, American Journal of Mathematics, vol. 18 (1896), p. 265.

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letters belonging to the latter are contained among the $p+\nu-r+1$ letters belonging to the former.

Suppose, on the other hand, that one starts with the complete $n$-copoint in $S_\nu$, and projects it upon an $S_r$, taking a co-$S_r$, or $S_{\nu-r+1}$, as the projection-center. By such a projection,* each point, line, plane, $\cdots$ or $S_{r-1}$ of the original figure is sent into a new point, line, plane, $\cdots$ or $S_{r-1}$ in the $S_r$. The $S_r$'s, $S_{r+1}$'s, $\cdots$ and $S_{\nu-1}$'s of the original figure are lost. The configuration thus obtained in the $S_r$ is evidently the dual of the $\Gamma^\nu_{n,r}$, and may be represented by the symbol, $C^\nu_{n,r}$.

The incidence relations of the $C^\nu_{n,r}$ may be shown by the matrix

$$(a_{pq}) \qquad (p, q=0, 1, \cdots, r-1)$$

where

$$a_{pp} = \binom{n}{\nu-p},$$

$$a_{pq} = \binom{n-\nu+q}{q-p}, \qquad (q>p).$$

$$a_{pq} = \binom{\nu-q}{\nu-p}, \qquad (p>q).$$

Any element, $S_p$, of a $C^\nu_{n,r}$ is named by a combination of $\nu-p$ letters out of $n$. If we re-name all the elements, assigning as a new name to each $S_p$ the $n-\nu+p$ letters *not contained in its old name*, we see at once that the $C^\nu_{n,r}$ under this new lettering becomes a $\Gamma^{n-\nu+r-1}_{n,r}$. A $\Gamma^\nu_{n,r}$, with a similar change of lettering, becomes a $C^{n-\nu+r-1}_{n,r}$. (It will be convenient hereafter to write $\mu$ for $n-\nu+r-1$.) Since a $\Gamma^\nu_{n,r}$ and a $C^\nu_{n,r}$ are dual figures, it follows that a $C^\nu_{n,r}$ and a $C^\mu_{n,r}$ are dual, as are also a $\Gamma^\nu_{n,r}$ and a $\Gamma^\mu_{n,r}$. Also a $\Gamma^\nu_{n,r}$ or a $C^\nu_{n,r}$ is self-dual if $n-\nu+r-1=\nu$, i. e., if $2\nu=n+r-1$.†

While the symbols $C^\nu_{n,r}$ and $\Gamma^\mu_{n,r}$ represent the same configuration, it is convenient to have both. One may use $\Gamma^\mu_{n,r}$, when considering the figure as a *section* of a $\mu$-dimensional figure, and $C^\nu_{n,r}$ when regarding it as a *projection* of a $\nu$-dimensional figure. ‡

THEOREM I. *If in a $C^\nu_{n,r}$ we separate all those elements whose names contain a certain letter from those whose names do not contain it, the latter form a $C^\nu_{n-1,r}$ and the former a $C^{\nu-1}_{n-1,r}$. Each $S_p$ of the $C^\nu_{n-1,r}$ $(p=1, 2, 3, \cdots, r-1)$ is incident with an $S_{p-1}$ of the $C^{\nu-1}_{n-1,r}$.*

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* This is what VERONESE calls an *eindeutige* projection. Cf. *Grundzüge der Geometrie*, p. 614.

† Cf. VERONESE, loc. cit., p. 164; also *Grundzüge der Geometrie*, p. 623.

‡ These configurations may, of course, be defined without reference to space of higher dimensions than the $S_r$ in which they lie; but it should be understood that *the matrix giving their incidence relations is not sufficient so to define them.* We must add the condition that they be so arranged that they can be lettered, $p+\nu-r+1$ letters to each $S_p$, in such a way that the $q+\nu-r+1$ letters of any $S_q$ incident with a given $S_p$ $(p>q)$ shall be contained among the $p+\nu-r+1$ letters of the $S_p$.

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When a $C^{\nu}_{n-1,r}$ and a $C^{\nu-1}_{n-1,r}$ are so related we may speak of them as being *chiastically incident*, or, for present purposes, simply as *chiastic*.*

THEOREM II. *With respect to two letters, say a and b, the $C^{\nu}_{n,r}$ breaks up into a $C^{\nu}_{n-2,r}$, the names of whose elements contain neither a nor b; two $C^{\nu-1}_{n-2,r}$'s, one containing a but not b, the other b but not a; and a $C^{\nu-2}_{n-2,r}$ containing both a and b.*

The two $C^{\nu-1}_{n-2,r}$'s are both chiastic with the $C^{\nu}_{n-2,r}$ and also with the $C^{\nu-2}_{n-2,r}$. Moreover each $S_p$ of the $C^{\nu}_{n-2,r}$ $(p = 2, 3, \cdots, r-1)$ is incident with an $S_{p-2}$ of the $C^{\nu-2}_{n-2,r}$; and we may speak of a $C^{\nu}_{n-2,r}$ and a $C^{\nu-2}_{n-2,r}$ so related as being chiastic.

In general, any two configurations whose symbols have the same subscripts but different superscripts, as $C^{\nu}_{n,r}$ and $C^{\nu-\kappa}_{n,r}$, are chiastic if each $S_p$ of the $C^{\nu}_{n,r}$ is incident (in a certain order) with an $S_{p-\kappa}$ of the $C^{\nu-\kappa}_{n,r}$. It is evident that we cannot have chiasm between these two figures unless $\kappa \leqq r-1$. If a $C^{\nu-\kappa}_{n,r}$ is chiastic with a $C^{\nu}_{n,r}$, and the $C^{\nu}_{n,r}$ is chiastic with a $C^{\nu+l}_{n,r}$, then the $C^{\nu-\kappa}_{n,r}$ is chiastic with the $C^{\nu+l}_{n,r}$.

THEOREM III. *The breaking up of a $C^{\nu}_{n,r}$ with respect to $S$ given letters may be symbolically expressed by the formula:*

$$C^{\nu}_{n,r} \equiv C^{\nu}_{n-s,r} + sC^{\nu-1}_{n-s,r} + \binom{s}{2} C^{\nu-2}_{n-s,r} + \cdots + \binom{s}{2} C^{\nu-s+2}_{n-s,r} + sC^{\nu-s+1}_{n-s,r} + C^{\nu-s}_{n-s,r}.$$

Any $C^{\nu-\kappa}_{n-s,r}$ of this expression is made up of those elements of the $C^{\nu}_{n,r}$ whose symbols contain a certain $\kappa$ of the $s$ letters but not the remaining $s-\kappa$. A $C^{\nu-\kappa}_{n-s,r}$ and a $C^{\nu-\kappa+l}_{n-s,r}$ are chiastic if the $\kappa - l$ letters which distinguish the latter are contained among the $\kappa$ letters which distinguish the former.

THEOREM IV. *There are $\binom{n}{s}\binom{s}{\kappa} C^{\nu-\kappa}_{n-s,r}$'s contained in every $C^{\nu}_{n,r}$.*

THEOREM V. *If $sC^{\nu-1}_{n-s,r}$'s are chiastic with a $C^{\nu}_{n-s,r}$, any two of the $C^{\nu-1}_{n-s,r}$'s determine a $C^{\nu-2}_{n-s,r}$ with which they are both chiastic; three of the $C^{\nu-1}_{n-s,r}$'s determine thus three $C^{\nu-2}_{n-s,r}$'s which are chiastic with and determine a $C^{\nu-3}_{n-s,r}$; $\cdots$ and finally, the $sC^{\nu-1}_{n-s,r}$'s determine $sC^{\nu-s+1}_{n-s,r}$'s which are chiastic with and determine a $C^{\nu-s}_{n-s,r}$; and the entire figure thus determined is a $C^{\nu}_{n,r}$.*

When $\nu = n - s$, this theorem is VERONESE's "perspective pyramid" theorem. †

THEOREM VI. *If a $C^{\nu}_{n,r}$ and a $C^{\nu-\kappa}_{n,r}$ are chiastic there are $\infty^{\kappa l - l^2}$ $C^{\nu-l}_{n,r}$'s $(\kappa > l)$ chiastic with both the $C^{\nu}_{n,r}$ and the $C^{\nu-\kappa}_{n,r}$. If we fix arbitrarily one $S_{r-l-1}$ of the $C^{\nu-l}_{n,r}$ lying in any $S_{r-1}$ of the $C^{\nu}_{n,r}$ and passing through the corresponding $S_{r-\kappa-1}$ of the $C^{\nu-\kappa}_{n,r}$, the $C^{\nu-l}_{n,r}$ is then completely determined.*

THEOREM VII. *There are $\infty^{\kappa(\nu-\kappa+1)}$ $C^{\nu-\kappa}_{n,r}$'s chiastic with any given $C^{\nu}_{n,r}$. We may fix arbitrarily the $\nu - \kappa + 1$ points of the $C^{\nu-\kappa}_{n,r}$ lying on the*

* The word chiastic is similarly used by Sir Robert Ball.

† Loc. cit., p. 171; also *Grundzüge der Geometrie*, p. 614.

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*$\nu - \kappa + 1$ $S_\kappa$'s passing through any $S_{\kappa-1}$ of the $C^\nu_{n,r}$, and the $C^{\nu-\kappa}_{n,r}$ is then completely determined.*

THEOREM VIII. *There are $\infty^{\kappa(n-\nu+r-\kappa)}$ $C^{\nu+\kappa}_{n,r}$'s chiastic with any given $C^\nu_{n,r}$. We may fix arbitrarily the $n - \nu + r - \kappa$ copoints of the $C^{\nu+\kappa}_{n,r}$ passing through the $n - \nu + r - \kappa$ $S_{r-\kappa-1}$'s lying in any $S_{r-\kappa}$ of the $C^\nu_{n,r}$, and the $C^{\nu+\kappa}_{n,r}$ will then be completely determined.*

THEOREM IX. *A $C^\nu_{n,r}$ is determined by*

$$r(n-\nu+r)+(\nu-r)(n-\nu+r-1)=n\nu-(\nu+1)(\nu-r)$$

*arbitrary constants.*

## § 2. *Constructions of Quadric Polar Systems.*

It is well known that the Desargues configuration in the plane, the $\Gamma^3_{5,2}$, determines a certain conic or polar system with respect to which it is self-polar. VERONESE shows,* more generally, that the complete figure of two perspective "fundamental pyramids" in $S_r$, i. e., the configuration $\Gamma^{r+1}_{r+3,r}$, determines a certain $(r-1)$ dimensional quadric spread, $\Phi_{r-1}$, with respect to which it is self-polar. Each point of the $\Gamma^{r+1}_{r+3,r}$, named by two letters is the pole, with respect to $\Phi_{r-1}$, of that particular copoint which is named by the remaining $r+1$ letters.

Let us consider first the case of the plane figure. CAYLEY shows† that the $\Gamma^3_{5,2}$ breaks up into two pentagons, each inscribed and circumscribed to the other. If we take any cyclic arrangement of the five letters, as $abcde$ ($= bcdea = edcba$, etc.), it will represent one of these pentagons, i. e., the pentagon whose vertices are $ab$, $bc$, $cd$, $de$, and $ea$, and whose sides are $abc$, $bcd$, $cde$, $dea$, and $eab$. The remaining five points and five lines of the configuration form the other pentagon, i. e., the pentagon $acebd$. In either of these pentagons, each vertex is the pole of the opposite side with respect to the polarity, $\Phi_1$, of the configuration.

To say that a quadric polarity sends each vertex of a pentagon into the opposite side places five linear conditions on the polarity, which is just sufficient to determine it uniquely.‡ A polarity may then be given by a pentagon.

Of the two pentagons which form a $\Gamma^3_{5,2}$, one may be taken arbitrarily, and any line or point of the second to satisfy a single condition. (A $\Gamma^3_{5,2}$ depends

* Loc. cit., p. 193.

† *Crelle's Journal*, vol. 31 (1846), p. 213; also *Collected Works*, vol. I, p. 317. This fact was probably first noticed by J. T. GRAVES. Cf. Philosophical Magazine, vol. 15 (1839), p. 131.

‡ Cf. REYE, *Geometrie der Lage* (3d edition), vol. 2, p. 125; and KOHN, Mathematische Annalen, vol. 46 (1895), p. 303.

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upon $3.5-(3+1)(3-2)=11$ arbitrary constants.* Ten of these constants are fixed in fixing the first pentagon, leaving one arbitrary constant to the second pentagon). But the polarity is determined as soon as the first pentagon is fixed, hence the second pentagon is necessarily self-polar with respect to the polarity determined by the first pentagon.

This gives a method for constructing the polar of a given point with respect to a polarity, when the polarity is given by a self polar pentagon.

Consider the given pentagon as the pentagon $abcde$ of a $\Gamma^3_{5,2}$. If we make any side, say $ace$, of the second pentagon, $acebd$, pass through the given point, $x$, this second pentagon is then determined, and the polar of $x$ must pass through the point $bd$. Now draw a different second pentagon, $a'c'e'b'd'$, making a different side, say $b'd'a'$, pass through $x$. Then the polar of $x$ must pass through the point $c'e'$ of this pentagon, and is thus determined.

The essential part of the construction may be remembered conveniently as follows: Number the vertices and sides of the pentagon consecutively $1, 2, 3, \cdots, 10$, assigning to the vertices the odd, and to the sides the even numbers. Then join $x$ to 1 to meet 4 at $A$, $A$ to 7 to meet 10 at $B$, $B$ to 3 to meet 6 at $C$, and $C$ is then on the polar of $x$. Shift the numbers, replacing 3 by 1, 4 by 2, 5 by 3, $\cdots$, 2 by 10, and repeat the above process, obtaining thus a point $C'$. Then $CC'$ is the polar of $x$.

A dual construction gives the pole of a given line.

Two pentagons, representing two polarities $\Phi$ and $\Phi'$, of which the first sends points into lines and the second sends lines into points, give a collineation. The common polar triangle of $\Phi$ and $\Phi'$ is the fixed triangle of the collineation. If $ABCDE$ is a pentagon giving the polarity $\Phi$, the triangle $B$, $D$, $(\overline{AB}\cdot\overline{DE})$ is evidently a polar triangle of $\Phi$.† If then a given triangle, with vertices $a$, $b$, and $c$, and sides $\alpha$, $\beta$, and $\gamma$, is to be the fixed triangle of a collineation which is in addition to send a given point, $P$, into a given point, $P'$, we can at once construct two pentagons which will give this collineation. Let $\delta$ be any line. Our two pentagons may be $P$, $a$, $(\beta\cdot\delta)$, $(\delta\cdot\alpha)$, $b$ and $P'$, $a$, $(\beta\cdot\delta)$, $(\delta\cdot\alpha)$, $b$.

The two pentagons $ABCDE$ and $ABCDE'$, where $E'$ lies on $DE$, give a collineation in which the point $D$ and all points on the line $AB$ are fixed points, while $\overline{AB}$, and all lines through the point $D$, are fixed lines. This collineation sends $E$ into $E'$, and it sends any point, $P$, into a point $P'$ on the line $PD$, such that the cross ratio $P$, $P'$, $D$, $\overline{AB}$ is equal to the cross ratio $E$, $E'$, $D$, $\overline{AB}$.

CLEBSCH ‡ calls attention to a collineation determined by a pentagon. With

* Cf. § 1, theorem IX.

† Cf. REYE, *Geometrie der Lage* (3d edition), vol. 2, p. 125.

‡ *Ueber die Anwendung der quadratischen Substitution auf die Gleichungen 5ten Grades und die geometrische Theorie des ebenen Fünfseits*, Mathematische Annalen, vol. 4 (1871).

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the pentagon $ABCDE$ is naturally associated the pentagon of the diagonals, $ACEBD$, and these two pentagons give the CLEBSCH collineation.

Passing now to the three-dimensional case,* we have the configuration, $\Gamma^4_{6,3}$, which consists of two perspective tetrahedra with their center and plane of perspection. The points, lines and planes are denoted respectively by combinations of two, three and four letters out of six. The point $ab$ is the pole of the plane $cdef$, and the lines $abc$ and $def$ are conjugate, with respect to a certain polarity $\Phi_2$.

Taking any cyclic arrangement of the six letters, as $abcdef$, they represent a skew-hexagon whose six vertices are $ab$, $bc$, $cd$, $de$, $ef$ and $fa$; six lines are $abc$, $bcd$, $cde$, $def$, $efa$ and $fab$; and six planes are $abcd$, $bcde$, $cdef$, $defa$, $efab$ and $fabc$.

In addition to the six vertices of this hexagon there are nine other points of the configuration, viz.,

$$ac,\ ce,\ ea,\ bd,\ df,\ fb,\ ad,\ be \text{ and } cf,$$

each of which lies in two planes of the hexagon, the first six lying on lines and the last three on the intersection of opposite planes of the hexagon. (For instance, $ac$ lies on the line $abc$, but $ad$ lies on the intersection of $abcd$ and $defa$.) Similarly there are nine planes of the configuration besides the six of the hexagon, viz.,

$$abce,\ bcdf,\ cdea,\ defb,\ efac,\ fabd,\ bcef,\ cdfa \text{ and } deab,$$

the first six passing through lines of the hexagon, and the last three through the joins of opposite vertices.

From these nine points and nine planes we can pick out, in three different ways, a second hexagon which, like the first, may be represented by a cyclic arrangement of the six letters. Such a hexagon is $adfbec$, and, like the first hexagon, each vertex is the pole of the opposite plane with respect to the polarity $\Phi_2$. Moreover the hexagons $abcdef$ and $adfbec$ are mutually related to each other. Of the six vertices of each, four lie on lines, and two on the intersection of opposite planes of the other. Of the six planes of each, four pass through the lines, and two through the joins of opposite vertices of the other. Also four out of the six lines of each lie in planes and pass through vertices of the other.

To say that each vertex of a hexagon is the pole of the opposite plane with respect to a polarity $\Phi_2$ is to place nine linear conditions upon the polarity, and this determines it uniquely. Hence, as soon as one of the hexagons of our configuration is given, the polarity $\Phi_2$ is determined.†

* Cf. the author's paper in The Johns Hopkins University Circular, April, 1904.

† Dr. KASNER, in a paper on *The double-six configuration*, American Journal, vol. 25 (1903), calls attention to a polarity, $\Omega$, connected with the double-six. In a letter to Professor Morley, he shows that a skew-hexagon, self-conjugate as to $\Omega$ (and thus determining $\Omega$) may be

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I wish now to show that the first hexagon, *abcdef*, can be taken arbitrarily, and any plane of the second hexagon, *adfbec*, can be taken to pass through any point in space, and that the second hexagon and the entire $\Gamma^4_{6,3}$ will then be determined.

Suppose *abcdef* to have been chosen arbitrarily. We then make any plane, say *adfb*, of the second hexagon pass through the line *fab* and any given point $x$. Let *adfb* cut *def* in *df* and *bcd* in *bd*. Then draw

*bdf* (join of *bd* and *df*) cutting *fab* in *fb*,

*fbe* ( " " *fb* " *ef*) " *bcde* " *be*,

[*abe* ( " " *be* " *ab*) " *efa* " *ae*],

*bec* ( " " *be* " *bc*) " *cde* " *ec*,

*eca* ( " " *ec* " *ae*) " *abc* " *ca*,

and *cad* ( " " *ca* " *cd*) " *defa* " *ad*;

and if our hexagon is now to close properly, *ad* should lie in the plane *adfb* on a line with *fa* and *df*; and I shall show that this will always be the case.

The figure thus far constructed is part of the figure of two perspective tetrahedra, which we proceed to complete.

(In Fig. 1 the hexagon *abcdef* is shown by dotted lines, the hexagon *adfbec* by heavy lines, and the rest of the configuration by light lines. The double figure viewed through a stereoscope shows the figure in relief.)

Draw *cef* (join of *ec* and *ef*). Then the three lines *fbc* ($fb \cdot bc$), *fac* ($fa \cdot ca$), and *cdf* ($cd \cdot df$) will cut *cef*; for they lie respectively in the three planes *fbce* ($fb \cdot be \cdot ef \cdot bc \cdot ec$), *face* ($fa \cdot ae \cdot ef \cdot ca \cdot ec$), and *fcde* ($df \cdot de \cdot ef \cdot cd \cdot ec$) passing through *cef*. Moreover these three lines all cut *cef* at the same point, *cf*, because each pair of them lies in a plane which does not contain *cef*; viz., *fbc* and *fac* lie in *fabc* ($fa \cdot fb \cdot ab \cdot bc \cdot ca$); *fac* and *cdf* lie in *facd* ($fa \cdot df \cdot ad \cdot cd \cdot ac$); and *cdf* and *fbc* lie in *fbcd* ($fb \cdot df \cdot bd \cdot cd \cdot bc$). Also the line *bde* ($be \cdot de$) passes through *bd* since it lies in two planes containing *bd*, viz., the planes *bdef* ($be \cdot de \cdot fb \cdot df \cdot bd$) and *bcde* ($be \cdot de \cdot bc \cdot dc \cdot bd$).

Examination of the figure now shows that *ca bc cd cf* and *ae be de ef* are two perspective tetrahedra with *ec* as perspection-center. Five pairs of corresponding edges meet in the five points *bd*, *fb*, *df*, *fa*, and *ab*, points in the plane

selected in twenty different ways from the lines of the double-six. If $L_i$ and $M_i$ ($i = 1, 2, \cdots, 6$) are the lines of the double-six, then any corresponding triples, as $L_1$, $L_2$, $L_3$ and $M_1$, $M_2$, $M_3$ form such a hexagon. In this same letter Dr. KASNER suggests a construction for a polarity when defined by a hexagon. He reduces it to the construction of several plane polarities defined by plane pentagons.

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*adfb*. The sixth pair of edges, *cad* and *ade* (*ae*·*de*) must also meet in a point of this plane. But *cad* was to meet the plane *defa* in *ad*, and the line *ade* is

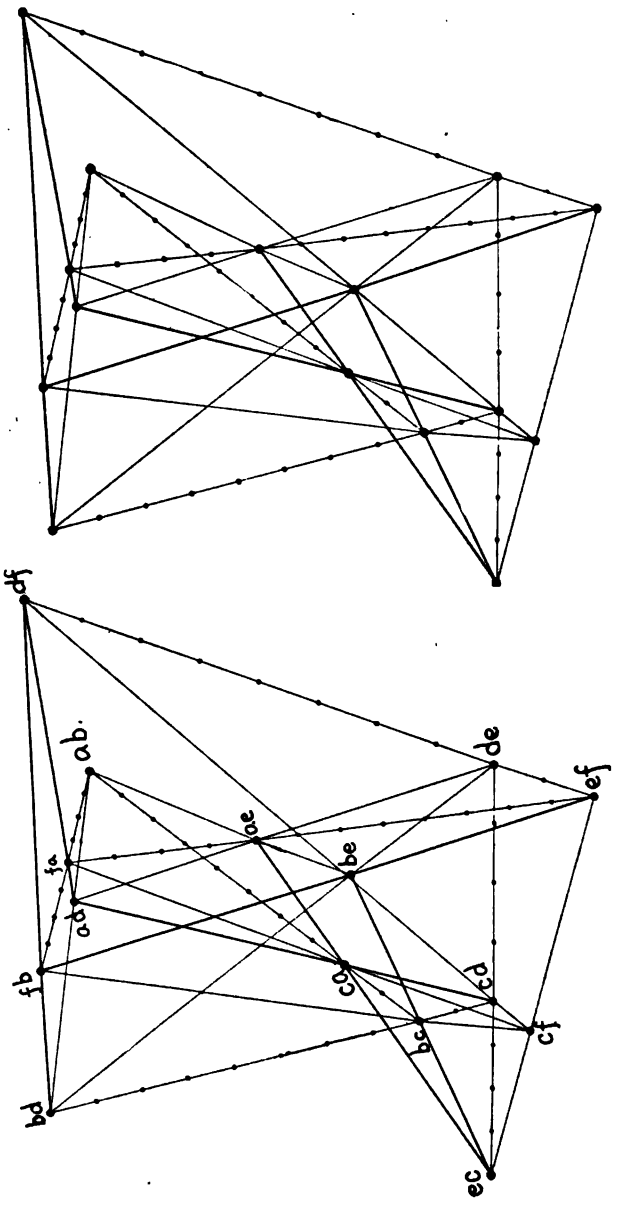


FIG. 1.

in the plane *defa*, and, therefore, *ad* lies in the plane *adfb* and on a line with *fa* and *df*. Hence our hexagon closes properly, and the entire $\Gamma^4_{6,3}$ is determined.

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This gives us now a method, similar to the method for the plane, of constructing the polar plane of any point $x$ with respect to a polarity when the latter is given by a self-polar hexagon. The vertex of the second hexagon opposite to the plane passed through $x$ (the vertex $ec$ in the case just described) must lie on the polar plane of $x$. The construction of this point $ec$ is in fact very simple, and may be stated as follows:

Let 1, 2, 3, 4, 5, 6 be the vertices of the given hexagon taken in order. Pass a plane through $x$, 1, and 2 and cutting $\overline{34}$ and $\overline{56}$ in $A$ and $A'$ respectively. Draw $\overline{AA'}$ cutting $\overline{12}$ in $B$, and $B6$ cutting the plane $\overline{345}$ in $C$. $\overline{C3}$ will then cut $\overline{45}$ in the point $D$ which is the point required on the polar plane of $x$. Now shift the digits around on the hexagon, replacing 2 by 1, 3 by 2, $\cdots$ 1 by 6, and repeat the above process, obtaining a second point $D'$, similarly for a third point $D''$. The plane $DD'D''$ will then be the polar of $x$ with respect to the polarity given by the hexagon.

A dual construction gives the polar point of a given plane.

The combination of two consecutive space polarities gives a space collineation, and hence a space collineation may be given by two skew-hexagons.

If a given tetrahedron with vertices $a$, $b$, $c$, and $d$, and faces $\alpha$, $\beta$, $\gamma$, and $\delta$ is to be the fixed tetrahedron of a collineation which is in addition to send a given point, $P$, into a given point, $P'$, this collineation may be given by the two skew hexagons whose vertices are

$$P,\ a,\ (\gamma\cdot\beta\cdot\epsilon),\ (\beta\cdot\alpha\cdot\epsilon),\ (\alpha\cdot\delta\cdot\epsilon),\ b$$

and

$$P',\ a,\ (\gamma\cdot\beta\cdot\epsilon),\ (\beta\cdot\alpha\cdot\epsilon),\ (\alpha\cdot\delta\cdot\epsilon),\ b$$

where $\epsilon$ is *any* plane.

The two hexagons $ABCDEF$ and $ABCDEF'$, where $F'$ lies on $\overline{EF}$, give a collineation in which the point $E$ and all points of the plane $\overline{ABC}$ are fixed points, while the plane $\overline{ABC}$ and all planes through $E$ are fixed planes. This collineation sends any point, $P$, into a point, $P'$, on the line $\overline{PE}$, such that the cross-ratio $P$, $P'$, $E$, $\overline{ABC}$ is equal to the cross-ratio $F$, $F'$ $E$, $\overline{ABC}$.

## § 3. *Associated systems of conics.*

The two dual configurations, $\Gamma^4_{6,2}$ and $C^4_{6,2}$ (which are respectively the section by a plane and the projection on a plane of the space configuration $\Gamma^4_{6,3}$) give rise to certain peculiar sets of conics which are worthy of notice.

All the elements of a $\Gamma^4_{6,2}$ containing a given letter of the six, say $a$, form a $\Gamma^3_{5,2}$* with which is associated a conic that may be called $F_a$. There are six such conics. The twenty points of the $\Gamma^4_{6,2}$ naturally fall into pairs such as $abc$

* Cf. § 1, theorem I.

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and *def*. The point *abc* is the pole of the line *adef* with respect to $F_a$, and since *def* lies on *adef*, *abc* and *def* are a pair of conjugate points with respect to $F_a$. Similarly, since *abc* and *bdef* are pole and polar with respect to $F_b$, *abc* and *def* are conjugate with respect to $F_b$. They are conjugate points, in fact, with respect to all six $F$'s. The same is true of *each of the ten pairs of points*. Regarding these ten point-pairs as degenerate line conics, and the $F$'s as point conics, we have then ten line conics each apolar with each of six point-conics.

This being the case, either (1) the ten conics belong to a range and the six to a 4-spread; or (2) the ten belong to a web and the six to a net; or (3) the ten belong to a 4-spread and the six to a pencil. The first supposition cannot be true, for there can be *only three degenerate* conics in a range. If the second alternative be true, the twenty points lie on a cubic curve; but this is impossible since they lie by fours on straight lines. Therefore the third supposition is true, and the six conics $F$ belong to a pencil, i. e., they pass through four points. Thus *every* $\Gamma^4_{6,2}$ *determines four covariant points.*

In the same way it may be shown that a $C^4_{6,2}$ determines six line conics, $\Phi_a, \cdots, \Phi_f$, belonging to a range; and hence *a* $C^4_{6,2}$ *determines four covariant lines.*

Every $\Gamma^\nu_{n,2}$ (where $\nu \geqq 3$ and $n \geqq \nu + 2$) contains $\binom{n}{n-5}\binom{n-5}{\nu-3}$ $\Gamma^3_{5,3}$'s, with each of which is associated a conic $F$.* We obtain one of these $\Gamma^3_{5,2}$'s by picking out all the elements of the $\Gamma^\nu_{n,2}$ which contain certain $\nu - 3$ letters, say $ab \cdots \kappa$, and do not contain certain $n - \nu - 2$ letters, say $mn \cdots s$. We may denote the conic associated with this $\Gamma^3_{5,2}$ by $F_{ab\ldots\kappa\cdot mn\ldots s}$. If $\nu \geqq 4$, the configuration $\Gamma^\nu_{n,2}$ contains $\binom{n}{n-6}\binom{n-6}{\nu-4}$ $\Gamma^4_{6,2}$'s, with each of which is associated a pencil of six of the conics $F$. The $\Gamma^4_{6,2}$, and therefore the associated pencil, may be denoted by the symbol $(b \cdots \kappa \cdot mn \cdots s)$ made up of two groups of $\nu - 4$ and $n - \nu - 2$ letters respectively. A given conic, as $F_{ab\ldots\kappa\cdot mn\ldots s}$ belongs to a given pencil, as $(b \cdots \kappa \cdot mn \cdots s)$, if the $n - \nu + 2$ letters in the two symbols are the same, and if the $\nu - 4$ letters in the symbol for the pencil are contained among the $\nu - 3$ letters in the symbol for the conic. We have then, connected with a $\Gamma^\nu_{n,2}$ (where $\nu \geqq 4$ and $n \geqq \nu + 2$), $\binom{n}{n-5}\binom{n-5}{\nu-3}$ conics lying in $\binom{n}{n-6}\binom{n-6}{\nu-4}$ pencils, six conics in each pencil, and each conic in $\nu - 3$ pencils.

Similarly, a $C^\mu_{n,2}$ (where $\mu \geqq 4$ and $n \geqq \mu + 2$) gives rise to $\binom{n}{n-5}\binom{n-5}{\mu-3}$ line-conics lying in $\binom{n}{n-6}\binom{n-6}{\mu-4}$ ranges, six conics in each range and each conic in $\mu - 3$ ranges.

Restating this last theorem in terms of the $\Gamma$'s instead of the $C$'s, and noting that $\mu = n - \nu + 1$, we have:

Associated with every $\Gamma^\nu_{n,2}$ (where $\nu \geqq 3$ and $n \geqq \nu + 3$) are $\binom{n}{n-5}\binom{n-5}{\nu-3}$ line-conics lying in $\binom{n}{n-6}\binom{n-6}{\nu-3}$ ranges, six conics in each range, and each conic in

* Cf. § 1, theorem IV.

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$n - \nu - 2$ ranges. But these $\binom{n}{n-5}\binom{n-5}{\nu-3}$ line-conics are the same as the point conics $F$ associated with the $\Gamma^{\nu}_{n,2}$. Hence we have the somewhat remarkable theorem:

*Associated with every* $\Gamma^{\nu}_{n,2}$ *(where* $\nu \geqq 4$ *and* $n \geqq \nu + 3$*) are* $\binom{n}{n-5}\binom{n-5}{\nu-3}$ *conics which lie by sixes in* $\binom{n}{n-6}\binom{n-6}{\nu-4}$ *pencils, each conic in* $\nu - 3$ *pencils, and which also lie by sixes in* $\binom{n}{n-6}\binom{n-6}{\nu-3}$ *ranges, each conic in* $n - \nu - 2$ *ranges.*

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# NOTES AND ERRATA: VOLUMES 5, 6

## VOLUME 5

E. D. ROE, JR.: *On the coefficients in the product* ⋯.

P. 197, l. 7. *For* p. 2 *read* p. 194.
In the table at the bottom of the page insert 00113 between 00023 and 00122.

P. 199. In the matrix of the $\beta$'s, for $m+n$ read $2m$. Later $n$ is taken equal to $m$.

P. 203, l. 1 up. *For* l. c. p. 2 *read* l. c. p. 194.
The recurrence formula stated in l. c. p. 194 gives coefficients as sums of those found in previous tables, as is seen by expressing the sums in the notation in which the lower series is also a partition.

H. F. BLICHFELDT: *Invariants of linear homogeneous groups* ⋯.

P. 466, l. 4. *For* $N = p^a m = kpl$ *read* $N = p^a m = p^a kl$.

P. 466, l. 6. *After* and then also *insert* by putting first $\alpha = 1$, then $\alpha = 2$, etc.

## VOLUME 6

E. V. HUNTINGTON: *A set of postulates for real algebra* ⋯.

Pp. 21, 22. The statement that postulates $R1$–$R8$ form a categorical set is clearly erroneous, as noted in the footnote on p. 211; a correct account is given on pp. 217–218. Since the statement in question was merely parenthetical, the rest of the paper is not affected by this correction.

P. 20 and p. 32. In postulate $R6$, the element $x$ in 2°) should be noted as "different from $X$," in order to make the proof of the independence of $R3$ conclusive.

E. D. ROE, JR.: *On the coefficients in the quotient* ⋯.

P. 63, formula (3). $\delta_r f(q_1 q_2 \cdots q_m) = \delta_r f(1^{p_n - p_{n-1}} 2^{p_{n-1} - p_{n-2}} \cdots n^{p_1})$, where $\delta_r$ is applied to the $p$'s and where $p_1 p_2 \cdots p_n$ is the partition conjugate to $q_1 q_2 \cdots q_m$. Thus any term of $\Sigma$ of (3) is excluded, which is inconsistent with the subtraction of $r$ single



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units from $r$ of the elements of the conjugate of the partition $p$.

P. 65, formula (10). Writing $p = 0^r p_1 p_2 \cdots p_n = z_1 z_2 \cdots z_{n+r}$, with $p$ conjugate to $q$,

$$\sigma_r f(\kappa_1 \kappa_2 \cdots \kappa_m) = \Sigma f(\kappa_1 + 1 \cdots \kappa_r + 1 \cdots \kappa_m)$$
$$= \Sigma f(\kappa_1' \kappa_2' \cdots \kappa_m'),$$

where any term presenting a greater before a lesser number in the complex $\kappa'$ is excluded,

$$\bar{\sigma}_r f(q_1 q_2 \cdots q_m) = \sigma_r f\left(0^\lambda 1^{z_{n+r} - z_{n+r-1}} \cdots (n+r)^{z_1}\right),$$

where $\sigma_r$ is applied to the $z$'s only, and where $\lambda$ is so chosen for a term that $\lambda + z'_{n+r} = m + 1$, the theorem of (10) is

$$a_r[q_1 q_2 \cdots q_m] = \bar{\sigma}_r[q_1 q_2 \cdots q_m]$$
$$= \sigma_r[0^\lambda 1^{z_{n+r} - z_{n+r-1}} \cdots (n+r)^{z_1}].$$

Thus any term is excluded from $\Sigma$ (10) which does not add $r$ single units to $r$ of the elements of $1^{q_m - q_{m-1}} \cdots m^{q_1}$ by the addition of $x_1 x_2 \cdots x_r$ to $r$ of the $q$'s.

In the left member of formula (10) *for* $q_r$ *read* $q_m$.

P. 68, l. 14 up. *For* 007 *read* 69.

P. 69, l. 4. " $t_2 t_4 - t_1 t_5$ " $t_2 t_{n-2} - t_1 t_{n-1}$.

P. 71, l. 9. The asterisk should be struck out, as also in the footnote at the bottom of the page, which is a continuation of the footnote of p. 70.

With respect to both product and quotient tables it is to be observed that when there is more than one self-conjugate among the partitions of a table, as in tables where $w > 7$, some of the coefficients are repeated in the table, since all the self-conjugates but one occur twice, equidistant from each end, according to the method of ordering partitions used in the tables. Where $w > 7$ the columns should also be numbered $0, 1, 2, 3, \cdots$, beginning with the center and proceding in each direction towards the ends, in order that conjugate columns may be immediately recognized, by the same number, without calculation or counting. —E. D. R.

L. E. Dickson: *Definitions of a group and a field* $\cdots$.

P. 203, l. 4. *For the heading* $[2^+]$ *read* $[2^\times]$.



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